ON SPACES WITH THE IDEAL CONVERGENCE PROPERTY

BY

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Abstract. Let $I \subseteq P(\omega)$ be an ideal. We continue our investigation of the class of spaces with the $I$-ideal convergence property, denoted $\mathcal{IC}(I)$. We show that if $I$ is an analytic, non-countably generated $P$-ideal then $\mathcal{IC}(I) \subseteq s_0$. If in addition $I$ is non-pathological and not isomorphic to $I_b$, then $\mathcal{IC}(I)$ spaces have measure zero. We also present a characterization of the $\mathcal{IC}(I)$ spaces using clopen covers.

1. Introduction. Throughout this paper $X$ is a separable metric space and $I \subseteq P(\omega)$ is an ideal on $\omega$ containing all finite subsets of $\omega$. The power set $P(\omega)$ is considered to be a topological space with the product topology induced from $2^\omega$ by identifying subsets of $\omega$ with their characteristic functions. We assume that $P(\omega)$ is a closed subset of the interval $[0,1]$. Recall that an ideal $I \subseteq P(\omega)$ is called a $P$-ideal if whenever $A_0, A_1, A_2, \ldots \in I$ is a sequence of sets then there exists a set $A_\infty \in I$ such that $A_n \subseteq^* A_\infty$ for all $n < \omega$, i.e., $|A_n \setminus A_\infty| < \omega$. We are especially interested in the analytic $P$-ideals $I \subseteq P(\omega)$ because of Solecki’s theorem stating in particular that for any such ideal there exists a finite lower semicontinuous submeasure $\varphi : P(\omega) \to [0,\infty]$ such that $I = \text{Exh}(\varphi) := \{A \subseteq \omega : \lim_{n \to \infty} \varphi(A \setminus n) = 0\}$ (see [6, Theorem 3.1]).

We say that a sequence of functions $f_n : X \to \mathbb{R}$ is $I$-convergent to a function $g : X \to \mathbb{R}$, denoted $I\text{-lim } f_n = g$, if for every $\varepsilon > 0$ and every $x \in X$ the set $\{n \in \omega : |f_n(x) - g(x)| \geq \varepsilon\} \in I$. In [3] the authors studied spaces $X$ where $I$-convergence of sequences of continuous functions implies pointwise convergence of a subsequence indexed by elements of a set from the dual filter $\mathcal{F}(I) := \{B \subseteq \omega : B^c \in I\}$. More specifically, recall

**Definition 1.** Let $I$ be an ideal on $\omega$ and let $X$ be a separable metric space. We say that $X$ has the $I$-ideal convergence property if whenever $f_n : X \to \mathbb{R}$ is a sequence of continuous functions $I$-convergent to the zero function then there exists a set $M = \{m_0 < m_1 < m_2 < \cdots\} \in \mathcal{F}(I)$ such that for all $x \in X, \lim_{i \to \infty} f_{m_i}(x) = 0$. The class of all spaces with the $I$-ideal

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convergence property is denoted by $IC(I)$ and the set of all subspaces of $X$ with the $I$-ideal convergence property is denoted by $IC_X(I)$.

In [3] we said that an ideal $I \subseteq \mathcal{P}(\omega)$ is generated by a set $C \subseteq \omega$ if $I = I_C := \{ A \subseteq \omega : A \subseteq^* C \}$. It is easy to show (see part $\Leftarrow$ of the proof of Proposition 2 of [3]) that for any space $X$, if $I$ is generated by a single set then $IC_X(I) = \mathcal{P}(X)$. Clearly, if $I$ is a $P$-ideal which is countably generated, i.e., there exists a sequence $A_0, A_1, A_2, \ldots \subseteq \omega$ such that $I = \{ A \subseteq \omega : A \subseteq^* \bigcup_{n<\omega} A_n \}$, then $I$ is also generated by a single set. For a characterization of countably generated ideals see Proposition 1.2.8 of [2].

