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WEAK AMENABILITY OF GENERAL MEASURE ALGEBRAS

BY

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Abstract. We study the weak amenability of a general measure algebra $M(X)$ on a locally compact space X. First we show that not all general measure multiplications are separately weak* continuous; moreover, under certain conditions, weak amenability of $M(X)^{**}$ implies weak amenability of $M(X)$. The main result of this paper states that there is a general measure algebra $M(X)$ such that $M(X)$ and $M(X)^{**}$ are weakly amenable without X being a discrete topological space.

1. Introduction. In a recent paper [4] Dales, Ghahramani and Helemskii studied the amenability and weak amenability of the measure algebra $M(G)$ on a locally compact group G. They have shown that $M(G)$ is amenable as a Banach algebra if and only if G is discrete and amenable as a group. They have proved that $M(G)$ is not amenable when the group G is not discrete; moreover, $M(G)$ is weakly amenable if and only if the group G is discrete.

In this paper we consider the same problem for a general measure algebra $M(X)$ on a locally compact Hausdorff space X. First we prove that if $M(X)^{**}$ is weakly amenable then so is $M(X)$. Then we give a general measure algebra $M(X)$ which is weakly amenable but the topology of X is not discrete.

1.1. *Notations and definitions.* Let X be a locally compact Hausdorff space. We denote by $C_{\mathrm{b}}(X)$, $C_0(X)$ and $C_{\mathrm{c}}(X)$ the spaces of all continuous functions on X which are bounded, vanish at infinity and have compact support, respectively. They are endowed with the uniform norm $\|\cdot\|_\infty$. We denote by $M(X)$, $M^+(X)$ and $M_{\mathrm{p}}(X)$ the spaces of all complex-valued bounded regular Borel measures, positive measures and probability measures on X, respectively. The total variation norm on each space is abbreviated by $\|\cdot\|$; if $\mu \in M(X)$ then

$$\|\mu\| = \int_X d|\mu| = |\mu|(1).$$

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The *support* of μ (or $\operatorname{supp}\mu$) is the smallest closed set F for which $|\mu|(F) = \|\mu\|$. By the Riesz representation theorem we have $M(X) = C_0(X)^*$ (see [10, 14.10 Theorem]).

We shall say that $M(X)$ has a *general measure multiplication* if there is a bilinear associative map on $M(X)$ which maps probability measures to probability measures; that is,

$$M_{\mathrm{p}}(X) * M_{\mathrm{p}}(X) \subseteq M_{\mathrm{p}}(X),$$

where we write $\mu\nu$ or $\mu * \nu$ for the product of μ and ν. A general measure multiplication on $M(X)$ makes it a Banach algebra (see [15, Proposition 2.1]). A subalgebra $\mathfrak{L}$ of $M(X)$ is an *L-space* if $\mu \in \mathfrak{L}$ and $|\nu| \ll |\mu|$ imply that $v \in \mathfrak{L}$. A subalgebra $\mathfrak{L}$ of $M(X)$ which is an L-space will be called a *general measure algebra* on X. Measure algebras are more general than hypergroups (see [6] or [11]) and than J. L. Taylor's convolution measure algebras (see [16]).

Let $\mathfrak{A}$ be a Banach algebra, and E a Banach $\mathfrak{A}$-bimodule. Then the dual space E^* of E is a Banach $\mathfrak{A}$-bimodule under the following actions:

$$\langle a.x^*, x\rangle = \langle x^*, x.a\rangle, \quad \langle x^*.a, x\rangle = \langle x^*, a.x\rangle \quad (a \in \mathfrak{A},\, x \in E,\, x^* \in E^*).$$

