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MIXING VIA FAMILIES FOR MEASURE PRESERVING TRANSFORMATIONS

BY

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Abstract. In topological dynamics a theory of recurrence properties via (Furstenberg) families was established in the recent years. In the current paper we aim to establish a corresponding theory of ergodicity via families in measurable dynamical systems (MDS). For a family $\mathcal{F}$ (of subsets of $\mathbb{Z}_+$) and a MDS $(X, \mathcal{B}, \mu, T)$, several notions of ergodicity related to $\mathcal{F}$ are introduced, and characterized via the weak topology in the induced Hilbert space $L^2(\mu)$.

T is $\mathcal{F}$-convergence ergodic of order k if for any $A_0, \dots, A_k$ of positive measure, $0 = e_0 < \dots < e_k$ and $\varepsilon > 0$, $\{n \in \mathbb{Z}_+ : |\mu(\bigcap_{i=0}^k T^{-ne_i} A_i) - \prod_{i=0}^k \mu(A_i)| < \varepsilon\} \in \mathcal{F}$. It is proved that the following statements are equivalent: (1) T is Δ^*-convergence ergodic of order 1; (2) T is strongly mixing; (3) T is Δ^*-convergence ergodic of order 2. Here Δ^* is the dual family of the family of difference sets.

1. Introduction. By a *topological dynamical system* (TDS) (X, T) we mean a compact metric space X together with a surjective continuous map T from X to itself. For a TDS (X, T) and non-empty open subsets U and V of X let $N(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n} V \neq \emptyset\}$, where $\mathbb{Z}_+$ is the set of non-negative integers. Note that we use $\mathbb{N}$ to denote the set of positive integers. It turns out that many recurrence properties of TDS can be described using the return time sets $N(U, V)$ (see [1], [8], [14], [12], [13] and [10]). For example, for a TDS (X, T) it is known that T is (topologically) *strongly mixing* iff $N(U, V)$ is cofinite, T is (topologically) *weakly mixing* iff $N(U, V)$ is thick [8], and T is (topologically) *mildly mixing* iff $N(U, V)$ is an $(IP\text{-}IP)^*$ set [14], [12] for each pair of non-empty open subsets U and V. Recently, Huang and Ye [14] showed that a minimal system (X, T) is weakly mixing iff the lower Banach density of $N(U, V)$ is 1, and (X, T) is mildly mixing iff $N(U, V)$ is an IP^*-set for each pair of non-empty open sets U and V.

By a *measurable dynamical system* (MDS) we mean $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a Lebesgue space and $T : X \to X$ is invertible and measure pre-

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serving. Many results on MDS and TDS share similar formulations, though the methods to prove them are quite different. For a MDS $(X, \mathcal{B}, \mu, T)$, let $\mathcal{B}^+ = \{B \in \mathcal{B} : \mu(B) > 0\}$ and $N(A, B) = \{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) > 0\}$ for $A, B \in \mathcal{B}^+$. The classical results in ergodic theory state that a transformation T is ergodic iff $N(A, B) \neq \emptyset$ for each pair of $A, B \in \mathcal{B}^+$; T is weakly mixing iff for each pair of measurable sets A, B there is a subset D of $\mathbb{Z}_+$ with density 1 such that $\lim_{n \in D, n \to \infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$; and T is mildly mixing iff IP^*-$\lim \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$ (see for example [19] and [9]).

We aim to establish a theory of ergodicity in MDS via families of subsets of $\mathbb{Z}_+$ as in topological dynamics. In the topological setup for a given family one naturally defines a notion of $\mathcal{F}$-transitivity. Unlike the topological case, we can associate several notions of ergodicity to a given family in the measure-theoretical case: $\mathcal{F}$-ergodicity, $\mathcal{F}$-positive ergodicity, $\mathcal{F}$-uniform positive ergodicity and $\mathcal{F}$-convergence ergodicity. We characterize these concepts via the weak topology in the associated Hilbert space $L^2(\mu)$. Moreover, high order mixing related to a family is discussed. In particular, it is proved that the following statements are equivalent: (1) T is Δ^*-convergence ergodic (of order 1); (2) T is strongly mixing; (3) T is Δ^*-convergence ergodic of order 2. Here $\Delta := \{F - F : F \subset \mathbb{Z}_+ \text{ is infinite}\}$ with $F - F := \{a - b > 0 : a, b \in F\}$ and Δ^* is the collection of subsets of $\mathbb{Z}_+$ which have non-empty intersection with each element in Δ.

As a by-product it is shown that for any MDS $(X, \mathcal{B}, \mu, T)$, any $A \in \mathcal{B}$ with positive measure and $\varepsilon > 0$, $\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon\} \in \Delta^*$; this strengthens a well known result of Khinchin, since a Δ^*-set is syndetic. We mention that in general $\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A \cap T^{-2n}A) > \mu(A)^3 - \varepsilon\} \in \Delta^*$ does not hold ([9, p. 177]) even for ergodic MDS, but the set $\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A \cap T^{-2n}A) > \mu(A)^3 - \varepsilon\}$ is syndetic [5] when T is ergodic.

After submission of the paper we got to know that Bergelson and Downarowicz have a paper [3] submitted to the same special volume and dealing with a similar topic. Though the results in both papers are almost complementary, they also have a strong connection. First, the stronger version of Khinchin's result is observed in both papers. Second, the results in this paper and in [17] answer some questions asked in the preliminary version of [3]. For details see Section 5.

The paper is organized as follows. In Section 2, we introduce necessary notations and ergodic concepts associated to a given family. In the following section we obtain some characterizations of the concepts via the weak topology in $L^2(\mu)$. In Section 4, we discuss high order mixing for the family Δ^*, and in the final section we outline how our results answer some questions asked in the preliminary version of [3].

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2. Some definitions. It was Furstenberg [8], [9] who first used subsets of $\mathbb{Z}_+$ to describe dynamical properties in a systematic way. For the recent results, see [1], [12], [10], [13] and [14].

