COLLOQUIUM MATHEMATICUM

VOL. 111 2008 NO. 1

ON STRONGLY l_p-SUMMING m-LINEAR OPERATORS

BY

LAHCÈNE MEZRAG (M'sila)

Abstract. We introduce and study a new concept of strongly l_p-summing m-linear operators in the category of operator spaces. We give some characterizations of this notion such as the Pietsch domination theorem and we show that an m-linear operator is strongly l_p-summing if and only if its adjoint is l_p-summing.

1. Introduction. The development of the theory of polynomials and multilinear operators can be divided into two periods. The first starts in the thirties of the last century, essentially motivated externally through holomorphic and differential functions on infinite-dimensional spaces. The second begins in the eighties, mainly due to Pietsch [Pie83] where the idea to generalize the theory of ideals to the multilinear setting appears. Motived by the importance of this theory, several authors have developed and studied many concepts relating to summability; see [Ale85, AM89, Dia03, Mat96, Mat03, MT99, Sch91] among so many others. In this note we introduce a new concept concerning summability of multilinear operators.

The concept of strongly p-summing linear operators $(1 \leq p < \infty)$ was introduced by J. S. Cohen [Coh73] in order to obtain a characterization of the conjugates of absolutely p^*-summing linear operators. In [AM07], we have generalized this concept to the multilinear case. It is natural to try to develop the same concept in the non-commutative case.

In the present work, we introduce a new notion of summability for multilinear operators, which we call strongly l_p-summing m-linear operators. Using this notion we prove some properties of multilinear operators in the non-commutative case. Our motivation is that the adjoint of a strongly l_p-summing m-linear operator is an l_p-summing operator as studied in [Mez02].

This paper is organized as follows.

In Section 1, we recall some basic definitions and properties concerning the theory of operator spaces.

2000 *Mathematics Subject Classification*: 46B15, 46B28, 47D15.

Key words and phrases: completely bounded operator, operator space, strongly l_p-summing multilinear operator.

In Section 2, we adapt the concept of Cohen strongly p-summing operators introduced in [AM07] to the non-commutative case. We characterize this type of operators by giving the Pietsch domination theorem. The proof is different from that used in [AM07] where we have used Ky Fan's lemma. We introduce the adjoint of a multilinear operator as in [PS05] and we show that a multilinear operator T is strongly l_p-summing if and only if T^* is l_p-summing. The notion of l_p-summing operators has been introduced in [Mez02].

2. Basic definitions and properties. We assume that the reader is familiar with the class of operator spaces. If $\mathcal{H}$ is a Hilbert space, we let $B(\mathcal{H})$ denote the space of all bounded operators on $\mathcal{H}$ and for every n in $\mathbb{N}$ we let M_n denote the space of all $n \times n$ matrices of complex numbers, i.e., $M_n = B(l_2^n)$. If X is a subspace of some $B(\mathcal{H})$ and $n \in \mathbb{N}$, then $M_n(X)$ denotes the space of all $n \times n$ matrices with X-valued entries which we consider in the natural manner as a subspace of $M_n(B(\mathcal{H})) = B(l_2^n(\mathcal{H})) = B(l_2^n \otimes_2 \mathcal{H})$ ($\otimes_2$ is the Hilbert space tensor product).

Definition 1.1. An *operator space* X is a norm closed subspace of some $B(\mathcal{H})$ equipped with a matrix norm inherited by the spaces $M_n(X)$, $n \in \mathbb{N}$.

Let $\mathcal{H}$ be a Hilbert space. We denote by $S_p(\mathcal{H})$ ($1 \le p < \infty$) the Banach space of all compact operators $u : \mathcal{H} \to \mathcal{H}$ such that $\mathrm{Tr}(|u|^p) < \infty$, equipped with the norm

$$\|u\|_{S_p(\mathcal{H})} = (\mathrm{Tr}(|u|^p))^{1/p}.$$

If $\mathcal{H} = l_2$ (resp. l_2^n), we denote $S_p(l_2)$ simply by S_p (resp. $S_p(l_2^n)$ by S_p^n). We also denote by $S_\infty(\mathcal{H})$ (resp. S_∞) the Banach space of all compact operators equipped with the norm induced by $B(\mathcal{H})$ (resp. $B(l_2)$) ($S_\infty^n = B(l_2^n)$). Recall that if $1/p = 1/q + 1/r$ ($1 \le p, q, r < \infty$), then $u \in B_{S_p(\mathcal{H})}$ if and only if there are $u_1 \in B_{S_q(\mathcal{H})}$ and $u_2 \in B_{S_r(\mathcal{H})}$ such that $u = u_1 u_2$, where $B_{S_p(\mathcal{H})}$ is the closed unit ball of $S_p(\mathcal{H})$. We also write $S_p^+(\mathcal{H}) = \{a \in S_p(\mathcal{H}) : a \ge 0\}$.

