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A NOTE ON THE THEOREMS OF LUSTERNIK–SCHNIRELMANN AND BORSUK–ULAM

BY

T. E. BARROS and C. BIASI (São Carlos)

Abstract. Let p be a prime number and X a simply connected Hausdorff space equipped with a free $\mathbb{Z}_p$-action generated by $f_p : X \to X$. Let $\alpha : S^{2n-1} \to S^{2n-1}$ be a homeomorphism generating a free $\mathbb{Z}_p$-action on the $(2n-1)$-sphere, whose orbit space is some lens space. We prove that, under some homotopy conditions on X, there exists an equivariant map $F : (S^{2n-1}, \alpha) \to (X, f_p)$. As applications, we derive new versions of generalized Lusternik–Schnirelmann and Borsuk–Ulam theorems.

1. Introduction. Let X be a simply connected Hausdorff space equipped with a free $\mathbb{Z}_p$-action (p is a prime number) generated by f_p : $X \to X$. Given $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ such that for each $j = 1, \ldots, n$, p does not divide l_j, consider the free $\mathbb{Z}_p$-action on S^{2n-1} generated by $\alpha_{p,l}$: $S^{2n-1} \to S^{2n-1}$,

$$\alpha_{p,l}(z_1, \ldots, z_n) = (e^{2\pi i l_1/p} \cdot z_1, \ldots, e^{2\pi i l_n/p} \cdot z_n).$$

We recall that a path connected space Y is *j-simple* (for $j \geq 1$) if the canonical action of the fundamental group of Y on the group $\pi_j(Y)$ is trivial. Our main result is the following

THEOREM 1. *Suppose that for all j with $2 \leq j < m = 2n-1$ the orbit space X/f_p is j-simple and*

(i) $\pi_j(X) = p \cdot \pi_j(X)$ *if j is odd,*
(ii) $\pi_j(X)$ *does not have elements of order p if j is even.*

Then there exists an equivariant map $F : (S^m, \alpha_{p,l}) \to (X, f_p)$.

If $p = 2$, Theorem 1 remains valid for any m odd or even ($\alpha_{2,l}$ is the antipodal map for any choice of l). This theorem provides the following versions of the Borsuk–Ulam and Lusternik–Schnirelmann theorems.

THEOREM 2. *Let X, $f_p : X \to X$ and m satisfy the hypotheses of Theorem 1. Then for each family $\mathcal{F} = \{C_0, \ldots, C_k\}$ of $k+1$ sets covering X, each of which is either open or closed, and such that either*

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(1) *$p = 2$ and $k \leq m$, or*
(2) *$p = 3$, m is odd and $k \leq m + 1$, or*
(3) *$p > 3$, m is odd and $(p-1)(k-2)/2 + 2 \leq m$,*

there exists $C_{j_0} \in \mathcal{F}$ such that $f_p(C_{j_0}) \cap C_{j_0} \neq \emptyset$.

THEOREM 3. *Let X, $f_p : X \to X$ and m satisfy the hypotheses of Theorem 1.*

(i) *If $m \geq k(p-1)$, then for each continuous map $f : X \to \mathbb{R}^k$ there exists $x \in X$ such that $f(x) = f \circ (f_p)^j(x)$ for all $1 \leq j \leq p-1$.*
(ii) *If $m \geq (k-1)(p-1)+1$, then for each continuous map $f : X \to \mathbb{R}^k$ there exists $x \in X$ such that $f(x) = f \circ f_p(x)$.*

THEOREM 4. *Let X be a paracompact, Hausdorff and simply connected space and let $f_2 : X \to X$ be an involution without fixed points, both satisfying the hypotheses of Theorem 1. If Y is a separable metric space with topological dimension $\dim(Y) \leq (m-1)/2$, then for any map $f : X \to Y$, there exists $x \in X$ such that $f(x) = f \circ f_2(x)$.*

In [6] M. Izydorek and J. Jaworowski constructed for each k and $n \leq 2k - 1$ a map f from the n-sphere S^n into a specific contractible k-dimensional complex Y such that $f(x) \neq f(-x)$ for all $x \in S^n$. Thus the upper bound for the dimension in Theorem 4 is sharp in the general case.