A *derivation* from $\mathfrak{A}$ to E is a bounded linear map $D : \mathfrak{A} \to E$ such that

$$D(ab) = D(a).b + a.D(b) \quad (a, b \in \mathfrak{A}).$$

The derivation D from $\mathfrak{A}$ to E is an *inner derivation* if it is of the form

$$D_x(a) = a.x - x.a \quad (x \in E, a \in \mathfrak{A})$$

A Banach algebra $\mathfrak{A}$ is called *amenable* if every continuous derivation from $\mathfrak{A}$ into E^* is inner for every Banach $\mathfrak{A}$-bimodule E. A Banach algebra $\mathfrak{A}$ is *weakly amenable* if every continuous derivation from $\mathfrak{A}$ into the dual module $\mathfrak{A}^*$ is inner. Weak amenability for commutative Banach algebras was introduced in [2], and in the general case in [12]. In [2] it was shown that a commutative Banach algebra $\mathfrak{A}$ is weakly amenable if and only if every continuous derivation from $\mathfrak{A}$ into E is zero, for all symmetric Banach $\mathfrak{A}$-bimodules E.

The second dual space $\mathfrak{A}^{**}$ of a Banach algebra $\mathfrak{A}$ admits two Banach algebra multiplication known as the *first and second Arens multiplications.* Each of them extends the multiplication of $\mathfrak{A}$ canonically embedded in $\mathfrak{A}^{**}$ ($\mathfrak{A}^\wedge$ is the image of $\mathfrak{A}$ in $\mathfrak{A}^{**}$ under the canonical mapping). Throughout this paper, the first and second Arens multiplications are denoted by $\square$ and $\diamondsuit$, respectively. They can be defined by

$$F \square G = w^*\text{-}\lim_\alpha w^*\text{-}\lim_\beta (a_\alpha b_\beta)^\wedge, \qquad F \diamondsuit G = w^*\text{-}\lim_\beta w^*\text{-}\lim_\alpha (a_\alpha b_\beta)^\wedge,$$

where (a_α) and (b_β) are nets of elements of $\mathfrak{A}$ such that $a_\alpha^\wedge \to F$ and $b_\beta^\wedge \to G$ in the weak* topology. We note that $\mathfrak{A}$ (or the multiplication) is *Arens* *regular* if and only if $F \,\square\, G = F \diamondsuit G$ for all F, G in $\mathfrak{A}^{**}$. See [1] and [14] for the properties of Arens multiplications.

2. General measure algebras. A general measure multiplication need not be separately weak* continuous, as the following example shows.

EXAMPLE 2.1. *There is a commutative general measure multiplication*

(i) *which is not weak* separately continuous* (*is not a hypergroup*),
(ii) *which is not Arens regular.*

This example serves two purposes. First, it shows that not all general measure algebras are hypergroups (see [6] and [11] for the properties of hypergroups). Second, it shows that weak* separately continuous multiplications need not have the non-regularity property for general measure algebras (see [14, Theorem 3.1]).

CONSTRUCTION. Start with the set $X = \{1, \dots, \frac{n+1}{n}, \dots, \frac{3}{2}, 2\}$ with its usual compact topology as a sequence with its limit point. Then

$$M(X) = \Big\{a_0\delta_1 + \sum_{n=1}^{\infty} a_n\delta_{(n+1)/n} : \sum_{n=0}^{\infty} |a_n| < \infty\Big\}.$$

A bilinear map from $M(X) \times M(X)$ to $M(X)$ is given by

$$\delta_x * \delta_y = \delta_y * \delta_x = \begin{cases} \delta_{\min\{x,y\}} & (x \neq 1,\, y \neq 1), \\ \delta_y & x = 1, \end{cases}$$

for x and y in X. It is obvious that this multiplication maps probability measures to probability measures. If x, y and z belong to X then

$$(\delta_x * \delta_y) * \delta_z = \delta_x * (\delta_y * \delta_z), \qquad \delta_1 * \delta_x = \delta_x.$$

