COLLOQUIUM MATHEMATICUM

VOL. 110 2008 NO. 1

THE $M/M/1$ QUEUE IS BERNOULLI

BY

MICHAEL KEANE (Middletown, CT) and NEIL O'CONNELL (Cork)

Abstract. The classical output theorem for the $M/M/1$ queue, due to Burke (1956), states that the departure process from a stationary $M/M/1$ queue, in equilibrium, has the same law as the arrivals process, that is, it is a Poisson process. We show that the associated measure-preserving transformation is metrically isomorphic to a two-sided Bernoulli shift. We also discuss some extensions of Burke's theorem where it remains an open problem to determine if, or under what conditions, the analogue of this result holds.

1. Introduction. The classical output theorem for the $M/M/1$ queue, due to Burke [1], states that the departure process from a stationary $M/M/1$ queue, in equilibrium, has the same law as the arrivals process, that is, it is a Poisson process. To be more precise, let A and S be Poisson processes on $\mathbb{R}$ with respective intensities $\lambda < \xi$ and define, for $t \in \mathbb{R}$,

$$Q(t) = \sup_{s \le t}(A(s,t] - S(s,t]).$$

For each t, $Q(t)$ should be interpreted as the number of customers in the queue at time t. Customers arrive according to the Poisson process A (the arrivals process) and at times given by the points of S, if the queue is non-empty, a customer is served and departs from the queue. The departure process D is defined to be the point process of times at which customers depart from the queue. More precisely, we define, for $s < t$,

$$D(s,t] = Q(s) + A(s,t] - Q(t).$$

Burke's theorem states that D is a Poisson process with intensity λ, and moreover that $(D(0,t], t > 0)$ is independent of $Q(0)$. The standard proof of this fact, due to Reich [10], is a reversibility argument which exploits the dynamical symmetry of the queue and the fact that Q is a stationary, reversible Markov chain. For more background on queueing theory, see, for example, Kelly [5].

The nature of Burke's theorem suggests that there may be a measure-preserving transformation somewhere nearby. It is not immediately obvious

2000 *Mathematics Subject Classification*: Primary 60K25, 37A50; Secondary 60J25, 60J65, 37H99.

Key words and phrases: $M/M/1$ queue, Bernoulli shift.

how to find it, since D is not only a function of A, it also depends on S. However, it was shown in [8] that, if we define $R = A + S - D$, then the pair (D, R) has the same joint law as (A, S), thus exhibiting a measure-preserving transformation; moreover, the restriction of (D, R) to $(-\infty, 0]^2$ is independent of $Q(0)$. We can restate this as follows. For $t \in \mathbb{R}$, set

$$X(t) = \begin{cases} S(0,t] - A(0,t], & t > 0, \\ A(t,0] - S(t,0], & t \leq 0, \end{cases}$$

and

$$Y(t) = \begin{cases} R(0,t] - D(0,t], & t > 0, \\ D(t,0] - R(t,0], & t \leq 0. \end{cases}$$

Note that we can write

$$Y(t) = 2M(t) - X(t) - 2M(0), \qquad M(t) = \sup_{-\infty < s \leq t} X(s).$$

Then X is a two-sided continuous-time simple random walk with positive drift $\xi - \lambda$, and the transformation which maps X to Y is measure-preserving; moreover, $(Y(t), t \leq 0)$ is independent of $Q(0) \equiv M(0)$.

This statement can be further simplified by considering only the times at which events occur (i.e. the times at which the random walk X jumps). Denote these times (which are almost surely distinct) by

$$\cdots < \tau_{-2} < \tau_{-1} < 0 < \tau_1 < \tau_2 < \cdots$$

and set $x_n = X(\tau_n)$ and $y_n = Y(\tau_n)$ for $n \in \mathbb{Z}$. Note that, for $n \in \mathbb{Z}$,

$$y_n = 2s_n - x_n - 2s_0, \qquad s_n = \sup_{m \leq n} x_m.$$

Then $(x_n, n \in \mathbb{Z})$ is a two-sided, discrete-time simple random walk, as is $(y_n, n \in \mathbb{Z})$, and $(y_n, n \leq 0)$ is independent of s_0. Finally, let $\Omega = \{-1, 1\}^{\mathbb{Z}}$ be equipped with Bernoulli product measure with parameter $p = \xi/(\lambda + \xi)$. Set $\varepsilon_n = x_n - x_{n-1}$ and $\sigma_n = y_n - y_{n-1}$. Then we can write $\sigma = T\varepsilon$, where T, defined almost everywhere on Ω, is a measure-preserving transformation.