Definition 1.2. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Let $X \subset B(\mathcal{H})$ and $Y \subset B(\mathcal{K})$ be two operator spaces. A linear map $u : X \to Y$ is *completely bounded* (c.b. for short) if the maps

$$u_n : M_n(X) \to M_n(Y), \qquad (x_{ij})_{1 \le i,j \le n} \mapsto (u(x_{ij}))_{1 \le i,j \le n},$$

are uniformly bounded as $n \to \infty$, i.e., $\sup\{\|u_n\| : n \ge 1\} < \infty$.

In this case we put $\|u\|_{\mathrm{cb}} = \sup\{\|u_n\| : n \ge 1\}$ and we denote by $\mathrm{cb}(X, Y)$ the Banach space of all c.b. maps from X into Y which is also an operator space $(M_n(\mathrm{cb}(X,Y)) = \mathrm{cb}(X, M_n(Y)))$ (see [BP91] and [ER91]). We denote by $X \otimes_{\min} Y$ the corresponding subspace of $B(H \otimes_2 K)$ with the induced norm.

Before continuing we briefly mention some properties of completely bounded operators. Let $\mathcal{OH}$ be the operator Hilbert space introduced by Pisier in [Pis96]). We recall that $\mathcal{OH}$ is homogeneous, in other words, every bounded linear operator $u : \mathcal{OH} \to \mathcal{OH}$ is c.b. and

$$\|u\| = \|u\|_{\mathrm{cb}}. \tag{1.1}$$

Note also that S_2 is completely isometric to $\mathcal{OH} \times \mathcal{OH}$. We denote by $\mathcal{OH}_n$ the n-dimensional version of the Hilbert operator space $\mathcal{OH}$. If now S_2^N ($N \in \mathbb{N}$) is equipped with the operator space structure $\mathcal{OH}_{N^2}$, then for any linear map $u : S_2^N \to \mathcal{OH}_n$, by homogeneity of $\mathcal{OH}$ we have

$$\|u\| = \|u\|_{\mathrm{cb}}. \tag{1.2}$$

Finally, let us recall one more property. Let Y be an operator space such that $Y \subset A$ (a commutative C^*-algebra) $\subset B(\mathcal{H})$. Let X be an arbitrary operator space. Then, for any bounded linear operator $u : X \to Y$, we have

$$\|u\| = \|u\|_{\mathrm{cb}}. \tag{1.3}$$

Let X be an operator space. As usual we denote by $l_p(X)$ (resp. $l_p^n(X)$) for $1 \leq p < \infty$ the space of all sequences (x_i) (resp. finite sequences $(x_1, \ldots, x_n)$) in X equipped with the norm

$$\|(x_i)\|_{l_p(X)} = \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{1/p} < \infty$$

$$\text{(resp. } \|(x_i)_{1\leq i\leq n}\|_{l_p^n(X)} = \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p}\text{)},$$

which becomes an operator space.

Let now $X \subset B(\mathcal{H})$. By Pisier [Pis98, p. 32] we have

$$l_\infty(X) = l_\infty \otimes_{\min} X = B(l_1, X).$$

We can show that for all n in $\mathbb{N}$ and $1 \leq p \leq \infty$,

$$\begin{aligned} \|v\|_{\mathrm{cb}} &= \sup_{a,b \in B^+_{S_{2p}(H)}} \Big(\sum_{j=1}^{n} \|a x_j b\|^p_{S_p(H)}\Big)^{1/p} \\ &= \Big\|\sum_{j=1}^{n} e_j \otimes x_j\Big\|_{l_p^n \otimes_{\min} X} \end{aligned} \tag{1.4}$$

if p is finite, and

$$\|v\|_{\mathrm{cb}} = \Big\| \sum_{j=1}^{n} e_j \otimes x_j \Big\|_{l_\infty^n \otimes_{\min} X} = \Big\| \sum_{j=1}^{n} e_j \otimes x_j \Big\|_{l_\infty^n \otimes_\varepsilon X} = \|v\| \tag{1.5}$$

if $p = \infty$. Here $v : l_{p^*}^n \to X$ is such that $v(e_i) = x_i$.

3. Non-commutative strongly p-summing multilinear operators. We extend to multilinear operators the class of strongly p-summing operators defined in 1973 by Cohen [Coh73]. In the non-commutative case, we prove directly the principal result of this section, which is the Pietsch domination theorem. For the linear case, Cohen deduces it by duality because the adjoint of a strongly p-summing operator is absolutely p^*-summing. In the commutative case, in the category of multilinear operators, to show the Pietsch domination theorem we have used Ky Fan's lemma (see [AM07]). We also give the relationship between the spaces $\mathcal{D}_{l_p}^m(X_1, \ldots, X_m; Y)$ and $\pi_{l_p}(Y^*, \mathcal{L}(X_1, \ldots, X_m; \mathbb{R}))$. We recall that $\pi_{l_p}(X, Y)$ (see [Mez02] for more details) is the space of all l_p-summing operators from X into Y, where X is an operator space.

In what follows, for $m \in \mathbb{N}$ we consider $X_1 \subset B(\mathcal{H}_1), \ldots, X_m \subset B(\mathcal{H}_m)$, $Y \subset B(\mathcal{K})$ $((\mathcal{H}_j)_{1 \le j \le m}$ and $\mathcal{K}$ are arbitrary Hilbert spaces) as operator spaces.