So, the multiplication is commutative, associative and distributive. Therefore, $M(X)$ is a general measure algebra. But the multiplication is not separately weak* continuous. In fact, set $\mu_n = \delta_{(n+1)/n}$ and take $f \in C_0(X)$ with $f(1) \neq f(2)$. Then we have

$$\begin{aligned} \lim_n \mu_n * \delta_2(f) &= \lim_n \mu_n(f) = \delta_1(f) = f(1), \\ \delta_1 * \delta_2(f) &= \delta_2(f) = f(2). \end{aligned}$$

Hence,

$$w^*\text{-}\lim_n(\mu_n * \delta_2) \neq \delta_1 * \delta_2,$$

and the map $x \mapsto \delta_x * \delta_2$ is not continuous from X into $M(X)$ with the weak* topology. Hence, X is not a hypergroup (see [6], [10]).

Now we show that $M(X)$ is not Arens regular. Write

$$\psi = \sum_{n=1}^{\infty} (-1)^n \chi_{(n+1)/n}$$

(where χ_A is the characteristic function of A), and consider the double sequence

$$\mu_{2m-1} * \mu_{2n}(\psi).$$

First assume that $m > n$, and so $2m - 1 > 2n$. Thus

$$\mu_{2m-1} * \mu_{2n}(\psi) = \mu_{2m-1}(\psi) = (-1)^{2m-1} = -1,$$

and we deduce that

$$\lim_n \lim_m \mu_{2m-1} * \mu_{2n}(\psi) = -1.$$

Similarly, when $n > m$, we get $2n > 2m - 1$ and

$$\mu_{2m-1} * \mu_{2n}(\psi) = \mu_{2n}(\psi) = (-1)^{2n} = 1$$

for each $n > m$, and so

$$\lim_m \lim_n \mu_{2m-1} * \mu_{2n}(\psi) = 1.$$

Thus, $M(X)$ is not Arens regular (see [9, Lemma 1.1]).

3. Amenability and weak amenability of $M(X)^{**}$. The notion of amenable algebra was introduced by B. E. Johnson in [12] and extended to weak amenability in [2]. It is known that if the Banach algebra $\mathfrak{A}^{**}$ is amenable then $\mathfrak{A}$ is amenable [7], [9]. The question of whether the Banach algebra $\mathfrak{A}$ is weakly amenable when $\mathfrak{A}^{**}$ is weakly amenable seems to be still open. In [7] and [8] weak amenability of $\mathfrak{A}^{**}$ was considered under certain conditions. We also find a condition under which the weak amenability of $M(X)^{**}$ implies the weak amenability of $M(X)$. In general, multiplication in a general measure algebra is not separately w^*-continuous (see Example 2.1). We assume this continuity in the following theorem.

THEOREM 3.1. *Suppose that $M(X)$ on a locally compact Hausdorff space X has a general measure multiplication that is separately weak* continuous. If $(M(X)^{**}, \Box)$ is weakly amenable then so is $M(X)$.*

Proof. Let η be the natural map of $C_0(X)$ into $C_0(X)^{**} = M(X)^*$ and let η^* denote the adjoint mapping from $M(X)^{**}$ to $C_0(X)^* = M(X)$. First we show that η^* is an algebra homomorphism from $M(X)^{**}$ onto $M(X)$. It is clear that η^* is weak*-to-weak* continuous. If $\mu \in M(X)$ then for all $f \in C_0(X)$, we have

$$\langle \eta^*(\widehat{\mu}), f \rangle = \langle \widehat{\mu}, \eta(f) \rangle = \langle \widehat{\mu}, \widehat{f} \rangle = \langle \mu, f \rangle.$$