Theorem 3 generalizes, in a certain sense, the result of Cohen–Connet [4]. Generalizations of the same nature of the Borsuk–Ulam theorem with "nice" topological spaces X and Y satisfying some homological conditions can be found in [8] and [3].

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2. Proof of Theorem 1. Let us consider the free action of $\mathbb{Z}_p$ on the sphere S^{2n-1} generated by the map $\alpha_{p,l}$ as defined in the Introduction. If X is a simply connected Hausdorff space equipped with a free $\mathbb{Z}_p$-action, then the diagonal action $\mathbb{Z}_p \times (S^{2n-1} \times X) \to (S^{2n-1} \times X)$, given by $g\cdot(q, x) = (g\cdot q, g\cdot x)$ is a free $\mathbb{Z}_p$-action. The projection $\pi : S^{2n-1} \times X \to S^{2n-1}$ onto the first coordinate is an equivariant map and induces a map

$$\xi_{n,l} : S^{2n-1} \times_{\mathbb{Z}_p} X \to L_p(l)$$

on orbit spaces. Since the $\mathbb{Z}_p$-action over S^{2n-1} is free, the quotient map $\pi_S : S^{2n-1} \to L_p(l)$ is a covering map. If $\widetilde{U} \subset L_p(l)$ and $U \subset S^{2n-1}$ are open sets such that $\pi_S^{-1}(\widetilde{U}) = \bigcup_{g \in \mathbb{Z}_p} g \cdot U$ (disjoint union) and $\pi_S|_{g\cdot U}$ is a homeomorphism from $g \cdot U$ onto $\widetilde{U}$, then the map $\phi_{\widetilde{U}} : \widetilde{U} \times X \to \xi_{n,l}^{-1}(\widetilde{U})$ given by $\phi_{\widetilde{U}}([q], x) = [(q, g \cdot x)]$, where $g \in \mathbb{Z}_p$ is such that $q \in g \cdot U$, provides a local trivialization of $\xi_{n,l}$. Thus $\xi_{n,l}$ is a locally trivial fibration over the lens space $L_p(l)$ with X as typical fiber.

With these notations we have the following

Lemma 1. *There exists an equivariant map $F : S^{2n-1} \to X$ if and only if there exists a cross-section of the fibration $\xi_{n,l}$.*

Proof. If $F : S^{2n-1} \to X$ is equivariant, then $\sigma_F : L_p(l) \to S^{2n-1} \times_{\mathbb{Z}_p} X$ defined by $\sigma_F([q]) = [(q, F(q))]$ is a cross-section of $\xi_{n,l}$.

Conversely, given a cross-section $\sigma : L_p(l) \to S^{2n-1} \times_{\mathbb{Z}_p} X$, we have $\sigma([q]) = [(q, F_\sigma(q))]$ for some continuous map $F_\sigma : S^{2n-1} \to X$. The continuity of F_σ follows from the fact that for $q \in g \cdot U$ we have $F_\sigma(q) = g \circ \pi_X \circ \phi_{\widetilde{U}}^{-1} \circ \sigma \circ \pi_S(q)$, where $\pi_X : \widetilde{U} \times X \to X$ is the projection onto X and we identify the element $g \in \mathbb{Z}_p$ with the homeomorphism $g : X \to X$ induced by the action over X.

Finally, to see that F_σ is equivariant, let $q' = g \cdot q$ for some $g \in \mathbb{Z}_p$. Then for the element $[g \cdot q, F_\sigma(g \cdot q)] \in S^{2n-1} \times_{\mathbb{Z}_p} X$ we have

$$[g \cdot q, F_\sigma(g \cdot q)] = [q', F_\sigma(q')] = \sigma([q']) = \sigma([q]) = [q, F_\sigma(q)] = [g \cdot q, g \cdot F_\sigma(q)].$$

Thus there exists $h \in \mathbb{Z}_p$ such that $q' = h \cdot q'$ and $F_\sigma(g \cdot q) = h \cdot (g \cdot F_\sigma(q))$. But the action on S^{2n-1} is free, therefore $h = 1$ and $F_\sigma(g \cdot q) = g \cdot F_\sigma(q)$.