Let $m \in \mathbb{N}$. If $X_1, \ldots, X_m, Y$ are simply Banach spaces over the real numbers, we denote by $\mathcal{L}(X_1, \ldots, X_m; Y)$ the space of all continuous m-linear operators from $X_1 \times \cdots \times X_m$ into Y. If $X_1 = \cdots = X_m = X$, we write simply $\mathcal{L}(^m X; Y)$. The vector space of all bounded linear operators from X into Y will be denoted by $\mathcal{B}(X, Y)$.

Definition 2.1. Let $1 \le p < \infty$ and $m \in \mathbb{N}$. Let $X_1, \ldots, X_m$ be Banach spaces and Y be an operator space. An m-linear operator $T : X_1 \times \cdots \times X_m \to Y$ is *strongly l_p-summing* if there is a constant $C > 0$ such that for any $n \in \mathbb{N}, x_1^j, \ldots, x_n^j \in X_j$ $(j = 1, \ldots, m)$ and $y_1^*, \ldots, y_n^* \in Y^*$, we have

$$\|(\langle T(x_i^j)_j, y_i^* \rangle)\|_{l_1^n} \le C \Big(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|_{X_j}^p \Big)^{1/p} \sup_{a,b \in B_{S_{2p^*}}^+} \|(a y_i^* b)\|_{l_{p^*}^n(S_{p^*}(\mathcal{L}))}. \tag{2.1}$$

Here $\mathcal{L}$ is a Hilbert space such that $Y^* \subset B(\mathcal{L})$, because by [Ble92], Y^* is an operator space.

The class of strongly l_p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into Y, denoted by $\mathcal{D}_{l_p}^m(X_1, \ldots, X_m; Y)$, is a Banach space when equipped with the norm $d_{l_p}^m(T)$, which is the smallest constant C such that the inequality (2.1) holds.

In the commutative case the supremum in (2.1) is replaced by $\sup_{y\in B_Y}\|(y_i^*(y))\|_{l_{p^*}^n}$.

Let $m\in\mathbb{N}$ and $T\in\mathcal{L}(X_1,\dots,X_m;Y)$. The operator T is strongly l_p-summing if and only if, for all $n\in\mathbb{N}$ and all $v\in\mathcal{B}(l_{p^*}^n,Y^*)$ ($v(e_i)=y_i^*$ or $v=\sum_{i=1}^n e_i\otimes y_i^*$), we have (by (2.1) and (1.4))

$$\sum_{i=1}^n|\langle T(x_i^1,\dots,x_i^m),v(e_i)\rangle|\le C\Big(\sum_{i=1}^n\prod_{j=1}^m\|x_i^j\|_{X_j}^p\Big)^{1/p}\|v\|_{\mathrm{cb}}. \tag{2.2}$$

For $p=1$, we have $\mathcal{D}_{l_1}^m(X_1,\dots,X_m;Y)=\mathcal{L}(X_1,\dots,X_m;Y)$.

Remark. 1. If $Y=\mathcal{OH}$, then by (1.1) and (2.2) strongly l_2-summing and Cohen strongly 2-summing coincide.

2. If $Y=L_1$, then by (1.3) and (2.2) strongly l_p-summing and Cohen strongly p-summing coincide.

Proposition 2.2. *Let $X_1,\dots,X_m,E_1,\dots,E_m$ be Banach spaces and let Y,Z be operator spaces. Let $T\in\mathcal{L}(X_1,\dots,X_m;Y)$, $R\in\mathrm{cb}(Y,Z)$ and $S_j\in\mathcal{B}(E_j,X_j)$ $(1\le j\le m)$.*

- (i) *If T is strongly l_p-summing, then RT is strongly l_p-summing and $d_{l_p}^m(RT)\le\|R\|_{\mathrm{cb}}d_{l_p}^m(T)$.*
- (ii) *If T is strongly l_p-summing, then $T\circ(S_1,\dots,S_m)$ is strongly l_p-summing and $d_{l_p}^m(T\circ(S_1,\dots,S_m))\le d_{l_p}^m(T)\prod_{j=1}^m\|S_j\|$.*

Proof. (i) Let $n\in\mathbb{N}$, $x_1^j,\dots,x_n^j\in X_j$ and $z_1^*,\dots,z_n^*\in Z^*$. It suffices by (2.2) to show that

$$\sum_{i=1}^n|\langle RT(x_i^1,\dots,x_i^m),z_i^*\rangle|\le\|R\|_{\mathrm{cb}}d_{l_p}^m(T)\Big(\sum_{i=1}^n\prod_{j=1}^m\|x_i^j\|_{X_j}^p\Big)^{1/p}\|v\|_{\mathrm{cb}}$$

where $v:Z\to l_p^n$ is such that $v(z)=\sum_{i=1}^n z_i^*(z)e_i$. By (2.2) we have

$$\begin{aligned}\sum_{i=1}^n|\langle RT(x_i^1,\dots,x_i^m),z_i^*\rangle|&=\sum_{i=1}^n|\langle T(x_i^1,\dots,x_i^m),R^*(z_i^*)\rangle|\\&\le d_{l_p}^m(T)\Big(\sum_{i=1}^n\prod_{j=1}^m\|(x_i^j)\|_{X_j}^p\Big)^{1/p}\|w\|_{\mathrm{cb}}\end{aligned}$$