The fact that $(\sigma_n, n \leq 0)$ is independent of s_0 can now be interpreted as saying that T has a *factor* which is Bernoulli, that is, a factor which is metrically isomorphic to a two-sided Bernoulli shift (see Section 2 for details). The main result of this paper is that T is, in fact, Bernoulli. This will be presented in Section 2. In Section 3 we discuss the Brownian analogue of Burke's theorem where it is only possible to show that the corresponding transformation has a Bernoulli factor. The difficulty here is similar to that encountered in the open question, posed by Marc Yor, of determining whether Lévy's transformation of Brownian motion is ergodic. Dubins and Smorodinsky [3] proved that there is a discrete version of Lévy's transformation which is isomorphic to a one-sided Bernoulli shift. In Section 4 we describe a natural extension of Burke's theorem to the more general setting of iterated random functions, and leave it as an open problem to determine under what conditions the corresponding transformation is Bernoulli.

2. The main result. Let μ be a Bernoulli product measure on $\Omega = \{-1,1\}^{\mathbb{Z}}$ with

$$\mu\{\omega \in \Omega : \omega_0 = 1\} = p > 1/2.$$

Define a two-sided simple random walk $x = (x_n, n \in \mathbb{Z})$ by $x_0 = 0$,

$$x_n = \begin{cases} x_{n-1} + \omega_n, & n > 0, \\ x_{n+1} - \omega_{n+1}, & n < 0. \end{cases}$$

For $n \in \mathbb{Z}$, set $s_n = \sup_{m \le n} x_m$ and $\Omega' = \{s_0(\omega) < \infty\}$. Note that $\mu(\Omega') = 1$. Write $y = 2s - x$ and define $T : \Omega' \to \Omega$ by setting $(T\omega)_n = y_n - y_{n-1}$ for each $n \in \mathbb{Z}$. In order to discuss the inverse transformation we further define

$$\Omega'' = \{\omega \in \Omega' : \liminf_n (s_n - x_n) = 0\}$$

and note that $\mu(\Omega'') = 1$. Let $R : \Omega \to \Omega$ be the "time-reversal" operator defined by $(R\omega)_n = \omega_{-n}$ for $n \in \mathbb{Z}$, and set $\varrho = p^{-1}(1-p)$. We first recall the analogue of Burke's theorem in this discrete setting.

THEOREM 2.1.

(i) $\mu \circ T^{-1} = \mu$.
(ii) *For* $x \ge 0$, $\mu\{\omega : s_0(\omega) = x\} = (1-\varrho)\varrho^x$.
(iii) *The random variable* s_0 *is independent of* $((T\omega)_n, n \le 0)$.
(iv) *If* $\omega \in \Omega''$ *then* $\omega = (RTR)(T\omega)$.

Proof. The measure-preserving property (i) is essentially equivalent to the output theorem for the stationary $M/M/1$ queue, as discussed in Section 1, which follows easily from the fact that the Markov chain $q = s - x$ is stationary and reversible. Property (ii) is well-known. Properties (iii) and (iv) follow from (i) and the fact that, for $\omega \in \Omega''$, $s_n = \min_{l \ge n} y_l$ for all n. ∎

An immediate consequence of (iv) is that there exists $\Omega^* \subset \Omega$ with $\mu(\Omega^*) = 1$ and on which T^k is defined for all $k \in \mathbb{Z}$. Define a mapping $\varphi : \Omega^* \to \mathbb{N}^{\mathbb{Z}}$ by putting $(\varphi\omega)_k = s_0(T^k\omega)$ for each $k \in \mathbb{Z}$. Denote the shift operator on $\mathbb{N}^{\mathbb{Z}}$ by θ and let γ be the θ-invariant product measure on $\mathbb{N}^{\mathbb{Z}}$ with

$$\gamma\{\alpha \in \mathbb{N}^{\mathbb{Z}} : \alpha_0 = x\} = (1-\varrho)\varrho^x, \qquad x \ge 0.$$

THEOREM 2.2.