In the light of the above lemma, to prove Theorem 1 it is enough to prove the existence of a cross-section of $\xi_{n,l} : S^{2n-1} \times_{\mathbb{Z}_p} X \to L_p(l)$. To do this, let us consider the fibration $\xi_{n+1,l'} : S^{2n+1} \times_{\mathbb{Z}_p} X \to L_p(l')$, where $l' = (l, 1) \in \mathbb{Z}^{n+1}$. Then it can be easily checked that $\xi_{n,l}$ is isomorphic to the pull-back fibration of $\xi_{n+1,l'}$, induced by the inclusion $J : L_p(l) \to L_p(l')$, $J([z_1, \dots, z_n]) = [z_1, \dots, z_n, 0]$. A lift $\widetilde{\sigma} : L_p(l) \to S^{2n+1} \times_{\mathbb{Z}_p} X$ of J is a partial cross-section of $\xi_{n+1,l'}$, defined on the $(2n-1)$-skeleton of $L_p(l')$. If we succeed in constructing $\widetilde{\sigma}$, then by the universal property of the pull-back, there exists a unique continuous map $\sigma : L_p(l) \to S^{2n-1} \times_{\mathbb{Z}_p} X$ such that $\xi_{n,l} \circ \sigma = 1_{L_p(l)}$ and $\hat{J} \circ \sigma = \widetilde{\sigma}$.

$$\begin{tikzcd}
L_p(l) \arrow[dr, "\sigma"] \arrow[drr, "\widetilde{\sigma}"] \arrow[ddr, "1_{L_p(l)}"'] & & \\
& S^{2n-1} \times_{\mathbb{Z}_p} X \arrow[r, "\hat{J}"] \arrow[d, "\xi_{n,l}"] & S^{2n+1} \times_{\mathbb{Z}_p} X \arrow[d, "\xi_{n+1,l'}"] \\
& L_p(l) \arrow[r, "J"] & L_p(l')
\end{tikzcd}$$

In particular, σ is a cross-section of $\xi_{n,l}$.

Let us now prove the existence of $\widetilde{\sigma}$. Since the fiber X of the fibration $\xi_{n+1,l'}$ is path connected, there exists a cross-section over the 1-skeleton of $L_p(l')$. The hypothesis that X is simply connected implies X is j-simple for all positive integers j, so the obstruction to the existence of a cross-section over the j-skeleton of $L_p(l')$ is an element of the cohomology with local coefficients, $H^j(L_p(l');\, \pi_{j-1})$; here the system of local coefficients is formed by the groups $\pi_{j-1}(X_{[q]})$, where $X_{[q]}$ is the fiber over $[q]$, and $[q]$ runs over all points of the base space $L_p(l')$. The assumption that X/f_p is j-simple for all $2 \le j < 2n-1$ (together with the 1-connectedness of X) guarantees that the local system of groups π_{j-1} is simple for $2 \le j \le 2n-1$, i.e., the cohomology with local coefficients $H^j(L_p(l'); \pi_{j-1})$ reduces to the ordinary singular cohomology $H^j(L_p(l'); \pi_{j-1}(X))$ for all $2 \le j \le 2n-1$ (cf. [10]).