where $w^*:l_{p^*}^n\to Y^*$ is such that $w^*(e_i)=R^*(z_i^*)$. We have $w^*=R^*\circ v^*$, where $v:l_{p^*}^n\to Z^*$, $v(e_i)=z_i^*$. This implies that

$$\|w\|_{\mathrm{cb}}=\|w^*\|_{\mathrm{cb}}=\|R^*\circ v^*\|_{\mathrm{cb}}\le\|R^*\|_{\mathrm{cb}}\|v\|_{\mathrm{cb}}.$$

(ii) Let $n \in \mathbb{N}$, $e_1^j, \ldots, e_n^j \in E_j$ and $y_1^*, \ldots, y_n^* \in Y^*$. By (2.2) we have

$$\begin{aligned}
\sum_{i=1}^{n} |\langle T \circ (S_1, \ldots, S_m)(e_i^1, \ldots, e_i^m), y_i^* \rangle| \\
&= \sum_{i=1}^{n} |\langle T(S_1(e_i^1), \ldots, S_m(e_i^m)), y_i^* \rangle| \\
&\leq d_{l_p}^m(T) \Big(\sum_{i=1}^{n} \prod_{j=1}^{m} \|S_j(e_i^j)\|_{X_j}^p \Big)^{1/p} \|v\|_{\mathrm{cb}} \quad \Big(v(y) = \sum_{i=1}^{n} y_i^*(y) e_i\Big) \\
&\leq d_{l_p}^m(T) \Big(\prod_{j=1}^{m} \|S_j\|^p \sum_{i=1}^{n} \|e_i^j\|_{E_j}^p \Big)^{1/p} \|v\|_{\mathrm{cb}} \\
&\leq d_p^m(T) \prod_{j=1}^{m} \|S_j\| \Big(\sum_{i=1}^{n} \prod_{j=1}^{m} \|e_i^j\|_{E_j}^p \Big)^{1/p} \|v\|_{\mathrm{cb}}.
\end{aligned}$$

This implies that $d_p^m(T \circ (S_1, \ldots, S_m)) \leq d_p^m(T) \prod_{j=1}^{m} \|S_j\|$ and concludes the proof. ∎

This class satisfies a Pietsch domination theorem which is our main result:

THEOREM 2.3. *Let $m \in \mathbb{N}$. Let $X_1, \ldots, X_m$ be Banach spaces and let Y be an operator space. If a m-linear operator T in $\mathcal{L}(X_1, \ldots, X_m; Y)$ is strongly l_p-summing $(1 < p < \infty)$ then there is an ultrafilter $\mathcal{U}$ over an index set I and families a_α, b_α in $B^+_{S_{2p}(\mathcal{L})}$ such that for all $(x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m$,*

$$|\langle T(x^1, \ldots, x^m), y^* \rangle| \leq d_{l_p}^m(T) \prod_{j=1}^{m} \|x^j\| \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{L})}. \tag{2.3}$$

Conversely, if there is a positive constant C, an ultrafilter $\mathcal{U}$ over a set I and families a_α, b_α in $B^+_{S_{2p^*}(\mathcal{L})}$ such that for all $(x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m$,

$$|\langle T(x^1, \ldots, x^m), y^* \rangle| \leq C \prod_{j=1}^{m} \|x^j\| \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{L})}, \tag{2.4}$$

then $T \in \mathcal{D}_{l_p}^m(X_1, \ldots, X_m; Y)$ and $d_{l_p}^m(T) \leq C$.

Proof. We prove the first implication by using the Hahn–Banach theorem in the same spirit as in [Pis98]. Let

$$S = \{s = (a, b) \in B_{S_{2p^*}(\mathcal{L})} \times B_{S_{2p^*}(\mathcal{L})} : a, b \geq 0\}$$

and K be the set of all real-valued functions on S of the form

$$f_{((x_i^j),(y_i^*))}(s) = \frac{C}{p}\sum_{i=1}^{n}\prod_{j=1}^{m}\|x_i^j\|^p + \frac{C}{p^*}\|ay_i^*b\|^{p^*}_{l^n_{p^*}(S_{p^*}(\mathcal{L}))} - \sum_{i=1}^{n}|\langle T((x_i^j)), y_i^*\rangle| \tag{2.5}$$

where $(x_i^j)_{1\le i\le n} \subset X_j$, $1 \le j \le m$ and $(y_i^*)_{1\le i\le n} \subset Y^*$.