In this note we show that it is consistent that for all analytic, non-countably generated, non-pathological $P$-ideals $I$ the class $IC(I)$ contains countable spaces only. This result should be viewed in the context of Corollary 5 of [3] stating that under CH, for any analytic $P$-ideal $I$ there exists an uncountable space $X$ in $IC(I)$. Note that the class of analytic, non-countably generated, non-pathological $P$-ideals is very broad and includes most ideals discussed in the literature including summable and Erdős–Ulam ideals (see comments following Corollary 1.9.4, p. 31 of [2]). In Theorem 3 we give a characterization of spaces with $I$-ideal convergence property using clopen covers.

2. Main results. For the readers’ convenience we recall a few more definitions and simple facts. A set $X \subseteq \mathbb{R}$ is called a $\sigma$-set (or $X \in \sigma$) if every relative $G_\delta$ subset of $X$ is also a relative $F_\sigma$ set in $X$ (see [5, p. 210]). A mapping $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is called a submeasure if $\varphi(\emptyset) = 0$ and $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ (see [2, p. 20]). A submeasure $\varphi$ is called lower semicontinuous if for all $A \subseteq \omega$, $\lim_{n \to \infty} \varphi(A \cap n) = \varphi(A)$. Since $\mathbb{R}$ contains a closed subset homeomorphic to $\mathcal{P}(\omega)$, to simplify the notation we assume that $\mathcal{P}(\omega) \subseteq [0, 1] \subseteq \mathbb{R}$. For any two integers $n_1 \leq n_2 < \omega$, $\langle n_1, n_2 \rangle := \{ k < \omega : n_1 \leq k \leq n_2 \} = [n_1, n_2] \cap \omega$, and for $m < \omega$ we identify $m$ with $\langle 0, m - 1 \rangle$. Following the idea in the proof of Proposition 2 of [3] we prove the following lemma.

**Lemma 1.** Let $I \subseteq \mathcal{P}(\omega)$ be an ideal and let $X \subseteq I$ be such that $X \in IC(I)$. Then there exists a set $C \in I$ such that $X \subseteq I_C \subseteq I$. In particular, if $X = I$ then $X = I_C$.

**Proof.** Let $g_n : X \to \mathbb{R}$ be defined as follows:

$$g_n(A) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A. \end{cases}$$

It is easy to verify that each $g_n$ is continuous as both inverse images, $g_n^{-1}(\{1\})$ and $g_n^{-1}(\{0\})$, are open in $X$. For $A \in X$ we have $\{ n : g_n(A) > 0 \} = \{ n : g_n(A) = 1 \} = \{ n : n \in A \} = A \in I$, so $I$-$\lim g_n = 0$. Since $X \in IC(I)$ there
exists a set \( M = \{m_0 < m_1 < m_2 < \cdots \} \in \mathcal{F}(I) \) such that \( \lim_{i \to \infty} g_{m_i}(A) = 0 \) for all \( A \in X \). So for all \( A \in X \), \( g_{m_i}(A) = 0 \) for sufficiently large \( i \), hence for those \( i \), \( m_i \notin A \). It follows that \( M \subseteq^* A^c \) and consequently \( A \subseteq^* M^c \). Setting \( C = M^c \) we obtain \( X \subseteq I_C \). ■

**Lemma 2.** If \( I \) is an analytic, non-countably generated \( P \)-ideal on \( \omega \) then \( IC_\mathbb{R}(I) \) does not contain intervals.

**Proof.** Let \( \varphi : \mathcal{P}(\omega) \to [0, \infty] \) be a finite lower semicontinuous submeasure on \( \mathcal{P}(\omega) \) such that

\[
I = Exh(\varphi) = \{ A \subseteq \omega : \lim_{n \to \infty} \varphi(A \setminus n) = 0 \}
\]

(see [6, Theorem 3.1]). Define a descending sequence of sets \( \{A_k : k < \omega\} \) as follows: \( A_0 = \omega \), \( A_k = \{n < \omega : \varphi(\{n\}) \leq 1/k\} \). We shall consider the limit \( \lim_{k \to \infty} \lim_{n \to \infty} \varphi(A_k \setminus n) \).

