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ABSOLUTELY CONTINUOUS, INVARIANT MEASURES FOR DISSIPATIVE, ERGODIC TRANSFORMATIONS

BY

JON AARONSON and TOM MEYEROVITCH (Tel Aviv)

Abstract. We show that a dissipative, ergodic measure preserving transformation of a σ-finite, non-atomic measure space always has many non-proportional, absolutely continuous, invariant measures and is ergodic with respect to each one of these.

0. Introduction. Let $(X, \mathcal{B}, m, T)$ be an invertible, ergodic measure preserving transformation of a σ-finite measure space. Then there are no other σ-finite, m-absolutely continuous, T-invariant measures other than constant multiples of m, because the density of any such measure is T-invariant, whence constant by ergodicity.

When T is not invertible, the situation becomes more complicated.

If $(X, \mathcal{B}, m, T)$ is a conservative, ergodic, measure preserving transformation of a σ-finite measure space, then (again) there are no other σ-finite, m-absolutely continuous, T-invariant measure other than constant multiples of m (see e.g. Theorem 1.5.6 in [A]). When T is not conservative, the situation is different.

In this note, we show (Proposition 1) that a dissipative measure preserving transformation has many non-proportional, σ-finite, absolutely continuous, invariant measures.

If the dissipative measure preserving transformation is ergodic (exact), then it is also ergodic (exact) with respect to each of these σ-finite, absolutely continuous, invariant measures (Proposition 2).

Proposition 1 was known for certain examples: the "Engel series transformation" (see [T], also [S1]); the one-sided shift of a random walk on a polycyclic group with centered, adapted jump distribution (ergodicity follows from [K], existence of non-proportional invariant densities follows from [B-E]); and the Euclidean algorithm transformation (see [D-N] which inspired this note). More details are given in §2.

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§1 is devoted to results (statements and proofs) and §2 has examples of ergodic, dissipative measure preserving transformations.

To conclude this introduction, we consider

An illustrative example. Fix $q \in (0,1)$ and consider the stochastic matrix $p : \mathbb{Z} \times \mathbb{Z} \to [0,1]$ defined by $p_{s,s} := 1-q$, $p_{s,s+1} := q$ and $p_{s,t} = 0$ for $t \neq s, s+1$. Let $(X, \mathcal{B}, m, T)$ be the *one-sided Markov shift* with $X := \mathbb{Z}^{\mathbb{N}}$, $\mathcal{B}$ the σ-algebra generated by *cylinders* (sets of the form $[a_1, \ldots, a_k] := \{x \in X : x_j = a_j \text{ for all } 1 \le j \le k\}$ and $m : \mathcal{B} \to [0,\infty]$ the measure satisfying $m([a_1,\ldots,a_k]) := \prod_{j=1}^{k-1} p_{a_j,a_{j+1}}$. It is not hard to check that $(X, \mathcal{B}, m, T)$ is a measure preserving transformation. By random walk theory (see §2 and [D-L]) it is *exact* in the sense that $\bigcap_{n\ge 0} T^{-n}\mathcal{B} \overset{m}{=} \{\emptyset, X\}$. It can be checked directly that $F : X \to [0,\infty)$ defined by

$$F(x_1, x_2, \ldots) = \begin{cases} 0, & N_0(x) := \sum_{n=1}^{\infty} \delta_{x_n,0} > 1, \\ 1, & N_0(x) = 1,\ x_1 < 0, \\ q, & \text{else}, \end{cases}$$

is the density of a σ-finite, m-absolutely continuous, T-invariant measure.

1. Results

Wandering sets. For a measure preserving transformation $(X, \mathcal{B}, m, T)$ let $\mathcal{W}_T := \{W \in \mathcal{B} : W \cap T^{-n}W = \emptyset \text{ for all } n \ge 1\}$, the collection of *wandering sets* for T. As is well known (see e.g. [A] or [Kr]), T is dissipative iff X is a countable union of wandering sets mod m.