The set K is a convex cone. Indeed, let f_1, f_2 be in K and $\lambda \ge 0$. Then

$$f_{1((x_i'^j),(y_i'^*))}(s) = \frac{C}{p}\sum_{i=1}^{n}\prod_{j=1}^{m}\|x_i'^j\|^p + \frac{C}{p^*}\|ay_i'^*b\|^{p^*}_{l^n_{p^*}(S_{p^*}(\mathcal{L}))} - \sum_{i=1}^{k}|\langle T(x_i'^1,\dots,x_i'^m), y_i'^*\rangle|$$

and

$$f_{2((x_i''^j),(y_i''^*))}(s) = \frac{C}{p}\sum_{i=1}^{n}\prod_{j=1}^{m}\|x_i''^j\|^p + \frac{C}{p^*}\|ay_i''^*b\|^{p^*}_{l^n_{p^*}(S_{p^*}(\mathcal{L}))} - \sum_{i=1}^{l}|\langle T(x_i''^1,\dots,x_i''^m), y_i^*\rangle|.$$

It follows that

$$\begin{aligned}\lambda f_{1((x_i'^j),(y_i'^*))}(a,b) &= \frac{\lambda C}{p}\sum_{i=1}^{n}\prod_{j=1}^{m}\|x_i'^j\|^p + \frac{\lambda C}{p^*}\|ay_i'^*b\|^{p^*}_{l^n_{p^*}(S_{p^*}(\mathcal{L}))} \\ &\quad - \sum_{i=1}^{k}\lambda|\langle T(x_i'^1,\dots,x_i'^m), y_i^*\rangle| \\ &= \frac{C}{p}\sum_{i=1}^{n}\prod_{j=1}^{m}\|\lambda^{1/mp}x_i'^j\|^p + \frac{C}{p^*}\|a\lambda^{1/p^*}y_i'^*b\|^{p^*}_{l^n_{p^*}(S_{p^*}(\mathcal{L}))} \\ &\quad - \sum_{i=1}^{k}|\langle T(\lambda^{1/mp}x_i'^j)_j, \lambda^{1/p^*}y_i^*\rangle| \\ &= f_{1((\lambda^{1/mp}x_i'^j),(\lambda^{1/p^*}y_i'^*))}(a,b)\end{aligned}$$

and finally we have

$$(f_1+f_2)(s) = \frac{C}{p}\prod_{j=1}^{m}\|x_i^j\|^p + \frac{C}{p^*}\|ay_i^*b\|^{p^*}_{l^n_{p^*}(S_{p^*}(\mathcal{L}))} - \sum_{i=1}^{n}|\langle T(x_i^1,\dots,x_i^m), y_i^*\rangle|$$

with $n = k + l$ and

$$x_i^j = \begin{cases} x_i'^j & \text{if } 1 \le i \le k, \\ x_i''^j & \text{if } k+1 \le i \le n, \end{cases} \qquad y_i^* = \begin{cases} y_i'^* & \text{if } 1 \le i \le k, \\ y_i''^* & \text{if } k+1 \le i \le n. \end{cases}$$

Using the elementary equality

$$\forall \alpha, \beta \in \mathbb{R}_+^* \quad \alpha\beta = \inf_{\varepsilon > 0} \left\{ \frac{1}{p}\left(\frac{\alpha}{\varepsilon}\right)^p + \frac{1}{p^*}(\varepsilon\beta)^{p^*} \right\}, \tag{2.6}$$

we have

$$\begin{aligned}
&\sup_{(a,b)\in S} f_{((x_i),(y_i^*))}(a,b) \\
&\quad = \sup_{(a,b)\in S} \left(\frac{C}{p} \sum_{i=1}^n \prod_{j=1}^m \|x_i^j\|^p + \frac{C}{p^*} \|a y_i^* b\|^{p^*}_{l^n_{p^*}(S_{p^*}(\mathcal{L}))} - \sum_{i=1}^n |\langle T(x_i^j), y_i^* \rangle| \right) \\
&\quad = \frac{C}{p} \sum_{i=1}^n \prod_{j=1}^m \|x_i^j\|^p + \frac{C}{p^*} \sup_{(a,b)\in S} \sum_{i=1}^n \|a y_i^* b\|^{p^*}_{S_{p^*}(\mathcal{L})} - \sum_{i=1}^n |\langle T(x_i^j)_j, y_i^* \rangle| \\
&\quad \ge C \Big(\sum_{i=1}^n \prod_{j=1}^m \|x_i^j\|^p \Big)^{1/p} \sup_{(a,b)\in S} \Big(\sum_{i=1}^n \|a y_i^* b\|^{p^*}_{S_{p^*}(\mathcal{L})} \Big)^{1/p^*} - \sum_{i=1}^n |\langle T(x_i^j)_j, y_i^* \rangle| \\
&\quad \ge 0 \quad \text{(by hypothesis, see (2.1))},
\end{aligned}$$

for all f in the convex cone K. Let $\mathcal{C}$ be the open set of all f in $l_\infty(S)$ such that $\sup_{(a,b)\in S} f_{((x_i^j),(y_i^*))}(a,b) < 0$. The sets K and $\mathcal{C}$ are disjoint in $l_\infty(S)$, which is isomorphically isometric to $C(\widehat{S})$, the space of all continuous real-valued functions on the Stone–Čech compactification $\widehat{S}$ of S. By the Hahn–Banach theorem, there exists a bounded linear functional on $C(\widehat{S})$ which separates K and $\mathcal{C}$. By the Riesz representation theorem, we obtain a probability λ on $\widehat{S}$ such that

$$\lambda(f) \ge 0 \quad \text{for all } f \text{ in } K.$$

Consequently, there is an ultrafilter $\mathcal{U}$ over an index set I and a family $\{\lambda_\alpha\}_{\alpha\in I}$ of finitely supported probability measures on S such that

$$\lambda_\alpha \to \lambda \quad \text{in } \sigma(l_\infty^*(S), l_\infty(S))$$

and

$$\forall f \in K, \quad \int_{\widehat{S}} f(a,b)\, d\lambda(a,b) = \lim_{\mathcal{U}} \int_S f(a,b)\, d\lambda_\alpha(a,b) \ge 0.$$