**Case 1.** Assume

\[
\lim_{k \to \infty} \lim_{n \to \infty} \varphi(A_k \setminus n) = 0
\]

but

\[
\lim_{n \to \infty} \varphi(A_k \setminus n) > 0.
\]

For every \( k < \omega \) there exists a \( t > k \) such that \( |A_k \setminus A_t| = \omega \). Otherwise there would be a \( k_0 \) such that \( \lim_{n \to \infty} \varphi(A_k \setminus n) = \lim_{n \to \infty} \varphi(A_{k_0} \setminus n) \) for all \( k \geq k_0 \), contradicting (2.2). Define an increasing sequence \( k_l, l < \omega \), as follows: \( k_0 = 0 \), \( k_{l+1} = \min\{k : |A_{k_l} \setminus A_k| = \omega\} \). Set \( B_l = A_{k_l} \setminus A_{k_{l+1}} \).

**Claim.** \( I = \{A : \forall l \ |B_l \cap A| < \omega\} \).

If \( A \in I \) then \( \lim_{n \to \infty} \varphi(A \setminus n) = 0 \) so \( |B_l \cap A| < \omega \) because otherwise we would have \( \lim_{n \to \infty} \varphi(A \setminus n) \geq 1/k_{l+1} \). On the other hand, if all sets \( B_l \cap A \) are finite then for every \( k \) there exists an \( n \) such that \( A \setminus n \subseteq A_k \). It follows that \( \lim_{n \to \infty} \varphi(A \setminus n) \leq \lim_{k \to \infty} \lim_{n \to \infty} \varphi(A_k \setminus n) = 0 \), which proves the Claim.

This shows that there exists a bijection \( \alpha : \omega \to \omega^2 \) such that \( \{\alpha[A] : A \in I\} = I_b \) where

\[
I_b := \{A \subseteq \omega^2 : \forall n < \omega \exists m < \omega \forall k < \omega ((n, k) \in A \Rightarrow k \leq m)\}.
\]

By Theorem 4 of [3] it follows that \( IC_\mathbb{R}(I) = IC_\mathbb{R}(I_b) \subseteq \sigma \). Intervals are not \( \sigma \)-sets [5], hence the assertion of Lemma 2 in Case 1 is proved.

**Case 2.** Assume that

\[
\lim_{k \to \infty} \lim_{n \to \infty} \varphi(A_k \setminus n) = 0
\]

and

\[
\text{there exists a } k_0 \text{ such that } \lim_{n \to \infty} \varphi(A_{k_0} \setminus n) = 0.
\]
Then $A_{k_0} \in I$ and $I$ is countably generated because $I = \{ A : A \subseteq^* A_{k_0} \}$. The inclusion $I \supseteq \{ A : A \subseteq^* A_{k_0} \}$ is clear while if $|A \setminus A_{k_0}| = \omega$ then $\liminf_{n \to \infty} \varphi(A \setminus n) \geq 1/k_0$ so $A \notin I$.

**Case 3.** Finally assume that

\[(2.7) \quad \lim_{k \to \infty} \lim_{n \to \infty} \varphi(A_k \setminus n) = \varepsilon > 0.\]

We recursively define a sequence $\{n_k : k < \omega\} \subseteq \omega$ as follows. Since $\lim_{n \to \infty} \varphi(A_0 \setminus n) \geq \varepsilon$ there exists an integer $n_0$ such that for all $n \geq n_0$, $\varphi(A_0 \setminus n) > \varepsilon/2$. Now suppose we have defined the terms $n_0, n_1, \ldots, n_k$ such that for all $j = 1, \ldots, k$, if $n > n_j$ then $\varphi(A_j \setminus \langle n_{j-1}, n \rangle) > \varepsilon/2$. Since $\lim_{n \to \infty} \varphi(A_{k+1} \setminus n) \geq \varepsilon$ and $\varphi(A_{k+1} \setminus n) \geq \varphi(A_{k+1} \setminus (n+1))$ for all $n \in \omega$, we have $\varphi(A_{k+1} \setminus n_k) > \varepsilon/2$. By the lower semicontinuity of $\varphi$, $\lim_{n \to \infty} \varphi((A_{k+1} \setminus n_k) \cap n) = \varphi(A_{k+1} \setminus n_k)$ so there exists an integer $n_{k+1} \geq k + 1$ such that for all $n \geq n_{k+1}$, $\varphi(A_{k+1} \cap \langle n, n_k \rangle) > \varepsilon/2$. Also by lower semicontinuity the set $Z = \{ A : \forall k > 0 (\varphi(A \setminus n_k) \leq 2/k) \}$ is a closed subset of $\mathcal{P}(\omega)$ and by (2.1), $Z \subseteq I$.