(i) $\mu \circ \varphi^{-1} = \gamma$.
(ii) *Almost every* $\omega \in \Omega^*$ *is uniquely determined by* $\varphi\omega$.
(iii) $T = \varphi^{-1} \circ \theta \circ \varphi$ *almost everywhere.*

Proof. Claim (i) follows from Theorem 2.1(iii). To prove (ii) we first note that $\omega_0 = (-1)^N$ where $N = \min\{k \ge 0 : s_0(T^k\omega) = 0\}$. Indeed, if $s_0(T^k\omega) > 0$, then $(T^{k+1}\omega)_0 = -(T^k\omega)_0$, whereas if $s_0(T^k\omega) = 0$, then $(T^k\omega)_0 = 1$. By the same reasoning, for any $k \geq 0$, we have $(T^k\omega)_0 = (-1)^{N_k}$, where $N_k = \min\{l \geq 0 : s_0(T^{k+l}\omega) = 0\}$. Thus, we can recover $((T^k\omega)_0, k \in \mathbb{Z})$ from $\varphi\omega$. In exactly the same way, for any $n \in \mathbb{Z}$, we can recover $((T^k\omega)_n, k \in \mathbb{Z})$ from the sequence $(q_n(T^k\omega), k \in \mathbb{Z})$, where $q = s - x$. Combining this observation with the identity

$$q_{n-1}(T^k\omega) = \max\{q_n(T^k\omega) + (T^{k+1}\omega)_n, 0\}$$

we see that, for any $n \leq 0$, we can recover $((T^k\omega)_n, k \in \mathbb{Z})$ from $\varphi\omega$. In particular, we recover $(\omega_n, n \leq 0)$ from $\varphi\omega$. A similar argument works in the other direction, starting with the observation that, if $s_0(T^k\omega) > 0$, then $(T^{k+1}\omega)_1 = -(T^k\omega)_1$, whereas if $s_0(T^k\omega) = 0$, then $(T^{k+1}\omega)_1 = 1$; this leads to the conclusion that $\{\omega_n, n \geq 1\}$ can be recovered from $\varphi\omega$, which completes the proof of (ii), and (iii) follows. ∎

3. Brownian version. Let $(X(t), t \in \mathbb{R})$ be a two-sided standard Brownian motion with drift $\nu > 0$ and with $X(0) = 0$. For $t \in \mathbb{R}$, set

$$Y(t) = 2M(t) - X(t) - 2M(0), \qquad M(t) = \sup_{-\infty < s \leq t} X(s).$$

The continuous analogue of Burke's theorem (see, for example, [9] and references therein) states that Y has the same law as X and, moreover, that $(Y(t), t \leq 0)$ is independent of $M(0)$, which is exponentially distributed with parameter 2ν. It follows that the measure-preserving transformation T, which maps X to Y, has a factor which is metrically isomorphic to the shift operator on $\mathbb{R}_+^{\mathbb{Z}}$, equipped with the product measure $\varepsilon^{\otimes\mathbb{Z}}$, where ε is the exponential distribution on $\mathbb{R}_+$ with parameter 2ν. However, it is not clear in this setting whether or not X can be recovered from the sequence $(\sup_{-\infty<s\leq 0}(T^kX)(s), k \in \mathbb{Z})$, so we cannot conclude that T is Bernoulli. The recovery map for the discrete case, defined in the proof of Theorem 2.2, does not have an obvious continuous analogue. It is thus an open problem to determine whether or not this transformation is Bernoulli, or even ergodic.

This is reminiscent of a (still open) question, originally posed by Marc Yor, in relation to the following transformation of Brownian motion. Let $(B_t, t \geq 0)$ be a standard one-dimensional Brownian motion. It is a classical result, due to Paul Lévy, that the process

$$(|B_t| - L_t^0(|B|),\ t \geq 0)$$

is also a standard Brownian motion, where $L_t^0(|B|)$ denotes the local time at zero of $|B|$ up to time t. Is this an ergodic transformation? Dubins and Smorodinsky [3] proved that there is a discrete version which is metrically isomorphic to a (one-sided) Bernoulli shift.

4. Iterated random functions. The classical output theorem for the $M/M/1$ queue extends quite naturally to the more general setting of iterated random functions. Loosely following [2], let S be a topological space equipped with its Borel σ-algebra, $\{f_\theta,\ \theta \in \Theta\}$ a family of continuous functions that map S to itself and μ a probability distribution on Θ. Let $(\theta_n, n \in \mathbb{Z})$ be a sequence of random variables with common law κ. Consider the Markov chain $x = (x_n, n \geq 0)$ with state space S defined by $x_0 = s$ and

$$x_n = f_{\theta_n}(x_{n-1}) = (f_{\theta_n} \circ \cdots \circ f_{\theta_1})(s), \qquad n > 0. \tag{1}$$