In particular, if we take

$$f_{((x),(y^*))}(s) = \frac{C}{p} \prod_{j=1}^m \|x^j\|^p + \frac{C}{p^*} \|a y^* b\|^{p^*}_{S_{p^*}(\mathcal{L})} - |\langle T(x^1, \dots, x^m), y^* \rangle|$$

we have

$$\lim_{\mathcal{U}}\int_S f(a,b)\,d\lambda_\alpha(a,b)$$
$$= \frac{C}{p}\prod_{j=1}^m \|x^j\|^p + \frac{C}{p^*}\lim_{\mathcal{U}}\int_S \|ay^*b\|^{p^*}_{S_{p^*}(\mathcal{L})}\,d\lambda_\alpha(a,b) - |\langle T((x^j)),y^*\rangle| \geq 0$$

($\lambda_\alpha = \sum_{k=1}^{n_\alpha}\lambda_{\alpha_k}\delta(a_{\alpha_k},b_{\alpha_k})$ with $\sum_{k=1}^{n_\alpha}\lambda_{\alpha_k} = 1$ and $\lambda_{\alpha_k} \geq 0$). Hence by [Pis98, Lemma 1.14] we obtain

$$|\langle T(x^1,\dots,x^m),y^*\rangle|$$
$$\leq \frac{C}{p}\prod_{j=1}^m \|x^j\|^p + \frac{C}{p^*}\lim_{\mathcal{U}}\sum_{k=1}^{n_\alpha}\lambda_{\alpha_k}\|a_{\alpha_k}y^*b_{\alpha_k}\|^{p^*}_{S_{p^*}(\mathcal{L})} \qquad (a_{\alpha_k},b_{\alpha_k}\geq 0)$$
$$\leq \frac{C}{p}\prod_{j=1}^m \|x^j\|^p + \frac{C}{p^*}\lim_{\mathcal{U}}\Big\|\Big(\sum_{k=1}^{n_\alpha}\lambda_{\alpha_k}a_{\alpha_k}^{2p^*}\Big)^{1/2p^*}y^*\Big(\sum_{j=1}^{n_\alpha}\lambda_{\alpha_k}b_{\alpha_k}^{2p^*}\Big)^{1/2p^*}\Big\|^{p^*}_{S_{p^*}(\mathcal{L})}$$
$$\leq C\bigg(\frac{1}{p}\prod_{j=1}^m \|x^j\|^p + \frac{1}{p^*}\lim_{\mathcal{U}}\|a_\alpha y^* b_\alpha\|^{p^*}_{S_{p^*}(\mathcal{L})}\bigg),$$

using once again the equality (2.6). Fix $\varepsilon > 0$. Replacing x^j by $(1/\varepsilon^{1/m})x^j$, y^* by εy^* and taking the infimum over all $\varepsilon > 0$ in (2.6), we find that

$$|\langle T(x^1,\dots,x^m),y^*\rangle| = \bigg|\bigg\langle T\bigg(\frac{1}{\varepsilon^{1/m}}\,x^1,\dots,\frac{1}{\varepsilon^{1/m}}\,x^m\bigg),\varepsilon y^*\bigg\rangle\bigg|$$
$$\leq C\bigg(\frac{1}{p}\prod_{j=1}^m\bigg\|\frac{x^j}{\varepsilon^{1/m}}\bigg\|^p + \frac{1}{p^*}\lim_{\mathcal{U}}\|a_\alpha\varepsilon y^*b_\alpha\|^{p^*}_{S_{p^*}(\mathcal{L})}\bigg)$$
$$\leq C\bigg(\frac{1}{p}\bigg(\frac{\prod_{j=1}^m\|x^j\|}{\varepsilon}\bigg)^p + \frac{1}{p^*}(\varepsilon(\lim_{\mathcal{U}}\|a_\alpha\varepsilon y^*b_\alpha\|^{p^*}_{S_{p^*}(\mathcal{L})})^{1/p^*})^{p^*}\bigg)$$
$$\leq C\prod_{j=1}^m\|x^j\|(\lim_{\mathcal{U}}\|a_\alpha y^*b_\alpha\|^{p^*}_{S_{p^*}(\mathcal{L})})^{1/p^*}.$$

This implies that

$$|\langle T(x^1,\dots,x^m),y^*\rangle| \leq C\prod_{j=1}^m\|x^j\|\lim_{\mathcal{U}}\|a_\alpha y^*b_\alpha\|^{p^*}_{S_{p^*}(\mathcal{L})}.$$