Hence, $\eta^*(\widehat{\mu}) = \mu$. For $F, G \in M(X)^{**}$, if we regard $M(X)$ as a subspace of $M(X)^{**}$, we can find two bounded nets $(\mu_\alpha), (\nu_\beta)$ in $M(X)$ with $\widehat{\mu}_\alpha \to F$ and $\widehat{\nu}_\beta \to G$ in the weak* topology $\sigma(M(X)^{**}, M(X)^*)$. Since the multiplication in $M(X)$ is separately weak* continuous, for $f \in C_0(X)$, we have

$$\begin{aligned}\langle \eta^*(F \,\Box\, G), f\rangle &= \langle F \,\Box\, G, \eta(f)\rangle = \lim_\alpha \lim_\beta \langle (\mu_\alpha \nu_\beta)^\wedge, \widehat{f}\rangle \\ &= \lim_\alpha \lim_\beta \langle \mu_\alpha \nu_\beta, f\rangle = \lim_\alpha \lim_\beta \langle \eta^*(\widehat{\mu}_\alpha)\eta^*(\widehat{\nu}_\beta), f\rangle \\ &= \langle \eta^*(F)\eta^*(G), f\rangle.\end{aligned}$$

Hence, $\eta^*(F \,\Box\, G) = \eta^*(F)\eta^*(G)$ and η^* is an algebra homomorphism.

Now, suppose that $D : M(X) \to M(X)^*$ is a derivation. We shall prove that so is $\overline{D} = \eta^{**} \circ D \circ \eta^* : M(X)^{**} \to M(X)^{***}$. Indeed if $F, G, H \in M(X)^{**}$ then

$$\begin{aligned}\langle \overline{D}(F \,\Box\, G), H\rangle &= \langle \eta^{**} \circ D \circ \eta^*(F \,\Box\, G), H\rangle \\ &= \langle D \circ \eta^*(F \,\Box\, G), \eta^*(H)\rangle = \langle D(\eta^*(F)\eta^*(G)), \eta^*(H)\rangle \\ &= \langle D(\eta^*(F)).\eta^*(G) + \eta^*(F).D(\eta^*(G)), \eta^*(H)\rangle \\ &= \langle D(\eta^*(F)).\eta^*(G), \eta^*(H)\rangle + \langle \eta^*(F).D(\eta^*(G)), \eta^*(H)\rangle \\ &= \langle D(\eta^*(F)), \eta^*(G)\eta^*(H)\rangle + \langle D(\eta^*(G)), \eta^*(H)\eta^*(F)\rangle \\ &= \langle D(\eta^*(F)), \eta^*(G \,\Box\, H)\rangle + \langle D(\eta^*(G)), \eta^*(H \,\Box\, F)\rangle \\ &= \langle \eta^{**} \circ D \circ \eta^*(F), G \,\Box\, H\rangle + \langle \eta^{**} \circ D \circ \eta^*(G), H \,\Box\, F\rangle \\ &= \langle \overline{D}(F).G + F.\overline{D}(G), H\rangle.\end{aligned}$$

Thus

$$\overline{D}(F \,\Box\, G) = \overline{D}(F)G + F\overline{D}(G).$$

Hence $\overline{D}$ is a derivation. By assumption, $M(X)^{**}$ is weakly amenable. Hence there exists $\phi \in M(X)^{***}$ such that

$$\overline{D}(F) = F.\phi - \phi.F \quad (F \in M(X)^{**}).$$

Now, let $\lambda : M(X) \to M(X)^{**}$ be the canonical mapping and let λ^* denote the adjoint mapping from $M(X)^{***}$ to $M(X)^*$. On the other hand, $M(X)^{**}$ is naturally a $M(X)$-bimodule and λ^* is an $M(X)$-bimodule morphism. In fact, for μ and ν in $M(X)$,

$$\begin{aligned}\langle \lambda^*(\widehat{\mu}.\phi), \nu\rangle &= \langle \widehat{\mu}.\phi, \lambda(\nu)\rangle = \langle \phi, \lambda(\nu\mu)\rangle \\ &= \langle \lambda^*(\phi), \nu\mu\rangle = \langle \mu.\lambda^*(\phi), \nu\rangle,\end{aligned}$$

so, $\lambda^*(\widehat{\mu}.\phi) = \mu.\lambda^*(\phi)$. Similarly $\lambda^*(\phi.\widehat{\mu}) = \lambda^*(\phi).\mu$. Therefore