Now towards a contradiction, based on Proposition 3 of [3], without loss of generality we may assume that $[0, 1] \in \mathcal{I}C_{\mathbb{R}}(I)$. By the same Proposition 3(2), $\mathcal{I}C_{\mathbb{R}}(I)$ also contains $Z$. By Lemma 1 there exists a set $C \in I$ such that $Z \subseteq \{ A : A \subseteq^* C \}$. As $C \in I$, by (2.1) let $l_0 \in \omega$ be such that $\varphi(C \setminus m) < \varepsilon/2$ for all $m \geq 2^{l_0}$. Since for $l < \omega$, $\varphi(A_{2^l+j+1} \cap \langle n_{2^l}, n_{2^l+1} \rangle) > \varepsilon/2$, we may select numbers $d_l \in (A_{2^l+j+1} \cap \langle n_{2^l}, n_{2^l+1} \rangle) \setminus C$ and define the set $D = \{ d_l : l \geq l_0 \}$. As $D$ is infinite and disjoint from $C$, clearly $D \notin \mathcal{I}C_{\mathbb{R}}(I)$. In particular

\[(2.8) \quad D \notin Z.\]

Now for $k > 0$ let $l_k = \max\{ l_0, \lfloor \log_2 n_k \rfloor \}$. We have

\[
\varphi(D \setminus n_k) \leq \sum_{l \geq l_k} \varphi(\{ d_l \}) \leq \sum_{l \geq l_k} \frac{1}{2^l + 1} \leq \frac{2}{2^{\lfloor \log_2 n_k \rfloor}} \leq \frac{2}{n_k} \leq \frac{2}{k}.
\]

It follows that $D \in Z$, contradicting (2.8). \(\blacksquare\)

The family $\mathcal{I}C_{\mathbb{R}}(I)$ is closed under continuous images (Proposition 3(1) of [3]). Propositions 2.6.1 and 2.6.13 of [7] imply the following:

**Remark 1.** If $I$ is as in Lemma 2 and $X \in \mathcal{I}C_{\mathbb{R}}(I)$ then $X$ is totally imperfect (i.e., does not contain any perfect sets).

A subset $X \subseteq \mathbb{R}$ is called an $s_0$-set (or $X \in s_0$) if for every perfect subset $P \subseteq \mathbb{R}$ there exists another perfect subset $Q \subseteq P \setminus X$ (see [5, p. 217]).

**Theorem 1.** If $I$ is an analytic, non-countably generated $P$-ideal on $\omega$ then $\mathcal{I}C_{\mathbb{R}}(I) \subseteq s_0$.

**Proof.** Suppose $X \in \mathcal{I}C_{\mathbb{R}}(I)$ and let $P \subseteq \mathbb{R}$ be a perfect set. Let $P_1 \subseteq P$ be a perfect set homeomorphic to the Cantor set (see [7, Theorem 2.6.3]).
Let $h : P_1 \to P_1 \times P_1$ be a homeomorphism and let $p : P_1 \times P_1 \to P_1$ be the projection, $p(x, y) = x$. It is well known that there is a continuous surjection $g : P_1 \to [0, 1]$ (see [7, Theorem 2.6.13]). The composition $q = g \circ p \circ h : P_1 \to [0, 1]$ is a continuous surjection so $q[P_1 \cap X] \in IC_R(I)$ by Proposition 3 of [3]. By Lemma 2 there exists $y \in [0, 1] \setminus q[P_1 \cap X]$. It is easy to see that $Q = q^{-1}\{\{y\}\}$ is a perfect subset of $P$ disjoint from $X$. 

Under the Continuum Hypothesis we have an example of a non-countably generated, non-analytic $P$-ideal $J$ such that $IC(J)$ contains large spaces.