If T is dissipative and invertible then

- there exists $W_{\max} \in \mathcal{W}_T$ with $\biguplus_{n\in\mathbb{Z}} T^n W_{\max} = X$ mod m (see e.g. [A] or [Kr]);
- if $W \in \mathcal{W}_T$, then $\biguplus_{n\in\mathbb{Z}} T^n W = X$ mod m only if $m(W) = m(W_{\max})$, the reverse implication holding when $m(W_{\max}) < \infty$ (see Theorem 1 in [H-K]). We denote the constant $m(W_{\max})$ by $\mathfrak{w}(T)$.

Proposition 1. *Let $(X, \mathcal{B}, m, T)$ be a dissipative measure preserving transformation of a standard, non-atomic, σ-finite measure space. Then there exists $c \in (0,\infty]$ so that for every $W \in \mathcal{W}_T$ with $m(W) < c$, there exists a non-zero, m-absolutely continuous, T-invariant measure μ with bounded density so that $\mu(W) = 0$.*

Proof. By Rokhlin's theorem (see [Ro] or Theorem 3.1.5 in [A]), there is an invertible, measure preserving transformation $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{m}, \widetilde{T})$ equipped with a measurable map $\pi : \widetilde{X} \to X$ satisfying

$$(\ddagger) \qquad \pi \circ \widetilde{T} = T \circ \pi, \qquad \widetilde{m} \circ \pi^{-1} = m.$$

It follows that $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{m}, \widetilde{T})$ is dissipative.

Given $p \in L^\infty(\widetilde{X})_+$ with $p \circ \widetilde{T} = p$, define $\mu_p \in \mathfrak{M}(X, \mathcal{B})$ by

$$\mu_p(A) := \int_X 1_A \circ \pi p \, d\widetilde{m}.$$

Evidently $\mu_p \ll m$ with $\|d\mu_p/dm\|_\infty \le \|p\|_\infty$ and $\mu_p(T^{-1}A) = \mu_p(A)$ $(A \in \mathcal{B})$.

Next we show, as advertised, that each wandering set of small enough measure is annihilated by some μ_p.

Let $c := \mathfrak{w}(\widetilde{T}) \in (0, \infty]$ and suppose that $W \in \mathcal{W}_T$ has $m(W) < c$. Then $\pi^{-1}W \in \mathcal{W}_{\widetilde{T}}$ and $\widetilde{m}(\widetilde{X} \setminus \biguplus_{n\in\mathbb{Z}} \widetilde{T}^n \pi^{-1}W) > 0$.

Set $Y := \widetilde{X} \setminus \biguplus_{n\in\mathbb{Z}} \widetilde{T}^n \pi^{-1}W$ (then $\widetilde{T}Y = Y$) and let $\mu := \mu_{1_Y}$. Then (as above) $\mu \ll m$ with $\|d\mu/dm\|_\infty \le 1$ and $\mu(T^{-1}A) = \mu(A)$ $(A \in \mathcal{B})$.

By construction, $\mu(W) = \widetilde{m}(\pi^{-1}W \cap Y) = 0$. ■

REMARKS. 1) The density F in the illustrative example above can be obtained as in the proof of Proposition 1 as

$$\int_A F\, dm = \widetilde{m}\Big(\pi^{-1}A \cap \biguplus_{n\in\mathbb{Z}} \widetilde{T}^n \pi^{-1}[-1,0,1]\Big)$$

or

$$F = \sum_{n\ge 0} 1_{[-1,0,1]} \circ T^n + \sum_{n\ge 1} \widehat{T}_m^n 1_{[-1,0,1]}$$

where $\widehat{T}_m$ denotes the *transfer operator* of the measure preserving transformation $(X, \mathcal{B}, m, T)$, which is the operator defined on the space $L(X)_+$ of non-negative, measurable functions by $\int_A \widehat{T}_m f\, dm = \int_{T^{-1}A} f\, dm$ $(f \in L(X)_+, A \in \mathcal{B})$.

2) Evidently, $p \in L(X)_+$ is the density of an m-absolutely continuous, T-invariant measure iff $\widehat{T}_m p = p$. Also $\widehat{T}_m(f \circ T) = f$.