Let us recall some notions related to Furstenberg families (for details see [1]). Let $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$ be the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is a *family* if it is upwards hereditary, that is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, i.e. neither empty nor all of $\mathcal{P}$. It is easy to see that $\mathcal{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Any subset $\mathcal{A}$ of $\mathcal{P}$ generates the family $[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A$ for some $A \in \mathcal{A}\}$. For a family $\mathcal{F}$, the *dual family* is

$$\mathcal{F}^* = \{F \in \mathcal{P} : \mathbb{Z}_+ \setminus F \notin \mathcal{F}\} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F}\}.$$

It is indeed a family, proper if $\mathcal{F}$ is. Clearly,

$$(\mathcal{F}^*)^* = \mathcal{F} \quad \text{and} \quad \mathcal{F}_1 \subset \mathcal{F}_2 \ \Rightarrow\ \mathcal{F}_2^* \subset \mathcal{F}_1^*.$$

Let $\mathcal{F}_{\text{inf}}$ be the family of all infinite subsets of $\mathbb{Z}_+$ and let $\mathcal{F}_{\text{c}} := \mathcal{F}_{\text{inf}}^*$. Note that $\mathcal{F}_{\text{c}}$ is the collection of all cofinite subsets of $\mathbb{Z}_+$. A family $\mathcal{F}$ is *full* if $F_1 \cap F_2 \in \mathcal{F}_{\text{inf}}$ for any $F_1 \in \mathcal{F}$ and $F_2 \in \mathcal{F}^*$. **All the families considered in this paper are assumed to be full**.

We say that a family $\mathcal{F}$ has the *Ramsey property* if whenever $F_1 \cup F_2 \in \mathcal{F}$, then either $F_1 \in \mathcal{F}$ or $F_2 \in \mathcal{F}$. If a proper family $\mathcal{F}$ is closed under intersection, then $\mathcal{F}$ is called a *filter*. One can show that $\mathcal{F}$ has the Ramsey property iff $\mathcal{F}^*$ is a filter [1]. Note that if $\mathcal{F}$ has the Ramsey property, then $F_1 \cap F_2 \in \mathcal{F}$ if $F_1 \in \mathcal{F}$ and $F_2 \in \mathcal{F}^*$. Since we need some special families to describe various ergodicity properties, we give some definitions.

Definition 2.1. Let S be a subset of $\mathbb{Z}_+$.

(1) The *lower density* and *upper density* of S are defined by

$$\underline{d}(S) = \liminf_{n\to\infty} \frac{1}{n}|S \cap [0, n-1]| \text{ and } \overline{d}(S) = \limsup_{n\to\infty} \frac{1}{n}|S \cap [0, n-1]|$$

respectively, where $[a, b]$ denotes the interval $\{a, a+1, a+2, \ldots, b\}$.

(2) If $\underline{d}(S) = \overline{d}(S) = d(S)$, then we say that the *density* of S is $d(S)$.

(3) The *lower Banach density* and *upper Banach density* of S are defined by

$$\mathrm{BD}_*(S) = \liminf_{|I|\to\infty} \frac{|S \cap I|}{|I|} \quad \text{and} \quad \mathrm{BD}^*(S) = \limsup_{|I|\to\infty} \frac{|S \cap I|}{|I|}$$

respectively, where I is taken over all finite intervals of $\mathbb{Z}_+$.

(4) $S = \{s_1 < s_2 < \cdots\}$ is *syndetic* if $\{s_{n+1} - s_n : n \in \mathbb{N}\}$ is bounded.

(5) S is *thick* if for any $L \in \mathbb{N}$ there exists some $N \in \mathbb{N}$ with $[N, N + L - 1] \subset S$.

From the definitions it is not hard to see that S is syndetic iff $\mathrm{BD}_*(S) > 0$, and S is thick iff $\mathrm{BD}^*(S) = 1$ (see [17]). We use $\mathcal{F}_{\mathrm{s}}$ and $\mathcal{F}_{\mathrm{t}}$ to denote the collections of syndetic sets and thick sets respectively, and $\mathcal{F}_{\mathrm{pud}}$, $\mathcal{F}_{\mathrm{d1}}$, $\mathcal{F}_{\mathrm{pubd}}$ and $\mathcal{F}_{\mathrm{lbd1}}$ to denote the collections of subsets of $\mathbb{Z}_+$ with positive upper density, density 1, positive upper Banach density and lower Banach density 1 respectively. It is clear that $\mathcal{F}_{\mathrm{s}}^* = \mathcal{F}_{\mathrm{t}}$, $\mathcal{F}_{\mathrm{pud}}^* = \mathcal{F}_{\mathrm{d1}}$ and $\mathcal{F}_{\mathrm{pubd}}^* = \mathcal{F}_{\mathrm{lbd1}}$. Also, it is easy to see that $\mathcal{F}_{\mathrm{d1}}$ and $\mathcal{F}_{\mathrm{lbd1}}$ are filters.

DEFINITION 2.2. Let S be a subset of $\mathbb{Z}_+$.

(1) S is called an *IP-set* if there is a subsequence $\{p_i\}_{i=1}^{\infty}$ in $\mathbb{N}$ such that all finite sums $p_{i_1} + \cdots + p_{i_j}$ with $i_1 < \cdots < i_j$, $j \in \mathbb{N}$, are in S. The collection of IP-sets is denoted by $\mathcal{F}_{\mathrm{ip}}$ and each element of $\mathcal{F}_{\mathrm{ip}}^*$ is called an *IP^*-set*.
(2) S is called a *Δ-set* if it contains an infinite difference set, i.e. there is a subsequence $F = \{p_1 < p_2 < \cdots\}$ of $\mathbb{Z}_+$ such that $S \supset \Delta(F) := \{p_i - p_j : i > j\}$. The collection of Δ-sets is denoted by Δ and each element of Δ^* is called a *Δ^*-set*.