Now to compute the cohomology groups we use the universal coefficient theorem to conclude that

$$H^j(L_p(l'); \pi_{j-1}(X)) \cong \begin{cases} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \pi_{j-1}(X)) & \text{if } j \text{ is odd,} \\ \mathrm{Ext}(\mathbb{Z}_p, \pi_{j-1}(X)) & \text{if } j \text{ is even.} \end{cases}$$

But from the definitions of Hom and Ext we have

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \pi_{j-1}(X)) \cong \{\alpha \in \pi_{j-1}(X) : p\alpha = 0\},$$
$$\mathrm{Ext}(\mathbb{Z}_p, \pi_{j-1}(X)) \cong \pi_{j-1}(X)/p\pi_{j-1}(X)$$

Hence, as X satisfies hypotheses (i) and (ii), we have $H^j(L_p(l'); \pi_{j-1}(X)) = 0$ for all $2 \le j \le 2n-1$. Therefore there exists a lift of J restricted to $L_p(l)$, the $(2n-1)$-skeleton of $L_p(l')$, and Theorem 1 follows.

Remark 1. Given integers $m > n$, it is well known that there is no equivariant map $F : (S^m, a_m) \to (S^n, a_n)$, where a_m and a_n are the antipodal maps. We note that for n even (S^n, a_n) satisfies all hypotheses of Theorem 1 except that $S^n/a_n = \mathbb{R}P^n$ is j-simple for each $2 \le j < m$. Thus the assumption that the orbit space X/f_p is j-simple cannot be dropped.

Remark 2. The procedure of extending the cross-section performed above does not apply directly to the fibration $\xi_{n,l}$, because in this case the obstruction to extending the cross-section σ to the $(2n-1)$-skeleton is an element of $H^{2n-1}(L_p(l); \pi_{2n}(X))$ but this is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(H_{2n-1}(L_p(l)), \pi_{2n}(X))$, and since $H_{2n-1}(L_p(l)) = \mathbb{Z}$ we have $H^{2n-1}(L_p(l); \pi_{2n}(X)) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \pi_{2n}(X))$, which, upon our assumptions about X, is not zero in general.

3. Proof of Theorem 2. The classical Lusternik–Schnirelmann theorem says the following:

Theorem LS. *Let $m \ge k$ and let $H_0, H_1, \dots, H_k$ be closed subsets of the sphere S^m such that $S^m = \bigcup_{j=0}^k H_j$. Then there exists $j_0 \in \{0, 1, \dots, k\}$ with $H_{j_0} \cap -H_{j_0} \neq \emptyset$.*

In 1979, Steinlein [11] proved the following

THEOREM S. *Let p be a prime number and $\alpha_p : S^m \to S^m$ a continuous map generating a free $\mathbb{Z}_p$-action on S^m. Let $m, k \in \mathbb{N}$ be such that m is odd and*

$$m \geq \begin{cases} k-1 & \text{if } p = 3, \\ (p-1)(k-2)/2+2 & \text{if } p > 3. \end{cases}$$

Then for each covering $S^m = \bigcup_{j=0}^k H_j$ by $k+1$ closed sets, there exists H_{j_0} such that $H_{j_0} \cap \alpha_p(H_{j_0}) \neq \emptyset$.

In [5], J. E. Greene proved that in Theorem LS each set H_j can be either open or closed. With a similar reasoning Theorem S can be improved to

THEOREM SG. *Let p be a prime number and $\alpha_p : S^m \to S^m$ a continuous map generating a free $\mathbb{Z}_p$-action on S^m. Let $m, k \in \mathbb{N}$ be such that m is odd and*

$$m \geq \begin{cases} k-1 & \text{if } p = 3, \\ (p-1)(k-2)/2+2 & \text{if } p > 3. \end{cases}$$

Then for each covering $S^m = \bigcup_{j=0}^k H_j$ by $k+1$ sets, each of which is either open or closed, there exists H_{j_0} such that $H_{j_0} \cap \alpha_p(H_{j_0}) \neq \emptyset$.