If $(X, \mathcal{B}, m, T)$ is a dissipative measure preserving transformation, then

$$\sum_{n\ge 0} f \circ T^n < \infty \ \&\ \sum_{n\ge 0} \widehat{T}_m^n f < \infty \quad \forall f \in L^1(X),\ f \ge 0.$$

It follows that $\widehat{T}_m F = F$ where $F = F(f) := \sum_{n\ge 0} f \circ T^n + \sum_{n\ge 1} \widehat{T}_m^n f$ whenever $f \in L^1$. This can be used to prove a less precise version of Proposition 1 without assuming standardness of $(X, \mathcal{B}, m)$: if $A, B \in \mathcal{B}$ are disjoint and $A \uplus B \in \mathcal{W}_T$, then $F(1_A)1_B = 0 \bmod m$.

3) Let $(X, \mathcal{B}, m, T)$ be a dissipative measure preserving transformation of a standard, non-atomic, σ-finite measure space and let $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{m}, \widetilde{T})$ be its *natural extension*, i.e. an invertible, measure preserving transformation $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{m}, \widetilde{T})$ equipped with a measurable map $\pi : \widetilde{X} \to X$ satisfying (‡) and

a minimality condition that

$$\bigvee_{n=1}^{\infty} \widetilde{T}^n \pi^{-1}\mathcal{B} = \widetilde{\mathcal{B}} \bmod \widetilde{m}.$$

Natural extensions are unique up to isomorphism, and exist by Rokhlin's theorem (mentioned above). We claim that any m-absolutely continuous, T-invariant measure μ with bounded density is of the form μ_p where $p \in L^\infty(\widetilde{X})$, $p \circ \widetilde{T} = p$.

To see this, let $\mu : \mathcal{B} \to [0,\infty]$ be such a measure. Now define the $\widetilde{T}$-invariant measure $\widetilde{\mu}$ on $(\widetilde{X}, \widetilde{\mathcal{B}})$ as in the proof of Theorem 3.1.5 in [A]. Evidently $\widetilde{\mu} \ll \widetilde{m}$, $p := d\widetilde{\mu}/d\widetilde{m}$ is a bounded, measurable, $\widetilde{T}$-invariant function and $\mu = \mu_p$.

Proposition 2. *Let $(X,\mathcal{B},m,T)$ be an ergodic (exact) measure preserving transformation of a standard, σ-finite measure space. If $\mu \ll m$ is a σ-finite, T-invariant measure, then $(X,\mathcal{B},\mu,T)$ also an ergodic (exact) measure preserving transformation.*

Remark. Proposition 2 applies mainly to dissipative, ergodic (exact) measure preserving transformations of standard, non-atomic σ-finite measure spaces.

Proof. By Theorem 2 in [D], $(X,\mathcal{B},\mu,T)$ is

- ergodic iff $\|n^{-1}\sum_{k=0}^{n-1}\widehat{T}_\mu^k u\|_{L^1(\mu)} \to 0$ for each $u \in L^1(\mu)_0$;
- exact iff $\|\widehat{T}_\mu^n u\|_{L^1(\mu)} \to 0$ for each $u \in L^1(\mu)_0$.

Here $L^1(\mu)_0 := \{u \in L^1(\mu) : \int_X u\,d\mu = 0\}$.

Suppose that $p \in L(X)_+$, $\widehat{T}_m p = p$. We will show that $(X,\mathcal{B},m,T)$ exact implies that $(X,\mathcal{B},\mu,T)$ is also exact where $d\mu = p dm$. The proof for ergodicity is analogous. We note first that

$$\widehat{T}_\mu f = 1_{[p>0]}\frac{1}{p}\widehat{T}_m(fp).$$

Suppose that $u \in L^1(\mu)_0$. Then $up \in L^1(m)_0$, and $\|\widehat{T}_m^n(up)\|_{L^1(m)} \to 0$ by exactness of $(X,\mathcal{B},m,T)$. Thus

$$\|\widehat{T}_\mu^n u\|_{L^1(\mu)} = \int_X 1_{[p>0]}|\widehat{T}_m^n(up)|\,dm \le \|T_m^n(up)\|_{L^1(m)} \to 0$$

and $(X,\mathcal{B},\mu,T)$ is exact. ∎

2. Examples of ergodic, dissipative measure preserving transformations

The Engel series transformation. This is the piecewise linear map $T : (0,1] \to (0,1]$ defined by $T(x) := ([1/x]+1)x - 1$ considered with respect to