It is well known that both $\mathcal{F}_{\mathrm{ip}}$ and Δ have the Ramsey property [4, 9], and

$$\mathcal{F}_{\mathrm{inf}} \supsetneq \Delta \supsetneq \mathcal{F}_{\mathrm{ip}} \supsetneq \mathcal{F}_{\mathrm{t}}, \qquad \mathcal{F}_{\mathrm{c}} \subsetneq \Delta^* \subsetneq \mathcal{F}_{\mathrm{ip}}^* \subsetneq \mathcal{F}_{\mathrm{s}}.$$

Recall that a MDS $(X, \mathcal{B}, \mu, T)$ is *ergodic* if $B \in \mathcal{B}$ and $T^{-1}B = B$ imply that $\mu(B) = 0$ or $\mu(B) = 1$; it is *weakly mixing* if the product system $T \times T$ is ergodic; it is *mildly mixing* if $B \in \mathcal{B}$ and $\liminf_n \mu((B \backslash T^{-n}B) \cup (T^{-n}B \backslash B)) = 0$ imply that $\mu(B) = 0$ or $\mu(B) = 1$; and it is *strongly mixing* if for any two sets $A, B \in \mathcal{B}$ we have $\mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$.

The other mixing properties we shall use are intermixing and partial mixing. Let $(X, \mathcal{B}, \mu, T)$ be a MDS. We define a function $\gamma : \mathcal{B}^+ \times \mathcal{B}^+ \to \mathbb{R}$ by

$$\gamma(A, B) := \liminf_n \frac{\mu(A \cap T^{-n}B)}{\mu(A)\mu(B)}$$

for $A, B \in \mathcal{B}^+$. A MDS $(X, \mathcal{B}, \mu, T)$ is called

- *intermixing* or *lightly mixing* if $\gamma(A, B) > 0$ for any $A, B \in \mathcal{B}^+$,
- *partially mixing* if $\inf_{A,B \in \mathcal{B}^+} \gamma(A, B) > 0$.

It is known (see for example [17]) that

strong mixing $\Rightarrow$ partial mixing $\Rightarrow$ intermixing
$\Rightarrow$ mild mixing $\Rightarrow$ weak mixing.

Recall that for a given family $\mathcal{F}$ a TDS is *$\mathcal{F}$-transitive* if $N(U,V) \in \mathcal{F}$ for each pair of non-empty open subsets U and V. In [17] the authors defined $\mathcal{F}$-ergodicity just as for a TDS. Studying this property we realized that, unlike the topological case, some other notions of ergodicity related to a given family are also useful, which we now introduce.

Definition 2.3. Let $(X, \mathcal{B}, \mu, T)$ be a MDS and let $\mathcal{F}$ be a family.

E1: T is *$\mathcal{F}$-ergodic* if for any $A, B \in \mathcal{B}^+$,

$$N(A,B) := \{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) > 0\} \in \mathcal{F};$$

E2: T is *$\mathcal{F}$-positively ergodic* ($\mathcal{F}$-p.ergodic) if for any $A, B \in \mathcal{B}^+$, there exists $\alpha = \alpha(A,B) > 0$ such that

$$\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) > \alpha\} \in \mathcal{F};$$

E3: T is *$\mathcal{F}$-uniformly positively ergodic* ($\mathcal{F}$-u.p.ergodic) if there exists $\alpha > 0$ such that for any $A, B \in \mathcal{B}^+$,

$$\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) > \alpha\mu(A)\mu(B)\} \in \mathcal{F};$$

E4: T is *$\mathcal{F}$-convergence ergodic* ($\mathcal{F}$-c.ergodic) if for any $A, B \in \mathcal{B}^+$ and $\varepsilon > 0$, $\{n \in \mathbb{Z}_+ : |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| < \varepsilon\} \in \mathcal{F}$, i.e.

$$\mathcal{F}\text{-}\lim_n \mu(A \cap T^{-n}B) = \mu(A)\mu(B).$$

It is clear that E1–E4 are successively stronger ergodic properties. In particular, for $\mathcal{F} = \mathcal{F}_{\mathrm{c}}$, it is known that E1 and E2 are both equivalent to intermixing (i.e. light mixing) [6, 17], E3 is equivalent to partial mixing, and E4 is just strong mixing. So E2, E3 and E4 are not equivalent [6, 7, 16, 18].

For $\mathcal{F} = \mathcal{F}_{\mathrm{inf}}$ it is clear that E1–E3 are equivalent to ergodicity, and E4 is strictly stronger than ergodicity. To see this, we note that a periodic system does not satisfy E4.

Recall that we have shown in [17] that T is weakly mixing iff $N(A,B) \in \mathcal{F}_t$ iff T is $\mathcal{F}_{\mathrm{lbd1}}$-c.ergodic; and T is mildly mixing iff $N(A,B) \in \mathcal{F}_{\mathrm{ip}}^*$ iff T is $\mathcal{F}_{\mathrm{ip}}^*$-c.ergodic. Thus, E1–E4 are all equivalent to weak mixing when $\mathcal{F} = \mathcal{F}_{\mathrm{t}}$, $\mathcal{F} = \mathcal{F}_{\mathrm{d1}}$ or $\mathcal{F} = \mathcal{F}_{\mathrm{lbd1}}$; and E1–E4 are all equivalent to mild mixing when $\mathcal{F} = \mathcal{F}_{\mathrm{ip}}^*$.

As $\mathcal{F}_{\mathrm{c}}$, $\mathcal{F}_{\mathrm{lbd1}}$ and IP^* are filters, many families we consider in this paper are filters or have the Ramsey property. Unfortunately, we do not know any family $\mathcal{F}$ for which E1 and E2 are not equivalent.

Finally, we give a simple property of E1 which was observed in [17].

Proposition 2.4. *Let $(X, \mathcal{B}, \mu, T)$ be a MDS and let $\mathcal{F}$ be a family. Then the following statements are equivalent*:

(1) *T is $\mathcal{F}^*$-ergodic.*
(2) *For any $F \in \mathcal{F}$ and any $A \in \mathcal{B}^+$, $\mu(\bigcup_{i \in F} T^{-i}A) = 1$.*

Proof. (1)⇒(2). Assume that there are $B \in \mathcal{B}^+$ and $F \in \mathcal{F}$ such that $\mu(\bigcup_{i\in F} T^{-i}B) < 1$. Let $A = (\bigcup_{i\in F} T^{-i}B)^{\mathrm{c}}$. Then $\mu(A) > 0$. Hence $\mu(A \cap T^{-i}B) = 0$ for any $i \in F$. As $F \cap N(A,B) \neq \emptyset$, there is $i \in N(A,B)$ such that $\mu(A \cap T^{-i}B) = 0$, a contradiction.