**Example 1.** (CH) There exists a maximal $P$-ideal $J$ such that $\mathbb{R} \in IC(J)$.

**Proof.** We will construct a sequence of subsets $X_\alpha \in [\omega]^{\omega}$, $\alpha < \mathfrak{c}$, such that $X_\alpha \subsetneq^* X_\beta$ whenever $\alpha < \beta < \mathfrak{c}$. The sequence $\{X_\alpha : \alpha < \mathfrak{c}\}$ will be such that the ideal $J$ dual to the filter $\mathcal{F} = \{B \subseteq \omega : \exists \alpha < \mathfrak{c} (X_\alpha \subsetneq^* B)\}$ will have the desired properties.

Let $\{A_{\alpha + 1} : \alpha < \mathfrak{c}\}$ be a sequence of all subsets of $\omega$ and $\{\langle f_{\alpha + 1}^n : n < \omega\rangle : \alpha < \mathfrak{c}\}$ be an indexed family of all sequences of continuous functions $f_{\alpha + 1}^n : \mathbb{R} \to \mathbb{R}$. Set $X_0 = \omega$. Suppose that for some $\alpha < \mathfrak{c}$ the sets $X_\beta$, $\beta \leq \alpha$, are defined. Define $X'_{\alpha + 1}$ as follows. If the sequence $\langle f_\alpha^n : n \in X_\alpha\rangle$ is pointwise convergent to the zero function then we set $X'_{\alpha + 1} = X_\alpha$. Otherwise there exists an $\varepsilon > 0$ and $x \in \mathbb{R}$ such that $E = \{n : |f_\alpha^n(x)| \geq \varepsilon\}$ is an infinite subset of $X_\alpha$. In that case we set $X'_{\alpha + 1} = E$. Now to obtain $X_{\alpha + 1}$ we consider the intersection $X'_{\alpha + 1} \cap A_\alpha$. If it is infinite then let $X_{\alpha + 1} = X'_{\alpha + 1} \cap A_\alpha$. Otherwise set $X_{\alpha + 1} = X'_{\alpha + 1} \cap (\omega \setminus A_\alpha)$.

To finish the proof we need to define the $X_\lambda$ for limit ordinals $0 < \lambda < \mathfrak{c}$. Let $\{\gamma_n : n < \omega\} \subseteq \lambda$ be a sequence of ordinals cofinal in $\lambda$. For each $n < \omega$ let $Y_n = \bigcap_{k \leq n} X_{\gamma_k}$. We have $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$ and each $Y_n$ is infinite. Recursively pick points $x_n \in Y_n \setminus \{x_m : m < n\}$ and let $X_\lambda = \{x_n : n < \omega\}$. It is easy to verify that $X_\lambda \subsetneq^* X_\alpha$ for all $\alpha < \lambda$.

Now having defined the sequence $\{X_\alpha : \alpha < \mathfrak{c}\}$, it is easy to show that the ideal $J$ dual to the filter $\mathcal{F} = \mathcal{F}(J) = \{B \subseteq \omega : \exists \alpha < \mathfrak{c} (X_\alpha \subsetneq^* B)\}$ is a maximal $P$-ideal such that $\mathbb{R} \in IC(J)$. 

Recall that a measure on $\mathcal{P}(\omega)$ is a submeasure which is additive, i.e., $\mu : \mathcal{P}(\omega) \to [0, \infty]$ is a measure if $\mu(\emptyset) = 0$, $\mu(A) \leq \mu(A \cup B)$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ for any two disjoint sets $A, B \in \mathcal{P}(\omega)$. Before stating our next theorem we recall a definition from [2, p. 21]. We say that a submeasure $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is non-pathological if

$$\varphi(A) = \sup\{\mu(A) : \mu \text{ is a measure on } \mathcal{P}(\omega) \text{ and } \forall B \subseteq \omega (\mu(B) \leq \varphi(B))\}.$$ 

An ideal $I \subseteq \mathcal{P}(\omega)$ is non-pathological if $I = Exh(\varphi)$ for some non-patho-
logical submeasure \( \varphi \). Let \( \lambda \) and \( \lambda^* \) be the Lebesgue measure and the outer Lebesgue measure respectively. Set \( \mathcal{N} = \{ Y \subseteq \mathbb{R} : \lambda(Y) = 0 \} \) and recall \( I_b \) is as in 2.4.