Lebesgue measure. Dissipation follows from $T^n x \downarrow 0$ for each $x \in (0,1) \setminus \mathbb{Q}$, ergodicity was shown in [S2] and invariant densities were given explicitly in [T]. This material is also in the book [S1].

Dissipative, ergodic, random walks. The (left) *random walk on LCP group* $\mathbb{G}$ with *jump probability* $p \in \mathcal{P}(\mathbb{G})$ ($\mathtt{RW}(\mathbb{G},p)$) is $(X,\mathcal{B},\mu,T)$, the stationary, one-sided shift of the Markov chain on $\mathbb{G}$ with transition probability $P(g,A) := p(Ag^{-1})$ ($A \in \mathcal{B}(\mathbb{G})$) defined by

$$X := \mathbb{G}^{\mathbb{N}}, \quad \mathcal{B} := \mathcal{B}(X), \quad T(x_1,x_2,\ldots) = (x_2,x_3,\ldots)$$

and

$$\mu([A_1,,\ldots,A_N]) := \int_{\mathbb{G}} P_x([A_1,\ldots,A_N])\,dm(x)$$

where m is a left Haar measure on $\mathbb{G}$ and for $A_1,\ldots,A_N \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned} &[A_1,\ldots,A_N] := \{x = (x_1,x_2,\ldots) \in X : x_k \in A_k \text{ for all } 1 \le k \le N\}; \\ &P_x([A_1]) := 1_{A_1}(x), \\ &P_x([A_1,A_2,\ldots,A_N]) := 1_{A_1}(x) \int_{\mathbb{G}} P_{gx}([A_2,\ldots,A_N])\,dp(g). \end{aligned}$$

For an Abelian group $\mathbb{G}$ it is shown in [D-L] (using [F]) that $\mathtt{RW}(\mathbb{G},p)$ is ergodic iff $\overline{\langle \operatorname{spt} p\rangle} = \mathbb{G}$, and exact iff $\overline{\langle \operatorname{spt} p - \operatorname{spt} p\rangle} = \mathbb{G}$. An exact random walk on $\mathbb{Z}^d$ can be conservative or dissipative when $d = 1,2$ but is always dissipative when $d \ge 3$.

Dissipative, exact inner functions. By Herglotz's theorem, any analytic endomorphism $F : \mathbb{R}^{2+} := \{x + iy \in \mathbb{C} : y > 0\} \to \mathbb{R}^{2+}$ has the form

$$F(z) = \alpha z + \beta + \int_{\mathbb{R}} \left(\frac{1+tz}{t-z}\right) d\mu(t) \tag{2}$$

where $\alpha \ge 0$, $\beta \in \mathbb{R}$ and μ is a positive measure on $\mathbb{R}$. The limits $\lim_{y\to 0+} F(x+iy)$ exist for a.e. $x \in \mathbb{R}$. The analytic endomorphism $F : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ is called an *inner function* if $T(x) := \lim_{y\to 0+} F(x+iy) \in \mathbb{R}$ for a.e. $x \in \mathbb{R}$, equivalently: μ is a singular measure on $\mathbb{R}$. A (referenced) discussion of inner functions can be found in Chapter 6 of [A].

It is known that the real restriction T of an inner function is Lebesgue non-singular: $m(T^{-1}A) = 0 \Leftrightarrow m(A) = 0$ ($A \in \mathcal{B}(\mathbb{R})$) where m is Lebesgue measure on $\mathbb{R}$ (see e.g. Proposition 6.2.2 in [A]) and that $m \circ T^{-1} = m$ when $\alpha = 1$ in (2) (see e.g. Proposition 6.2.4 in [A]). If $\beta = 0$ and μ is a symmetric measure ($\mu(-A) = \mu(A)$), then the real restriction T is odd, and exact by Theorem 6.4.5 in [A].