(2)⇒(1). If there are $A, B \in \mathcal{B}^+$ with $N(A,B) \notin \mathcal{F}^*$, then we have $F = \mathbb{Z}_+ \setminus N(A,B) \in \mathcal{F}$. Thus, $\mu(\bigcup_{i\in F} T^{-i}B) = 1$, and hence

$$\mu(A) = \mu\Big(A \cap \bigcup_{i\in F} T^{-i}B\Big) = \mu\Big(\bigcup_{i\in F} A \cap T^{-i}B\Big) = 0,$$

a contradiction. ∎

3. Characterizations of ergodicity related to a family. In this section we shall give characterizations of the four ergodic properties associated to a given family. Some of these characterizations will be used in the next section.

For a MDS $(X, \mathcal{B}, \mu, T)$ let $U_T : L^2(\mu) \to L^2(\mu)$ be the associated unitary operator. For a given $B \in \mathcal{B}$, a family $\mathcal{F}$ and $F \in \mathcal{F}$, we use $\mathrm{cl}_w^c U_B^F$ to denote the closure (with respect to the weak topology in $L^2(\mu)$, i.e. $f_n \to f$ if $\int f_n g\, d\mu \to \int fg\, d\mu$ for each $g \in L^2(\mu)$) of the convex set generated by $U_B^F := \{U_T^n 1_B : n \in F\}$. An element in the convex set has the form of $\sum_{i=1}^N \lambda_i U_T^{n_i} 1_B$ with $\lambda_i \geq 0$ and $\sum_{i=1}^N \lambda_i = 1$, where $n_i \in F$ and $N \in \mathbb{N}$. For each $f \in \mathrm{cl}_w^c U_B^F$, it is easy to see $0 \leq f \leq 1$ and $\int f\, d\mu = \mu(B)$. It turns out that we can use this kind of functions to characterize the different ergodic properties related to a given family. We start from the strongest property.

THEOREM 3.1. *Let $(X, \mathcal{B}, \mu, T)$ be a MDS and let $\mathcal{F}$ be a family with the Ramsey property. Then the following statements are equivalent*:

(1) *T is $\mathcal{F}^*$-c.ergodic.*
(2) *For each $B \in \mathcal{B}^+$ and $F \in \mathcal{F}$, there is a subsequence $\{n_i\}_{i=1}^\infty$ of F such that $U_T^{n_i} 1_B \to f_B = \mu(B)$.*
(3) *For each $B \in \mathcal{B}^+$ and $F \in \mathcal{F}$, there is a constant function $f_B \in \mathrm{cl}_w^c U_B^F$.*

Proof. (1)⇒(2). Let $\{A_i\}_{i=1}^\infty \subset \mathcal{B}$ be a countable base of $\mathcal{B}$, i.e. $\{A_i\}_{i=1}^\infty$ is dense in $\mathcal{B}$ with the metric $d(A,B) := \mu(A \triangle B)$. For a fixed $B \in \mathcal{B}^+$, let

$$D(i,\varepsilon) = \{n \in \mathbb{Z}_+ : |\mu(A_i \cap T^{-n}B) - \mu(A_i)\mu(B)| < \varepsilon\}.$$

It is clear that $D(i,\varepsilon) \in \mathcal{F}^*$. Fix $F \in \mathcal{F}$, and let $n_1 \in F \cap D(1,1)$. Since $\mathcal{F}^*$ is a filter, we can find $n_2 > n_1$ with $n_2 \in F \cap D(1,1/2) \cap D(2,1/2)$. If $n_1 < \cdots < n_i$ are defined, let $n_{i+1} > n_i$ with

$$n_{i+1} \in F \cap D\Big(1, \frac{1}{i+1}\Big) \cap \cdots \cap D\Big(i+1, \frac{1}{i+1}\Big).$$

So we get a subsequence $\{n_i\}$ of F. By choosing a subsequence again we can assume $U^{n_i}1_B \to f_B$ (weakly). It is clear that for each i,

$$\int 1_{A_i}(f_B - \mu(B))\, d\mu = 0.$$

This implies that $f_B = \mu(B)$ by a simple approximation argument.

(2)⇒(3) is obvious.

(3)⇒(1). It is easy to see that $f_B = \mu(B)$. If (1) is not true, then we have $\{n \in \mathbb{Z}_+ : |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| \geq \varepsilon\} \in \mathcal{F}$ for some $\varepsilon > 0$. As $\mathcal{F}$ has the Ramsey property, we may assume that $F := \{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) \geq \mu(A)\mu(B)+\varepsilon\} \in \mathcal{F}$. Then each $f \in \mathrm{cl}_w^c\, U_B^F$ satisfies $\int 1_A \cdot f d\mu \geq \mu(A)\mu(B)+\varepsilon$. This contradicts the assumption that $\mu(B) \in \mathrm{cl}_w^c\, U_B^F$. ∎

For the $\mathcal{F}$-u.p.ergodicity we have the analogous result and the proof is similar.

THEOREM 3.2. *Let $(X, \mathcal{B}, \mu, T)$ be a MDS and let $\mathcal{F}$ be a family with the Ramsey property. Then the following statements are equivalent*:

(1) *T is $\mathcal{F}^*$-u.p.ergodic.*
(2) *There exists $\alpha > 0$ such that for each $B \in \mathcal{B}^+$ and $F \in \mathcal{F}$, there is a subsequence $\{n_i\}_{i=1}^{\infty}$ of F such that $U_T^{n_i}1_B \to f_B \geq \alpha\mu(B)$.*
(3) *There exists $\alpha > 0$ such that for each $B \in \mathcal{B}^+$ and $F \in \mathcal{F}$, there is a function $f_B \in \mathrm{cl}_w^c\, U_B^F$ with $f_B \geq \alpha\mu(B)$.*

In the above theorems we need the assumption that $\mathcal{F}^*$ is a filter. For example, without this condition in Theorem 3.1, (3) can only imply that both $\{n \in \mathbb{Z}_+ : \mu(A\cap T^{-n}B) > \mu(A)\mu(B)-\varepsilon\}$ and $\{n \in \mathbb{Z}_+ : \mu(A\cap T^{-n}B) < \mu(A)\mu(B) + \varepsilon\}$ are in $\mathcal{F}^*$.