We will assume that this Markov chain has reversible transition probabilities with respect to a unique invariant probability measure. Now consider the *backward* iterations:

$$u_m = (f_{\theta_1} \circ \cdots \circ f_{\theta_m})(s).$$

Under certain regularity conditions, as discussed in [2], the sequence u_m converges almost surely, as $m \to \infty$, to a limiting random variable u_∞ which does not depend on s and which realises the invariant distribution of x. We will assume that this property holds. It follows that, for each $n \in \mathbb{Z}$, the limit

$$z_n = \lim_{m\to\infty} (f_{\theta_{1+n}} \circ \cdots \circ f_{\theta_m})(s) \tag{2}$$

exists almost surely and does not depend on s. By continuity, these random variables satisfy

$$z_n = f_{\theta_{n+1}}(z_{n+1}), \qquad n \in \mathbb{Z}, \tag{3}$$

from which it follows, recalling that x is has reversible transition probabilities, that the sequence $z = (z_n,\ n \in \mathbb{Z})$ is a two-sided stationary version of x.

Now suppose that, for each $s \in S$, the map $\theta \mapsto (s, f_\theta(s))$ is injective, and define $F(r, s) = \theta$ whenever $s = f_\theta(r)$. Then we can write

$$\theta_n = F(z_n, z_{n-1}), \qquad n \in \mathbb{Z}. \tag{4}$$

Define a sequence of random variables $\eta = \{\eta_n, n \in \mathbb{Z}\}$ by setting

$$\eta_n = F(z_{n-1}, z_n), \qquad n \in \mathbb{Z}, \tag{5}$$

so that

$$z_n = f_{\eta_n}(z_{n-1}), \qquad n \in \mathbb{Z}. \tag{6}$$

Reversibility ensures that η is well-defined.

THEOREM 4.1. *In the above context, η has the same distribution as θ and the sequence $\eta_1, \eta_2, \ldots$ is independent of z_0.*

Proof. The first claim follows from (4) and (5), and the fact that z is stationary and reversible. By (6) we can write, almost surely,

$$z_0 = f_{\eta_0}(f_{\eta_{-1}}(f_{\eta_{-2}}(\cdots,$$

which is independent of $\eta_1, \eta_2, \ldots$ as required. ∎

This defines a measure-preserving transformation (mapping θ to η) which has a Bernoulli factor. When is it Bernoulli? The $M/M/1$ queue corresponds to the special case where $\Theta = \{-1, 1\}$, $1 - \kappa\{-1\} = \kappa\{1\} = q < 1/2$ and $f_\theta(x) = \max\{x + \theta, 0\}$. Examples of iterated random functions where Theorem 4.1 applies can be found in [4] and [7]. Further examples which arise from taking products of random matrices, and for which the invariant measure is known explicitly, are discussed in the paper [6]; note however that not all of these are reversible.

Acknowledgements. This paper was completed while the first author was visiting University College Cork, supported by the Science Foundation Ireland grant number SFI 04-RP1-I512.

REFERENCES

[1] P. Burke, *The output of a queueing system*, Oper. Res. 4 (1956), 699–704.

[2] P. Diaconis and D. Freedman, *Iterated random functions*, SIAM Rev. 41 (1999), 45–76.

[3] L. E. Dubins and M. Smorodinsky, *The modified, discrete, Lévy-transformation is Bernoulli*, in: Séminaire de Probabilités, XXVI, Lecture Notes in Math. 1526, Springer, Berlin, 1992, 157–161.

[4] F. J. Dyson, *The dynamics of a disordered linear chain*, Phys. Rev. 92 (1953), 1331–1338.

[5] F. P. Kelly, *Reversibility and Stochastic Networks*, Wiley, Chichester, 1979.

[6] J. Marklof, Y. Tourigny and L. Wolowski, *Explicit invariant measures for products of random matrices*, Trans. Amer. Math. Soc., to appear.

[7] N. O'Connell, *Random matrices, non-colliding processes and queues*, in: Séminaire de Probabilités, XXXVI, Lecture Notes in Math. 1801, Springer, Berlin, 2003, 165–182,

[8] N. O'Connell and M. Yor, *A representation for non-colliding random walks*, Electron. Comm. Probab. 7 (2002), no. 1, 12 pp.

[9] —, —, *Brownian analogues of Burke's theorem*, Stochastic Process. Appl. 96 (2001), 285–304.

[10] E. Reich, *Waiting times when queues are in tandem*, Ann. Math. Statist. 28 (1957), 768–773.

Department of Mathematics
and Computer Science
Wesleyan University
Middletown, CT 06459, U.S.A.
E-mail: mkeane@wesleyan.edu

Department of Mathematics and BCRI
University College Cork
Cork, Ireland
E-mail: noc@ucc.ie

Received 4 December 2006 (4834)