If, in addition, $\mu([-x,x]^c) \propto 1/x^{\alpha}$ for some $0 < \alpha < 1$, then by Lemma 6.4.7 in [A], T is dissipative.

Dissipative, ergodic, number theoretical transformations. The *Euclidean algorithm* is the transformation $T : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ defined by

$$T(x,y) = \begin{cases} (x-y,y), & x > y, \\ (x, y-x), & x < y. \end{cases}$$

It is shown in [D-N] that $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2), \mu, T)$ is an ergodic, dissipative, measure preserving transformation where $d\mu(x,y) = dxdy/xy$. Exactness does not seem to be known.

The *Rauzy induction* transformations considered in [V] are also known to be ergodic, dissipative measure preserving transformations.

Dissipative S-unimodal maps. These are discussed in [B-H] in terms of their attractors. Conditions are given for ergodicity, exactness, dissipativity and existence of σ-finite invariant densities.

REFERENCES

[A] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Math. Surveys Monogr. 50, Amer. Math. Soc., Providence, RI, 1997.

[B-E] P. Bougerol and L. Élie, *Existence of positive harmonic functions on groups and on covering manifolds*, Ann. Inst. H. Poincaré Probab. Statist. 31 (1995), 59–80.

[B-H] H. Bruin and J. Hawkins, *Exactness and maximal automorphic factors of unimodal interval maps*, Ergodic Theory Dynam. Systems 21 (2001), 1009–1034.

[D-N] S. G. Dani and A. Nogueira, *On invariant measures for the Euclidean algorithm*, ibid. 27 (2007), 417–425.

[D] Y. Derriennic, *Lois "zéro ou deux" pour les processus de Markov. Applications aux marches aléatoires*, Ann. Inst. H. Poincaré Sect. B (N.S.) 12 (1976), 111–129.

[D-L] Y. Derriennic et M. Lin, *Sur la tribu asymptotique des marches aléatoires sur les groupes*, Publ. Sém. Math. (1983), Univ. Rennes I, Rennes.

[F] S. Foguel, *On iterates of convolutions*, Proc. Amer. Math. Soc. 47 (1975), 368–370.

[H-K] A. Hajian and S. Kakutani, *An example of an ergodic measure preserving transformation defined on an infinite measure space*, in: Contributions to Ergodic Theory and Probability, Lecture Notes in Math. 160, Springer, 1970, 45–52.

[K] V. A. Kaimanovich, *Poisson boundaries of random walks on discrete solvable groups*, in: Probability Measures on Groups, X (Oberwolfach, 1990), Plenum, New York, 1987, 93–95.

[Kr] U. Krengel, *Ergodic Theorems*, de Gruyter, Berlin, 1985.

[Ro] V. A. Rokhlin, *Selected topics from the metric theory of dynamical systems*, Uspekhi Mat. Nauk 4 (1949), no. 2, 57–128 (in Russian); English transl. in Amer. Math. Soc. Transl. Ser. 2, 49 (1966).

[S1] F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Clarendon Press, Oxford, 1995.

[S2] —, *Ergodische Theorie der Engelschen und Sylvesterschen Reihen*, Czechoslovak Math. J. 20 (1970), 243–245; Addendum, ibid. 21 (1971), 165.

[T] M. Thaler, *σ-endliche invariante Masse für die Engelschen Reihen*, Anz. Öst. Akad. Wiss. Math.-Natur. Kl. 116 (1979), no. 2, 46–48.

[V] W. Veech, *The metric theory of interval exchange transformations. III. The Sah–Arnoux–Fathi invariant*, Amer. J. Math. 106 (1984), 1389–1422.

School of Mathematical Sciences
Tel Aviv University
69978 Tel Aviv, Israel
E-mail: aaro@tau.ac.il
tomm@tau.ac.il

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