Now we turn to characterizations of $\mathcal{F}^*$-p.ergodicity and $\mathcal{F}^*$-ergodicity. Let $(X, \mathcal{B}, \mu)$ be a probability measure space. We call a collection $\mathcal{H} \subset \mathcal{B}$ *hereditary* if whenever $A \in \mathcal{H}$ and $A \supset B \in \mathcal{B}$ then also $B \in \mathcal{H}$. We say that the hereditary collection $\mathcal{H}$ *saturates* $\mathcal{B}$ if for every $A \in \mathcal{B}^+$, there exists $B \in \mathcal{H} \cap \mathcal{B}^+$ with $B \subset A$. There is an important property concerning this collection: If $\mathcal{H}$ is a hereditary collection which saturates $\mathcal{B}$ then there exists a countable measurable partition $\xi = \{A_i : i \in \mathbb{N}\}$ of X, with $A_i \in \mathcal{H}$ for every i. See [11, p. 69] for a proof. Using this result we can show:

THEOREM 3.3. *Let $(X, \mathcal{B}, \mu, T)$ be a MDS and let $\mathcal{F}$ be a family. Then the following statements are equivalent*:

(1) *T is $\mathcal{F}^*$-positively ergodic.*
(2) *For each $B \in \mathcal{B}^+$ and $F_i \in \mathcal{F}$ with $F_1 \supset F_2 \supset \cdots$, there exists $f_B \in \bigcap_i \mathrm{cl}_w^c\, U_B^{F_i}$ with $f_B > 0$ a.e. $x \in X$.*

Proof. (1)⇒(2). Let

$$\mathcal{H} = \Big\{ A \in \mathcal{B} : \text{there exists } f \in \bigcap_i \operatorname{cl}_w^c U_B^{F_i} \text{ with } f(x) > 0 \text{ a.e. } x \in A \Big\}.$$

Then $\mathcal{H}$ satisfies:

(i) If $A \in \mathcal{H}$ and $A \supset C \in \mathcal{B}$ then also $C \in \mathcal{H}$.
(ii) For each $A \in \mathcal{B}^+$, there exists $C \in \mathcal{H}$ with $C \subset A$ and $\mu(C) > 0$.

(i) is obvious. To see (ii), we consider sets $A, B \in \mathcal{B}^+$. Since T is $\mathcal{F}^*$-p.ergodic there is $\delta(A,B) > 0$ with $\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) > \delta\} \in \mathcal{F}^*$. Let

$$E_i = F_i \cap \{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) > \delta\} \subset F_i.$$

Then $\{E_i\}$ is a decreasing sequence. Choose $f \in \bigcap_i \operatorname{cl}_w^c U_B^{E_i} \neq \emptyset$. It is clear that $\int_A f\,d\mu \geq \delta > 0$. Let $C := \{x \in A : f(x) > 0\}$. Then $C \in \mathcal{B}^+ \cap \mathcal{H}$.

So there exists a countable partition $\xi = \{A_k : k \in \mathbb{N}\}$ of X with $A_k \in \mathcal{H}$ for every k. Assume f_k is the function corresponding to A_k. Then $f_B := \sum_k 2^{-k} f_k \in \bigcap_i \operatorname{cl}_w^c U_B^{F_i}$ and $f_B > 0$ for a.e. $x \in X$.

(2)⇒(1). Assume (1) is false. Then there are $A, B \in \mathcal{B}^+$ such that for any i we have

$$F_i := \{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) \leq 1/i\} \in \mathcal{F}.$$

It is clear that $F_1 \supset F_2 \supset \cdots$. By (2) we can find $f_B \in \bigcap_i \operatorname{cl}_w^c U_B^{F_i}$ with $f_B > 0$. So $0 < \int 1_A \cdot f_B\,d\mu = (1_A, f_B) \leq 1/i \to 0$, a contradiction. ∎

For $\mathcal{F}^*$-ergodicity we have:

THEOREM 3.4. *Let $(X, \mathcal{B}, \mu, T)$ be a MDS and let $\mathcal{F}$ be a family. Then the following statements are equivalent*:

(1) *T is $\mathcal{F}^*$-ergodic.*
(2) *For each $B \in \mathcal{B}^+$ and $F \in \mathcal{F}$, there exists $f_B \in \operatorname{cl}_w^c U_B^F$ with $f_B > 0$ for a.e. $x \in X$.*

Proof. (1)⇒(2). By Proposition 2.4, for each $B \in \mathcal{B}^+$ and $F = \{n_k : k \in \mathbb{N}\} \in \mathcal{F}$ we have $\mu(\bigcup_k T^{-n_k}B) = 1$. Let $f_B := \sum_k 2^{-k} 1_{T^{-n_k}B}$. It is easy to see $f_B \in \operatorname{cl}_w^c U_B^F$ and $f_B > 0$ a.e. $x \in X$.

(2)⇒(1). Assume (1) is false. Then there are $A, B \in \mathcal{B}^+$ such that

$$F := \{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) = 0\} \in \mathcal{F}.$$

By (2) we can find $f_B \in \operatorname{cl}_w^c U_B^F$ with $f_B > 0$. So $0 < \int 1_A \cdot f_B\,d\mu = (1_A, f_B) = 0$, a contradiction. ∎

4. Strong mixing and high order mixing related to Δ^*. In this section we consider the ergodicity related to Δ^* and the high order mixing property. It is shown that Δ^*-c.ergodicity, strong mixing and Δ^*-c.ergodicity of order 2 are equivalent. The questions whether Δ^*-ergodicity implies inter-mixing, or whether Δ^*-ergodicity and Δ^*-p.ergodicity are equivalent remain open.

Recall that a subset F of $\mathbb{Z}_+$ is a *Poincaré sequence* if for any MDS $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}^+$, there is $0 \neq n \in F$ with $\mu(A \cap T^{-n}A) > 0$. It is known that every Δ-set is a Poincaré sequence [9]. So $N(A, A)$ is a Δ^*-set for any $A \in \mathcal{B}^+$. Khinchin had shown that $\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon\}$ is syndetic [15]. Recently Bergelson, Host and Kra got a similar result for 3-fold and 4-fold cases: for any ergodic MDS $(X, \mathcal{B}, \mu, T)$, $A \in \mathcal{B}$ and $\varepsilon > 0$, the sets

$$\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A \cap T^{-2n}A) > \mu(A)^3 - \varepsilon\}$$

and

$$\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A \cap T^{-2n}A \cap T^{-3n}A) > \mu(A)^4 - \varepsilon\}$$

are both syndetic [5]. The referee pointed out that Proposition 4.1 below, which can be seen as a generalization of Khinchin's result, is in fact essentially contained in [2, p. 49] (see also [3]). To see the connection with Theorem 4.4 and for completeness we include a proof which is different from the one given in [2].