To prove the converse, fix $n\in\mathbb{N}$. Let $(x_i^1,\dots,x_i^m)\in X_1\times\cdots\times X_m$ $(1\leq i\leq n)$ and $y_1^*,\dots,y_n^*\in Y^*$. By (2.4) we have

$$|\langle T(x_i^1,\dots,x_i^m),y_i^*\rangle| \leq C\prod_{j=1}^m\|x_i^j\|\lim_{\mathcal{U}}\|a_\alpha y^*b_\alpha\|_{S_p(\mathcal{L})}$$

for all $1 \leq i \leq n$. Thus

$$\sum_{i=1}^{n} |\langle T(x_i^1,\dots,x_i^m), y_i^*\rangle| \leq C \sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\| \lim_{\mathcal{U}} \|a_\alpha y_i^* b_\alpha\|_{S_p(\mathcal{L})}$$

$$\leq C\Big(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|^p\Big)^{1/p} \Big(\sum_{i=1}^{n} \lim_{\mathcal{U}} \|a_\alpha y_i^* b_\alpha\|_{S_p(\mathcal{L})}^{p^*}\Big)^{1/p^*} \qquad \text{(by Hölder)}$$

$$\leq C\Big(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|^p\Big)^{1/p} \sup_{a,b\in B^+_{S_{2p}}} \|(\|a y_i^* b\|_{S_p(\mathcal{L})})_{1\leq i\leq n}\|_{l^n_{p^*}}.$$

This implies that $T \in \mathcal{D}^m_{l_p}(X_1,\dots,X_m;Y)$ and $d^m_{l_p}(T) \leq C$. ■

LEMMA 2.4. *Let $X \subset B(\mathcal{H})$ be an operator space. Let $a, b \in B^+_{S_{2r}(\mathcal{H})}$ and $1 \leq r \leq s < \infty$. Then*

$$\forall x \in X, \quad \|axb\|_{S_r(\mathcal{H})} \leq \|a^{r/s} x b^{r/s}\|_{S_s(\mathcal{H})}.$$

Proof. For x in X and a, b in $B^+_{S_{2r}}$, we have

$$\begin{aligned}\|axb\|_{S_r(\mathcal{H})} &= \|a^{1-r/s} a^{r/s} x b^{r/s} b^{1-r/s}\|_{S_r(\mathcal{H})}\\ &\leq \|a^{1-r/s}\|_{S_{2rs/(r-s)}} \|a^{r/s} x b^{r/s} b^{1-r/s}\|_{S_{2rs/(r-s)}}\\ &\leq \|a^{1-r/s}\|_{S_{2rs/(r-s)}} \|a^{r/s} x b^{r/s}\|_{S_s(\mathcal{H})} \|b^{1-r/s}\|^{1-r/s}_{S_{2rs/(r-s)}}\\ &\leq \|a^{r/s} x b^{r/s}\|_{S_s(\mathcal{H})}\end{aligned}$$

(because $\|a^{1-r/s}\|_{S_{2rs/(r-s)}} = \|a\|^{(r-s)/s}_{S_{2r}} \leq 1$). ■

COROLLARY 2.5. *Let $1 \leq p, q < \infty$ with $p \leq q$. If $T \in \mathcal{D}^m_{l_q}(X_1,\dots,X_m;Y)$ then $T \in \mathcal{D}^m_{l_p}(X_1,\dots,X_m;Y)$ and $d^m_{l_p}(T) \leq d^m_{l_q}(T)$.*

Proof. By inequality (2.3) there is an ultrafilter $\mathcal{U}$ and families a_α, b_α in $B^+_{S_{2q^*}(\mathcal{L})}$ such that

$$|\langle T(x^1,\dots,x^m), y^*\rangle| \leq d^m_{l_q}(T) \prod_{j=1}^{m} \|x^j\|_{X_j} \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{q^*}(\mathcal{L})}.$$

Using Lemma 2.4, we obtain

$$|\langle T(x^1,\dots,x^m), y^*\rangle| \leq d^m_{l_q}(T) \prod_{j=1}^{m} \|x^j\|_{X_j} \lim_{\mathcal{U}} \|a_\alpha^{q^*/p^*} y^* b_\alpha^{q^*/p^*}\|_{S_{p^*}(\mathcal{L})}.$$

The assertion follows because $a_\alpha^{q^*/p^*}, b_\alpha^{q^*/p^*}$ are in $B^+_{S_{2p^*}(\mathcal{L})}$. ■

Now, we give a natural definition, as stated in [PS05], of the adjoint of an m-linear operator.

DEFINITION 2.6. Let $X_1, \ldots, X_m, Y$ be Banach spaces. If T belongs to $\mathcal{L}(X_1, \ldots, X_m; Y)$, then the *adjoint* of T is

$$T^* : Y^* \to \mathcal{L}(X_1, \ldots, X_m; \mathbb{K})$$

with $(T^*(y^*))(x^1, \ldots, x^m) = y^*(T(x^1, \ldots, x^m))$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

The following theorem establishes the relationship between the spaces $\mathcal{D}^m_{l_p}(X_1, \ldots, X_m; Y)$ and $\pi_{l_p}(Y^*, \mathcal{L}(X_1, \ldots, X_m; \mathbb{K}))$. We recall that $\pi_{l_p}(X, Y)$ is the space of all l_p-summing operators from an operator space X into a Banach space Y. For more information on this notion we refer the reader to [Mez02]. In fact, Theorem 2.3 and [Mez02, Theorem 2.3] give us the following characterization result.