**Theorem 2.** If \( I \) is an analytic, non-pathological, non-countably generated \( P \)-ideal not isomorphic to \( I_b \), then \( \mathcal{I}_{\mathbb{R}}(I) \subseteq \mathcal{N} \).

**Proof.** Suppose \( I = \text{Exh}(\varphi) \) for some non-pathological submeasure \( \varphi \). Let \( X \subset \mathcal{I}_{\mathbb{R}}(I) \). By Lemma 2, \( X \) is zero-dimensional and we may assume that \( X \subseteq [0, 1] \). Similarly to (but not exactly as in) the proof of Lemma 2 we define a sequence of sets \( A_0, A_1, A_2, \ldots \) as \( A_k = \{ n < \omega : \varphi(\{ n \}) \leq 1/2^k \} \) and again consider the limit (2.2). By the proof of Lemma 2 our assumptions on \( I \) imply that

\[
\lim_{k \to \infty} \lim_{n \to \infty} \varphi(A_k \setminus n) = \varepsilon \quad \text{for some } \varepsilon > 0.
\]

Let \( \{ n_k : k < \omega \} \) be defined exactly as in Case 3 of Lemma 2. Define \( I_k = A_k \cap [n_k, n_{k+1}] \) and let \( \mu_k \) be a measure on \( \mathcal{P}(\omega) \) such that \( \mu_k \leq \varphi \) and \( \mu_k(I_k) > \frac{1}{2} \varphi(I_k) \) (see (2.9)). For each \( k \) we form a partition of \( X \) into open (in \( X \)) sets \( \{ U_{ki} : i \in I_k \} \) such that \( \lambda^*(U_{ki}) < 2\mu_k(\{ i \})/\mu_k(I_k) \). We define \( F : X \to \mathcal{P}(\omega) \) by \( F(x) = \{ i : \exists k (x \in U_{ki}) \} \). Since the sets \( I_k \) are finite and pairwise disjoint the function \( F \) is continuous and we have \( F[X] \subset \mathcal{I}_{\mathbb{R}}(I) \) by Proposition 3(1) of [3]. Also for each \( x \in X \), \( F(x) \in I \) because \( \lim_{n \to \infty} \varphi(F(x) \setminus n) = 0 \). This is due to the fact that we are using \( 1/2^k \) in the definition of \( A_k \) and \( |F(x) \cap (A_k \setminus A_{k+1})| \leq 1 \). So with \( F[X] \subset I \) and \( F[X] \subset \mathcal{I}_{\mathbb{R}}(I) \) by Lemma 1 there exists a set \( C \in I \) such that \( F(x) \subseteq C \) for all \( x \in X \). It follows that

\[
X \subseteq \bigcup_{n<\omega} \bigcap_{k>n} \bigcup_{i \in C \cap I_k} U_{ki}.
\]

Notice that for any \( k < \omega \),

\[
\lambda^* \left( \bigcup_{i \in C \cap I_k} U_{ki} \right) \leq \sum_{i \in C \cap I_k} \lambda^* (U_{ki}) \leq \sum_{i \in C \cap I_k} 2\mu_k(\{ i \})/\mu_k(I_k)
\]

\[
= 2\mu_k(C \cap I_k)/\mu_k(I_k) \leq 2\mu_k(C \cap I_k)/(\varepsilon/2)
\]

\[
\leq 4\varphi(C \setminus n_k)/\varepsilon.
\]

Since \( C \in I \) the last quantity converges to zero as \( k \to \infty \). It follows that

\[
\lambda^* \left( \bigcap_{k>n} \bigcup_{i \in C \cap I_k} U_{ki} \right) = 0
\]

and \( \lambda^*(X) = 0 \) as well. 

**Corollary 1.** It is consistent with ZFC that if \( I \) is an analytic, non-pathological, non-countably generated \( P \)-ideal, then \( \mathcal{I}_{\mathbb{R}}(I) \) contains countable sets only.
Proof. Suppose $X \in \mathcal{IC}_{\mathbb{R}}(I)$. If $I$ is not isomorphic to $I_b$ then by Theorem 2 and Proposition 3(1) of [3] all continuous images of $X$ are also in $\mathcal{IC}_{\mathbb{R}}(I)$ and they have Lebesgue measure zero. Bartoszyński and Shelah [1] showed that consistently such spaces may be countable only. If $I$ is isomorphic to $I_b$ then see Corollary 7 of [3].