PROPOSITION 4.1. *Let $(X, \mathcal{B}, \mu, T)$ be a MDS, $\varepsilon > 0$ and $A \in \mathcal{B}^+$. Then*

$$\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon\} \in \Delta^*.$$

Proof. Assume to the contrary that there are $A \in \mathcal{B}^+$ and $\varepsilon > 0$ such that

$$\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon\} \notin \Delta^*.$$

That is, there is a sequence $\{n_i\}$ with

$$\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A) \leq \mu(A)^2 - \varepsilon\} \supset \{n_j - n_i : i < j\}.$$

We may assume $U_T^{n_i} 1_A \to f_A$ (weakly). By the Cauchy–Schwarz inequality we have $(f_A, f_A) \geq (\int f_A \, d\mu)^2 = \mu(A)^2$. But at the same time,

$$(f_A, f_A) = \lim_i \lim_j (U_T^{n_i} 1_A, U_T^{n_j} 1_A) = \lim_i \lim_j (1_A, U_T^{n_j - n_i} 1_A) \leq \mu(A)^2 - \varepsilon,$$

contradiction. ■

REMARK 4.2. In [9] Furstenberg constructed a minimal TDS (X, T) and a non-empty open set A with

$$\{n \in \mathbb{Z}_+ : A \cap T^{-n}A \cap T^{-2n}A \neq \emptyset\} \notin \Delta^*.$$

Thus for any invariant probability Borel measure μ on (X, T) and $0 < \varepsilon < \mu(A)^3$,

$$\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}A \cap T^{-2n}A) > \mu(A)^3 - \varepsilon\} \notin \Delta^*.$$

If we strengthen the assumption in Proposition 4.1, we can conclude that T is strongly mixing. To do this, we need a property related to Δ whose proof can be found in [9].

PROPOSITION 4.3. *Let $F=\{p_1<p_2<\cdots\}\subset\mathbb{Z}_+$ and let $S=\Delta(F)\in\Delta$. If $S=S_1\cup S_2$, then there is a subsequence $F_1=\{p_{i_1}<p_{i_2}<\cdots\}$ of F such that $S_1\supset\Delta(F_1)$ or $S_2\supset\Delta(F_1)$. In particular, Δ has the Ramsey property.*

Now we are ready to show

THEOREM 4.4. *Let $(X,\mathcal{B},\mu,T)$ be a MDS. If for any $\varepsilon>0$ and $A\in\mathcal{B}^+$,*

$$\{n\in\mathbb{Z}_+ : |\mu(A\cap T^{-n}A)-\mu(A)^2|<\varepsilon\}\in\Delta^*,$$

then T is strongly mixing. In particular, Δ^*-c.ergodicity implies strong mixing.

Proof. By Theorem 3.1 it remains to show that for each $B\in\mathcal{B}^+$ and each $F\in\mathcal{F}_{\mathrm{inf}}$, there exists a subsequence $\{n_i\}_{i=1}^\infty\subset F$ with $U_T^{n_i}1_B\to f_B=\mu(B)$ (weakly). Thus we assume $\lim_{n\in F}U_T^n 1_B=f_B$ (weakly) and will show $f_B=\mu(B)$. By Proposition 4.3 and the assumption there exists $F_1\subset F$ with

$$F_1-F_1\subset(F-F)\cap\{n\in\mathbb{Z}_+ : |\mu(B\cap T^{-n}B)-\mu(B)^2|<1/2\}.$$

Now assume $F_1\supset\cdots\supset F_k$ have been chosen. We can find $F_{k+1}\subset F_k$ with

$$F_{k+1}-F_{k+1}\subset(F_k-F_k)\cap\{n\in\mathbb{Z}_+ : |\mu(B\cap T^{-n}B)-\mu(B)^2|<1/2^{k+1}\}.$$

Thus we have $|\mu(T^{-a}B\cap T^{-b}B)-\mu(B)^2|<1/2^k$ for any $a,b\in F_k$ with $a\neq b$. Denote F_k by $\{n_i^k\}_{i=1}^\infty$ and form a new subsequence $\{n_1^1,n_2^2,n_3^3,\ldots\}$. Write it as $\{n_i\}_{i=1}^\infty$ and assume $n_1<n_2<\cdots$ by deleting some elements. Since $\lim_i U_T^{n_i}1_B=f_B$ (weakly), we have

$$(f_B,f_B)=\lim_i\lim_j(U_T^{n_i}1_B,U_T^{n_j}1_B)\le\mu(B)^2+\lim_i\frac{1}{2^i}=\mu(B)^2=\Big(\int f_B\,d\mu\Big)^2.$$

This implies $f_B=\mu(B)$ by using the Cauchy–Schwarz inequality. ■

As there is a partially mixing system which is not strongly mixing [7], Δ^*-c.ergodicity is strictly stronger than Δ^*-u.p.ergodicity. Checking the example in [6], we see that it is intermixing but not Δ^*-u.p.ergodic. So Δ^*-u.p.ergodicity is strictly stronger than Δ^*-p.ergodicity. We do not know whether Δ^*-ergodicity and Δ^*-p.ergodicity are equivalent.

We have proved that Δ^*-c.ergodicity is equivalent to strong mixing. In the following we shall show that strong mixing implies Δ^*-c.ergodicity of order 2. We start from the following definition.