Proof. Following the reasoning of Greene we prove Theorem SG by induction on the number t of closed sets in the cover of S^m. The case $t = 0$ corresponds to a cover of S^m by open sets $H_0, H_1, \ldots, H_k$. Select a Lebesgue number for this cover, that is, a positive number λ such that for all $x \in S^m$, the closed ball $\bar{B}(x, \lambda)$ is contained in some H_j. By compactness, there exists a finite collection of points $\{x_i\}$ such that the open balls $B(x_i, \lambda)$ cover S^m. For each j, let F_j denote the union of those $\bar{B}(x_i, \lambda)$ contained in H_j. Then F_j is closed, $F_j \subset H_j$ for each j, and together the F_j cover S^m. Therefore, Theorem S implies that there exists F_{j_0} such that $F_{j_0} \cap \alpha_p(F_{j_0}) \neq \emptyset$, and hence there exists H_{j_0} such that $H_{j_0} \cap \alpha_p(H_{j_0}) \neq \emptyset$.

Thus we may assume that $0 < t < k+1$ and the assertion holds for fewer than t closed sets. We now show that it holds for t closed sets. Let $\mathcal{C}$ be a cover of S^m with $k+1$ sets, of which exactly t are closed and the remaining ones are open. Fix a closed set F in $\mathcal{C}$, and suppose that $F \cap \alpha_p(F) = \emptyset$. By normality, there exist open sets A and B such that $F \subset A$, $\alpha_p(F) \subset B$ and $A \cap B = \emptyset$. Let $U = A \cap \alpha_p^{-1}(B)$. Then U is open, $F \subset U$ and $U \cap \alpha_p(U) = \emptyset$. Therefore $\mathcal{C}' = (\mathcal{C} - \{F\}) \cup \{U\}$ is a cover of S^m with $k+1$ sets, of which exactly $t-1$ are closed and the remaining ones are open, so by the induction hypothesis some set H in the cover satisfies $H \cap \alpha_p(H) \neq \emptyset$ and by construction this H must be different from U, and hence some set H in the original cover must satisfy $H \cap \alpha_p(H) \neq \emptyset$. This completes the inductive step.

Now we are ready to prove Theorem 2. Suppose that $X = \bigcup_{j=0}^{k} C_j$ is a covering by $k+1$ sets, each of which is either open or closed, and k satisfies condition (1), (2) or (3) of Theorem 2.

By Theorem 1 for each $\alpha_{p,l} : S^m \to S^m$ generating a free $\mathbb{Z}_p$-action on S^m, and for each $f_p : X \to X$ generating a free $\mathbb{Z}_p$-action on X, there exists an equivariant continuous map from $(S^m, \alpha_{p,l})$ to (X, f_p). Note that $(f_p)^{-1} = (f_p)^{p-1}$ also generates a free $\mathbb{Z}_p$-action on X, and analogously if $l = (l_1, \ldots, l_n)$ then $\alpha_{p,l}^{-1} = \alpha_{p,l'}$ ($l' = (p - l_1, \ldots, p - l_n)$) generates a free $\mathbb{Z}_p$-action on S^m. Then there exists an equivariant continuous map $F : (S^m, \alpha_{p,l}) \to (X, (f_p)^{-1})$. Thus $S^m = \bigcup_{j=0}^{k} F^{-1}(C_j)$ is a covering by $k+1$ sets, each of which is either open or closed.

If $p \geq 3$ it follows from Theorem SG that there exists C_{j_0} such that

$$F^{-1}(C_{j_0}) \cap \alpha_{p,l'}(F^{-1}(C_{j_0})) \neq \emptyset.$$

This together with the facts that $\alpha_{p,l'} = \alpha_{p,l}^{-1}$ and $F : (S^m, \alpha_{p,l}) \to (X, f_p)$ is equivariant implies that $C_{j_0} \cap f_p(C_{j_0}) \neq \emptyset$.

If $p = 2$, m can be even or odd, and in any case α_2 is the antipodal map. The same reasoning applies to Greene's version of Theorem LS.

4. Proof of Theorem 3. Here we need the following theorem, which follows from the works of H. J. Munkholm [9] and E. L. Lusk [7].