$$\begin{aligned}\langle D(\mu), \nu\rangle &= \langle D(\eta^*(\widehat{\mu})), \eta^*(\widehat{\nu})\rangle = \langle \eta^{**} \circ D \circ \eta^*(\widehat{\mu}), \widehat{\nu}\rangle \\ &= \langle \overline{D}(\widehat{\mu}), \lambda(\nu)\rangle = \langle \widehat{\mu}.\phi - \phi.\widehat{\mu}, \lambda(\nu)\rangle \\ &= \langle \lambda^*(\widehat{\mu}.\phi - \phi.\widehat{\mu}), \nu\rangle = \langle \mu.\lambda^*(\phi) - \lambda^*(\phi).\mu, \nu\rangle.\end{aligned}$$

Set $\lambda^*(\phi) = f_0$. Then if $\mu \in M(X)$, we have $D(\mu) = \mu f_0 - f_0\mu$ and so $M(X)$ is weakly amenable. ■

In [4], Dales, Ghahramani and Helemskii studied some implications of amenability and weak amenability and proved a conjecture on $M(G)$. We summarize all these results in the following theorem.

THEOREM 3.2. *Let G be a locally compact group. Then*:

(1) *$M(G)$ is amenable if and only if G is discrete and amenable*;
(2) *$M(G)$ is weakly amenable if and only if G is discrete*;
(3) *if G is non-discrete then the Banach algebra $(L^1(G)^{**}, \Box)$ is not weakly amenable.*

Proof. For details and proof, see [4]. ■

Here we are going to study the above assertions for a general measure algebra. We shall show that they are not necessarily true.

Let X be a locally compact Hausdorff space. Define a multiplication on $M(X)$ by

$$\mu\nu = \nu(1)\mu \quad (\mu, \nu \in M(X)).$$

Now let $\mu, \nu, \lambda \in M(X)$. Then

$$(\mu\nu)\lambda = \lambda(1)(\mu\nu) = \lambda(1)\nu(1)\mu = (\nu\lambda)(1)\mu = \mu(\nu\lambda).$$

If $\mu, \nu \in M_{\mathrm{p}}(X)$ then

$$(\mu\nu)(1) = \nu(1)\mu(1) = 1 \cdot 1 = 1.$$

So, the multiplication is associative, and also, $\mu\nu$ is a probability measure if μ and ν are. Thus we have a general measure algebra.

Now, we are in a position to present our theorem which shows the difference from the group case. The motivation for this theorem is given in [5, Example 4.5]. See also [3].

THEOREM 3.3. *Let $M(X)$ have the above multiplication. Then the following statements hold for any topology of X*:

(i) *If X contains at least two points then $M(X)$ is not amenable.*
(ii) *$M(X)$ and $M(X)^{**}$ are weakly amenable.*
(iii) *There is a general measure algebra $L^1(\mu)$ for which the Banach algebra $(L^1(\mu)^{**}, \Box)$ is weakly amenable, but the topology of* $\operatorname{supp} L^1(\mu)$ *is not discrete.*

Here the *support,* $\operatorname{supp} \mathfrak{L}$, of $\mathfrak{L}$ is defined by $\operatorname{supp} \mathfrak{L} = \mathrm{cl}(\bigcup_{\mu\in\mathfrak{L}} \operatorname{supp} \mu)$.

Proof. (i) If X contains at least two points, then $M_{\mathrm{p}}(X)$ contains at least two elements. Suppose $M(X)$ is amenable. So, $M(X)$ has a bounded approximate identity (e_α). If $\mu \neq \nu$ and $\mu, \nu \in M_{\mathrm{p}}(X)$, then $\mu(1) = \nu(1)$ and

$$\mu = \lim_\alpha e_\alpha \mu = \lim_\alpha \mu(1) e_\alpha = \lim_\alpha \nu(1) e_\alpha = \lim_\alpha e_\alpha \nu = \nu.$$

So, $\mu = \nu$ contrary to our assumption. Hence, $M(X)$ is not amenable.