Recall $\text{add}^*(I) := \min\{|A| : A \subseteq I \land A \nsubseteq A (A \subseteq^* B)\}$ and $\text{non}(\mathcal{N}) := \min\{|X| : X \subseteq \mathbb{R}, \lambda^*(X) > 0\}$. By Proposition 4 of [3] for $P$-ideals $I$ if $X \subseteq \mathbb{R}$ with $|X| < \text{add}^*(I)$ then $X \in \mathcal{IC}_{\mathbb{R}}(I)$. By Theorem 2 above we obtain

**Corollary 2.** If $I$ is an analytic, non-pathological, non-countably generated $P$-ideal non-isomorphic to $I_b$, then $\text{add}^*(I) \leq \text{non}(\mathcal{N})$.

Our last theorem gives a characterization of spaces with the $I$-ideal convergence property using clopen covers.

**Theorem 3.** Suppose $I$ is a $P$-ideal on $\omega$ and let $X$ be a separable, zero-dimensional metric space. Then $X \in \mathcal{IC}(I)$ if and only if for every sequence of clopen sets $U_0, U_1, U_2, \ldots \subseteq X$ with $\{n : x \notin U_n\} \in I$ for any $x \in X$, there exists a set $A \in \mathcal{F}(I)$ such that

$$X \subseteq \bigcup_{m<\omega} \bigcap_{a \in A \setminus m} U_a.$$

**Proof.** Assume that $X \in \mathcal{IC}(I)$. Let $U_0, U_1, U_2, \ldots \subseteq X$ be a sequence of clopen sets such that for any $x \in X$,

$$\{n : x \notin U_n\} \in I. \quad (2.10)$$

Define a sequence of functions $h_n : X \to \mathbb{R}$ by

$$h_n(x) = \begin{cases} 0 & \text{if } x \in U_n, \\ 1 & \text{if } x \notin U_n. \end{cases}$$

Since $U_n$ is clopen, $h_n$ is continuous. By (2.10), $\text{I-lim} h_n(x) = 0$. As $X \in \mathcal{IC}(I)$, there exists a set $A = \{a_0 < a_1 < a_2 < \cdots\} \in \mathcal{F}(I)$ such that $\lim_{n \to \infty} h_{a_n}(x) = 0$ for all $x \in X$. It follows that for every $x \in X$ there exists $m < \omega$ such that for any $a_n > m$ we have $h_{a_n}(x) = 0$, so $x \in U_{a_n}$. Hence

$$X \subseteq \bigcup_{m<\omega} \bigcap_{a \in A \setminus m} U_a.$$

To prove the other implication suppose that $f_n : X \to \mathbb{R}$ is a sequence of continuous functions with $\text{I-lim} f_n = 0$. Since $X$ is a separable, zero-dimensional metric space, any two disjoint closed subsets of $X$ may be separated by a clopen set (see [4, pp. 35 and 357]). It follows that there exist clopen sets $U^k_n \subseteq X$ such $\{x : |f_n(x)| \leq 1/k\} \subseteq U^k_n \subseteq \{x : |f_n(x)| < 2/k\}$. Since $\text{I-lim} f_n(x) = 0$ for every $x \in X$, we have $\{n : x \notin U^k_n\} \in I$ for every
\( k < \omega \). By our assumption for each \( k < \omega \) there exists a set \( A^k \in \mathcal{F}(I) \) such that \( X \subseteq \bigcup_{m<\omega} \bigcap_{n \in A^k \setminus m} U_n^k \). Since \( I \) is a \( P \)-ideal there exists a set \( M = \{ m_0 < m_1 < m_2 < \cdots \} \in \mathcal{F}(I) \) with \( M \subseteq^* A^k \) for all \( k < \omega \). It follows that \( \lim_{i \to \infty} f_{m_i}(x) = 0 \) for all \( x \in X \). Hence \( X \in \mathcal{I}(I) \).

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