DEFINITION 4.5. A MDS $(X,\mathcal{B},\mu,T)$ is called *$\mathcal{F}$-c.ergodic of order k* if for any $k+1$ sets $A_0,A_1,\ldots,A_k\in\mathcal{B}$ and integers $0<e_1<\cdots<e_k$,

$$\mathcal{F}\text{-}\lim_n\mu(A_0\cap T^{-ne_1}A_1\cap\cdots\cap T^{-ne_k}A_k)=\mu(A_0)\mu(A_1)\cdots\mu(A_k).$$

If $\mathcal{F}$ is a filter we can get the following characterization by a similar argument to the proof of Theorem 3.1:

T is $\mathcal{F}$-c.ergodic of order k iff for any k sets $A_1, \dots, A_k \in \mathcal{B}$, integers $0 < e_1 < \dots < e_k$ and $F \in \mathcal{F}^*$, there is a subsequence $\{n_i\}_{i=1}^{\infty}$ of F such that

$$\lim_i (T^{n_i e_1} 1_{A_1}) \cdots (T^{n_i e_k} 1_{A_k}) = \mu(A_1) \cdots \mu(A_k) \quad \text{(weakly)}.$$

Does $\mathcal{F}$-c.ergodicity imply higher order $\mathcal{F}$-c.ergodicity? This is a long standing open question known as the Rokhlin conjecture for $\mathcal{F} = \mathcal{F}_{\mathrm{c}}$. In [9] the author proved that it is true for the families $\mathcal{F} = \mathcal{F}_{\mathrm{d1}}$ and $\mathcal{F} = \mathcal{F}_{\mathrm{ip}}^*$. Since Δ^* is a family close to $\mathcal{F}_{\mathrm{c}}$, it is natural to ask: What is the situation when $\mathcal{F} = \Delta^*$? For this family we have:

THEOREM 4.6. *Strong mixing implies Δ^*-c.ergodicity of order* 2.

We remark that due to the limitation of our method which is very close to the one used in [9] the proof cannot be used for the case of order $k \geq 3$. We start from the following two lemmas.

LEMMA 4.7. *Let $Q \in \Delta^*$ and $S \in \Delta$. For each $q \in Q$ let $R_q \in \mathcal{F}_{\mathrm{c}}$. Then for each given $k \geq 1$ there exist $n_1 < \dots < n_k$ in S such that $n_j - n_i \in Q$ for $i < j$ and $n_i \in R_{n_j - n_i}$.*

Proof. Since Δ has the Ramsey property, $Q \cap S \in \Delta$. There exist $m_1 < m_2 < \cdots$ such that $m_j - m_i \in Q \cap S$ for $i < j$. For fixed $q_1 = m_{i_2} - m_{i_1} \in Q \cap S$ we choose $i_3 > i_2$ and $q_2 = m_{i_3} - m_{i_2}$ such that $q_2, q_2 + q_1 \in R_{q_1}$. It is clear that $q_2, q_2 + q_1 \in Q \cap S$. Assume $i_1, \dots, i_r$ and $q_1, \dots, q_{r-1}$ have been found. We choose $i_{r+1} > i_r$ and $q_r = m_{i_{r+1}} - m_{i_r}$ such that

$$q_r, q_r + q_{r-1}, \dots, q_r + q_{r-1} + \dots + q_1 \in \bigcap_{1 \leq s \leq t \leq r-1} R_{q_s + q_{s+1} + \dots + q_t}.$$

It is clear that $q_r, q_r + q_{r-1}, \dots, q_r + q_{r-1} + \dots + q_1 \in Q \cap S$. Continuing in this way we find $q_1, \dots, q_k$. Now we set $n_1 = q_k,\ n_2 = q_k + q_{k-1},\ \dots,\ n_k = q_k + q_{k-1} + \dots + q_1$. It is clear that $\{n_1 < \dots < n_k\} \subset S$. At the same time, we have

$$n_j - n_i = q_{k-i} + \dots + q_{k-j+1} \in Q \cap S \quad \text{for } 1 \leq i < j \leq k,$$

$$n_i = q_k + \dots + q_{k-i+1} \in \bigcap_{1 \leq s \leq t \leq k-1} R_{q_s + q_{s+1} + \dots + q_t} \subset R_{q_{k-i} + \dots + q_{k-j+1}} = R_{n_j - n_i}.$$

This completes the proof. ∎

LEMMA 4.8. *Let $\{x_n\}$ be a bounded sequence of vectors in Hilbert space and suppose that*

$$\Delta^*\text{-}\lim_m (\mathcal{F}_{\mathrm{c}}\text{-}\lim_n \langle x_{n+m}, x_n \rangle) = 0.$$

Then with respect to the weak topology, $\Delta^*\text{-}\lim_n x_n = 0$.

Proof. Let x be some vector and suppose that $S := \{n : \langle x_n, x\rangle > \varepsilon\} \in \Delta$ for some $\varepsilon > 0$. We assume for convenience that the Hilbert space is over the reals. We have $x \neq 0$ and for $\delta < \varepsilon^2/\|x\|^2$, let

$$Q = \{m : \mathcal{F}_{\rm c}\text{-}\lim_n \langle x_{n+m}, x_n\rangle) < \delta/2\}.$$

Then $Q \in \Delta^*$ and for each $q \in Q$, $R_q = \{n : \langle x_{n+q}, x_n\rangle) < \delta\} \in \mathcal{F}_{\rm c}$. Apply Lemma 4.7 to these sets with k to be specified later. If $n_1, \ldots, n_k$ satisfy the conclusion of Lemma 4.7, then

(i) $\langle x_{n_i}, x\rangle > \varepsilon$, $1 \le i \le k$,
(ii) $\langle x_{n_i}, x_{n_j}\rangle < \delta$, $1 \le i < j \le k$.