Theorem ML. *Let p be a prime number, $k, m \in \mathbb{N}$ and $\alpha : S^m \to S^m$ a continuous map generating a free $\mathbb{Z}_p$-action on S^m.*

(a) *If $m \geq k(p-1)$, then for each continuous map $h : S^m \to \mathbb{R}^k$, there exists an $x \in S^m$ with $h(x) = h(\alpha^j(x))$ for all $1 \leq j \leq p-1$.*
(b) *If $m \geq (k-1)(p-1)+1$, then for each continuous map $h : S^m \to \mathbb{R}^k$, there exists an $x \in S^m$ with $h(x) = h(\alpha(x))$.*

Now, to prove Theorem 3 let (X, f_p) be a pair satisfying the hypotheses of Theorem 3 and let $f : X \to \mathbb{R}^k$ be a continuous map. Then by Theorem 1 there exists a continuous equivariant map $F : (S^m, \alpha_{p,l}) \to (X, f_p)$. Thus $h = f \circ F : S^m \to \mathbb{R}^k$ is a continuous map.

If $m \geq (k-1)(p-1)+1$, it follows from item (b) of Theorem ML that there exists $y \in S^m$ such that $h(y) = h(\alpha_{p,l}(y))$. Then if $x = F(y) \in X$ we have

$$f(x) = h(y) = h(\alpha_{p,l}(y)) = f(F(\alpha_{p,l}(y))) = f(f_p(F(y))) = f(f_p(x)).$$

If $m \geq k(p-1)$, it follows from item (a) of Theorem ML that there exists $y \in S^m$ such that $h(y) = h((\alpha_{p,l})^j(y))$ for all $j = 0, 1, \ldots, p-1$. Then if $x = F(y) \in X$ we deduce in a similar way that $f(x) = f((f_p)^j(x))$ for all $j = 1, \ldots, p-1$.

5. Proof of Theorem 4. Here we use the following theorem due to Aarts, Fokkink and Vermeer [1]:

THEOREM AFV. *Let W be a paracompact, Hausdorff space such that* $\dim(W) \leq m$. *Suppose that α is a fixed point free involution of W. Then there exists a closed cover $\mathcal{C} = \{C_0, C_1, \ldots, C_k\}$ of W with $k \leq m+1$ sets such that $C_j \cap \alpha(C_j) = \emptyset$ for each $j = 0, 1, \ldots, m+1$.*

To prove Theorem 4, suppose by contradiction that $f(x) \neq f(f_2(x))$ for all $x \in X$. Let (W, τ) be a pair such that $W = Y \times Y - \triangle$ where $\triangle$ is the diagonal, and τ is the involution $\tau(x, y) = (y, x)$. Then W is paracompact, Hausdorff, $\dim(W) \leq m - 1$ and τ is a free continuous involution of W. By Theorem AFV there exists a covering $W = \bigcup_{j=0}^{k} H_j$ by $k+1$ closed sets such that $H_j \cap \tau(H_j) = \emptyset$ for all $j = 0, 1, \ldots, k$ and $k \leq m$. By Theorem 1 there exists an equivariant map $F : (S^m, \alpha_2) \to (X, f_2)$, and the map $g : (X, f_2) \to (W, \tau)$ given by $g(x) = (f(x), f(f_2(x)))$ is also equivariant, so $h = g \circ F : (S^m, \alpha_2) \to (W, \tau)$ is equivariant; therefore $S^m = \bigcup_{j=0}^{k} h^{-1}(H_j)$ is a covering by $k+1$ closed sets and since $H_j \cap \tau(H_j) = \emptyset$ for all $j = 0, 1, \ldots, k$, it follows that $h^{-1}(H_j) \cap \alpha_2(h^{-1}(H_j)) = \emptyset$ for all $j = 0, 1, \ldots, k$. This contradicts the classical Lusternik–Schnirelmann theorem (Theorem LS). Thus we conclude that there exists $x \in X$ such that $f(x) \neq f(f_2(x))$.

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Tomas Edson Barros
DM-UFSCar, CP 676
13565-905 São Carlos-SP, Brazil
E-mail: dteb@dm.ufscar.br

Carlos Biasi
ICMC-USP, CP 668
13560-970 São Carlos-SP, Brazil
E-mail: biasi@icmc.usp.br

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