(ii) Let $D : M(X) \to M(X)^*$ be a continuous derivation and $e \in M_{\mathrm{p}}(X)$. Then $\mu e = \mu$ and $e.D(\mu) = D(\mu)$ for all $\mu \in M(X)$. Now, let $\mu, \nu \in M(X)$. Then

$$\begin{aligned} \langle D(e\mu), \nu\rangle &= \langle D(\mu(1)e), \nu\rangle = \langle D(e), \mu(1)\nu\rangle \\ &= \langle D(e), \nu\mu\rangle = \langle \mu.D(e), \nu\rangle. \end{aligned}$$

Therefore, $D(e\mu) = \mu.D(e)$ and

$$\begin{aligned} D(e\mu) &= e.D(\mu) + D(e).\mu = D(\mu) + D(e).\mu, \\ D(\mu) &= D(e\mu) - D(e).\mu = \mu.D(e) - D(e).\mu. \end{aligned}$$

Hence, D is the inner derivation implemented by $D(e)$ and so $M(X)$ is weakly amenable. Similarly, $M(X)^{**}$ is weakly amenable.

(iii) Let $\mu \in M_{\mathrm{p}}(X)$ and let $a_n \in (0, \infty)$ for each n, with $\sum_n |a_n| < \infty$. Put $\lambda = \sum_n a_n \mu^n$. Then $L^1(\lambda)$ is a general measure algebra (see [15, Proposition 2.1(iii)]). Without loss of generality we can assume that $\operatorname{supp} \lambda = X$.

Take $F, E \in L^1(\lambda)^{**}$, and let $(\mu_\alpha), (\nu_\beta)$ be nets in $L^1(\lambda)$ such that $F = w^*\text{-}\lim_\alpha \mu_\alpha^\wedge, E = w^*\text{-}\lim_\beta \nu_\beta^\wedge$. Then the first Arens multiplication in $L^1(\lambda)^{**}$ is determined by

$$\begin{aligned} F \,\Box\, E &= w^*\text{-}\lim_\alpha w^*\text{-}\lim_\beta (\mu_\alpha \nu_\beta)^\wedge \\ &= w^*\text{-}\lim_\alpha w^*\text{-}\lim_\beta \nu_\beta(1) \mu_\alpha^\wedge = E(1)F. \end{aligned}$$

Take $E \in L^1(\lambda)^{**}$ with $E(1) = 1$, so $F \,\Box\, E = F$.

Now, let $D : L^1(\lambda)^{**} \to L^1(\lambda)^{***}$ be a derivation. Then $D(E \,\Box\, F) = F.D(E)$ and $E.D(F) = D(F)$. Thus, for $F \in L^1(\lambda)^{**}$, we have

$$D(F) = E.D(F) = D(E \,\Box\, F) - D(E).F = F.D(E) - D(E).F.$$

Hence, D is an inner derivation and $(L^1(\lambda)^{**}, \Box)$ is weakly amenable for any topology on X. ∎

In [13], B. E. Johnson proved that $L^1(G)$ of a locally compact group G is weakly amenable. The authors in [4] have shown that if the group G is discrete then $M(G)$ is weakly amenable. Now we prove that this is not necessarily true in a general measure algebra.