Set $y_i = x_{n_i} - \varepsilon x/\|x\|^2$. Then

$$\langle y_i, y_j\rangle < \delta - \frac{2\varepsilon^2}{\|x\|^2} + \frac{\varepsilon^2}{\|x\|^2} = \delta - \frac{\varepsilon^2}{\|x\|^2} < 0, \quad 1 \le i < j \le k.$$

But since the y_i are bounded independently of k, and

$$0 \le \Big\| \sum_{i=1}^k y_i \Big\|^2 = \sum_{i=1}^k \|y_i\|^2 + 2\sum_{i<j}\langle y_i, y_j\rangle \le k \max \|y_i\|^2 - k(k-1)\left(\frac{\varepsilon^2}{\|x\|^2} - \delta\right),$$

we arrive at a contradiction if k is chosen sufficiently large. ∎

Now we are ready to give

Proof of Theorem 4.6. It remains to show that

$$\Delta^*\text{-}\lim_n \mu(A_0 \cap T^{-ne_1}A_1 \cap T^{-ne_2}A_2) = \mu(A_0)\mu(A_1)\mu(A_2).$$

Let

$$a_n(x) = 1_{A_1}(T^{ne_1}x)1_{A_2}(T^{ne_2}x) - \mu(A_1)\mu(A_2).$$

We will show that $\Delta^*\text{-}\lim_n a_n = 0$ with respect to the weak topology. Since T is strongly mixing we have

$$\begin{aligned}
&\lim_m \lim_n \langle a_{n+m}, a_n\rangle \\
&\quad= \lim_m \lim_n \int 1_{A_1}(T^{(n+m)e_1}x)1_{A_2}(T^{(n+m)e_2}x)1_{A_1}(T^{ne_1}x)1_{A_2}(T^{ne_2}x)\, d\mu \\
&\qquad - \mu(A_1)^2\mu(A_2)^2 \\
&\quad= \lim_m \lim_n \int 1_{A_1}(T^{me_1}x)1_{A_1}(x)1_{A_2}(T^{n(e_2-e_1)+me_2}x)1_{A_2}(T^{n(e_2-e_1)}x)\, d\mu \\
&\qquad - \mu(A_1)^2\mu(A_2)^2 \\
&\quad= \lim_m \Big(\int 1_{A_1}(T^{me_1}x)1_{A_1}(x)\, d\mu\Big)\Big(\int 1_{A_2}(T^{me_2}x)1_{A_2}(x)\, d\mu\Big) \\
&\qquad - \mu(A_1)^2\mu(A_2)^2 \\
&\quad= \int 1_{A_1} d\mu \int 1_{A_1}\, d\mu \int 1_{A_2}\, d\mu \int 1_{A_2}\, d\mu - \mu(A_1)^2\mu(A_2)^2 = 0.
\end{aligned}$$

Thus, $\mathcal{F}_{\mathrm{c}}\text{-}\lim_m(\mathcal{F}_{\mathrm{c}}\text{-}\lim_n\langle a_{n+m}, a_n\rangle) = 0$. By Lemma 4.8 we know $\Delta^*\text{-}\lim_n a_n = 0$ in the weak topology. This proves the theorem. ∎

We remark that by similar arguments we can prove that strong mixing of order k implies Δ^*-c.ergodicity of order $k+1$ for any $k \geq 1$. We do not know whether strong mixing implies Δ^*-c.ergodicity of order k for any $k \geq 3$.

QUESTION 4.9. *Does Δ^*-c.ergodicity of order 2 imply strong mixing of order 2? Generally, does Δ^*-c.ergodicity of order k imply strong mixing of order k for each $k \geq 2$?*

Affirmative answers to these questions will answer the Rokhlin conjecture affirmatively by the above remark.

5. Applications. In this section we will use the results of Section 4 and of [17] to answer some questions asked in the preliminary version of [3].

Let $(X, \mathcal{B}, \mu, T)$ be a MDS. Given $\varepsilon > 0$ and $A, B \in \mathcal{B}^+$, the set of *fat intersection* is defined in [3] as follows:

$$R^{\varepsilon}_{A,B} = \{n \in \mathbb{Z} : \mu(A \cap T^n B) > \mu(A)\mu(B) - \varepsilon\}.$$

A simple observation is that if $R^{\varepsilon}_{A,B} \in \mathcal{F}$ for any $A, B \in \mathcal{B}^+$ with $\mathcal{F}$ given, then $\{n \in \mathbb{Z} : \mu(A \cap T^n B) < \mu(A)\mu(B) + \varepsilon\} \in \mathcal{F}$.

For a given family $\mathcal{F}$ let $\mathcal{F}_+ = \bigcup_{k\in\mathbb{Z}}(\mathcal{F} + k)$ and $\mathcal{F}_\bullet = \bigcap_{k\in\mathbb{Z}}(\mathcal{F} + k)$. To simplify the notations let $\mathcal{F}^*_+ = (\mathcal{F}^*)_+$ and $\mathcal{F}^*_\bullet = (\mathcal{F}^*)_\bullet$.

One of the questions asked in the preliminary version of [3] is whether the requirement that all sets $R^{\varepsilon}_{A,B}$ are in $\Delta^*_\bullet$ yields a class of systems situated strictly between mild mixing and strong mixing. By Theorem 4.4 we see that the requirement is equivalent to strong mixing since $\Delta^*_\bullet \subset \Delta^*$. So we have the following observation communicated to us by T. Downarowicz.

PROPOSITION 5.1. *The requirement that all sets $R^{\varepsilon}_{A,B}$ are in $\Delta^*_\bullet$ does not yield a class of systems situated strictly between mild mixing and strong mixing. In fact, the requirement is equivalent to strong mixing.*

Let $\mathcal{C}$ be the family consisting of central sets [9], [3]. Since $\mathcal{C}$ has the Ramsey property and $\mathcal{C} \subset \mathcal{F}_{\mathrm{ip}} \subset \Delta$ (see [9]) we have

$$\mathcal{C}^*_\bullet \subset \mathcal{C}^* \subset \mathcal{C} \subset \mathcal{F}_{\mathrm{ip}} \subset \Delta.$$

Theorem 3.1 in [17] states that T is weakly mixing iff $\{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) > 0\}$ is a recurrence set for any $A, B \in \mathcal{B}^+$. Recall that a subset S of $\mathbb{N}$ is a *recurrence set* if for any TDS (X, T) there are $x \in X$ and a subsequence $\{s_i\}$ of S with $T^{s_i}x \to x$ ([20]). Another question asked in the preliminary version of [3] is whether the requirement that all sets $R^{\varepsilon}_{A,B}$ are in $\mathcal{C}_\bullet^*$ generates a notion of "mixing" weaker than weak mixing. Since a Δ-set is a recurrence set, we have

PROPOSITION 5.2. *The requirement that all sets $R^\varepsilon_{A,B}$ are in $\mathcal{C}_\bullet^*$ does not generate a class of systems weaker than weak mixing. In fact, the requirement is equivalent to weak mixing.*

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