Example 3.4. *There is a general measure algebra $M(X)$ which is not weakly amenable but the topology of X is discrete.*

Construction. Let (x_n) be a set of different points of real numbers with $\lim_n x_n = x$. Suppose that $X = \{x_1, x_2, \dots\}$ and $x \notin X$. So the topology of X is discrete (not compact) and

$$M(X) = \Big\{ \sum_n a_n \delta_{x_n} : \sum_n |a_n| < \infty \Big\}.$$

Let $S = \{s_1, \ldots, s_r\}$ be any finite commutative semigroup and set $X_r = \{x_1, \ldots, x_r\}$. Define a multiplication on $M(X_r)$ by

$$\delta_{x_i} * \delta_{x_j} = \delta_{x_r} \quad \text{where} \quad s_i s_j = s_r.$$

It is commutative and associative. So, $M(X_r)$ is a general measure algebra. Define $\varphi : M(X) \to M(X_r)$ by

$$\varphi\Big(\sum_{n=1}^{\infty} a_n \delta_{x_n}\Big) = \sum_{n=1}^{r-1} a_n \delta_{x_n} + \Big(\sum_{n=r}^{\infty} a_n\Big)\delta_{x_r}.$$

It is a positive linear operator and maps probability measures to probability measures. In fact:

(i) If $\mu \in M_{\mathrm{p}}(X)$ then $\varphi(\mu) \in M_{\mathrm{p}}(X_r) \subseteq M_{\mathrm{p}}(X)$. In general, $\|\varphi(\mu)\| \leq \|\mu\|$.

(ii) If $\mu \in M(X_r)$ then $\varphi(\mu) = \mu$.

(iii) For $\mu \in M(X)$, $\varphi(\varphi(\mu)) = \varphi(\mu)$. So φ is a linear projection on $M(X)$. If the range and null space of φ are denoted by $R(\varphi)$ and $N(\varphi)$ respectively, then

$$R(\varphi) = M(X_r), \quad M(X) = M(X_r) \oplus N(\varphi).$$

Now, let $\mu, \nu \in M(X)$. Define a multiplication on $M(X)$ by

$$\mu\nu = \varphi(\mu) * \varphi(\nu).$$

So for μ, ν, λ in $M(X)$, $\mu\nu = \nu\mu$ and

$$\begin{aligned} \mu(\nu\lambda) &= \varphi(\mu) * \varphi(\varphi(\nu) * \varphi(\lambda)) = \varphi(\mu) * (\varphi(\nu) * \varphi(\lambda)) \\ &= (\varphi(\mu) * \varphi(\nu)) * \varphi(\lambda) = (\mu\nu)\lambda \end{aligned}$$

Hence, this multiplication is a commutative, associative and symmetric bilinear map from $M(X) \times M(X)$ to $M(X_r)$ which maps probability measures to probability measures. Thus $M(X)$ is a general measure algebra and $M(X_r)$ is an ideal of $M(X)$, i.e. $M(X_r)M(X) \subseteq M(X_r)$.

Now we prove that $M(X)$ is not weakly amenable. It is sufficient to show that there is a continuous derivation on $M(X)$ which is not inner. Define $f_0 \in M(X)^*$ by

$$\langle f_0, \mu\rangle = (\mu - \varphi(\mu))(1) = \int_X d(\mu - \varphi(\mu)),$$

so $f_0 \neq 0$ and $f_0|M(X_r) = 0$. In fact, if $\mu, \nu \in M(X)$ then

$$\langle f_0, \mu\nu\rangle = (\mu\nu - \varphi(\mu\nu))(1) = (\varphi(\mu) * \varphi(\nu) - \varphi(\varphi(\mu) * \varphi(\nu)))(1) = 0.$$

On the other hand, the map $D : M(X) \to M(X)^*$ given by

$$D(\mu) = \langle f_0, \mu\rangle f_0 \quad (\mu \in M(X))$$

is a continuous derivation. Indeed, $D(\mu\nu) = 0$ and

$$\langle \mu.D(\nu), \lambda\rangle = \langle D(\nu), \lambda\mu\rangle = \langle f_0, \mu\rangle\langle f_0, \lambda\mu\rangle = 0$$

for any $\mu, \nu, \lambda \in M(X)$. So, $\mu.D(\nu) = D(\mu).\nu = 0$. Thus D is a non-zero continuous derivation, but it is not inner. ∎

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