THEOREM 2.7. *Let $1 \leq p < \infty$. Let $X_1, \ldots, X_m$ be Banach spaces and Y be an operator space. An operator T belongs to $\mathcal{D}^m_{l_p}(X_1, \ldots, X_m; Y)$ if and only if its adjoint T^* belongs to $\pi_{l_p}(Y^*, \mathcal{L}(X_1, \ldots, X_m; \mathbb{R}))$. In this case $\mathcal{D}^m_{l_p}(T) = \pi_{l_{p^*}}(T^*)$.*

Proof. Let $T \in \mathcal{D}^m_{l_p}(X_1, \ldots, X_m; Y)$. If $(x^j)_{1 \leq j \leq m} \in X_1 \times \cdots \times X_m$ and $y^* \in Y^*$, by (2.3) we have

$$|\langle T(x^1, \ldots, x^m), y^* \rangle| \leq d^m_{l_p}(T) \prod_{j=1}^{m} \|x^j\| \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{L})}.$$

Hence

$$|\langle (x^1, \ldots, x^m), T^*(y^*) \rangle| \leq d^m_{l_p}(T) \prod_{j=1}^{m} \|x^j\| \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{L})}.$$

Taking the supremum, we obtain

$$\|T^*(y^*)\| \leq d^m_{l_p}(T) \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{L})}.$$

Therefore $T^* \in \pi_{l_{p^*}}(Y^*, \mathcal{L}(X_1, \ldots, X_m; \mathbb{R}))$ and

$$\pi_{l_{p^*}}(T^*) \leq d^m_{l_p}(T). \tag{2.7}$$

Conversely, assume that $T^* \in \pi_{l_{p^*}}(Y^*, \mathcal{L}(X_1, \ldots, X_m; \mathbb{R}))$. Let x^j be in the unit ball of X_j ($1 \leq j \leq m$). It follows by [Mez02, Theorem 2.3] that

$$|\langle (x^1, \ldots, x^m), T^*(y^*) \rangle| \leq \pi_{l_{p^*}}(T^*) \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{L})}.$$

We conclude that T is strongly l_p-summing and

$$d^m_{l_p}(T) \leq \pi_{l_{p^*}}(T^*). \tag{2.8}$$

Combining (2.7) and (2.8) we obtain $d^m_{l_p}(T) = \pi_{l_{p^*}}(T^*)$. The theorem is proved. ∎

REFERENCES

[AM07] D. Achour and L. Mezrag, *On the Cohen strongly p-summing multilinear operators*, J. Math. Anal. Appl. 327 (2007), 550–563.

[Ale85] R. Alencar, *Multilinear mappings of nuclear and integral type*, Proc. Amer. Math. Soc. 94 (1985), 33–38.

[AM89] R. Alencar and M. C. Matos, *Some classes of multilinear mappings between Banach spaces*, Publ. Departamento de Análisis Matemático, Univ. Complutense de Madrid 12, 1989.

[Ble92] D. Blecher, *The standard dual of an operator space*, Pacific J. Math. 153 (1992), 15–30.

[BP91] D. Blecher and V. Paulsen, *Tensor products of operator spaces*, J. Funct. Anal. 99 (1991), 262–292.

[Coh73] J. S. Cohen, *Absolutely p-summing p-nuclear operators and their conjugates*, Math. Ann. 201 (1973), 177–200.

[Dia03] V. Dimant, *Strongly p-summing multilinear mappings*, J. Math. Anal. Appl. 278 (2003), 182–193.

[ER91] E. Effros and Z. J. Ruan, *A new approach to operator spaces*, Canad. Math. Bull. 34 (1991), 329–337.

[Mat96] M. C. Matos, *Absolutely summing holomorphic mappings*, An. Acad. Brasil. Ciênc. 68 (1996), 1–13.

[Mat03] —, *Fully absolutely summing and Hilbert–Schmidt multilinear mappings*, Collect. Math. 54 (2003), 1–111.

[MT99] Y. Meléndez and A. Tonge, *Polynomials and the Pietsch domination theorem*, Math. Proc. Roy. Irish Acad. 99 (1999), 195–212.

[Mez02] L. Mezrag, *Comparison of non-commutative* 2 *and p-summing operators from* $B(l_2)$ *into* OH, Z. Anal. Anwendungen 21 (2002), 709–717.

[PS05] D. Pellegrino and M. Souza, *Fully summing multilinear and holomorphic mappings into Hilbert spaces*. Math. Nachr. 278 (2005), 877–887.

[Pie83] A. Pietsch, *Ideals of multilinear functionals (designs of a theory)*, in: Proc. Second Internat. Conf. on Operator Algebras, Ideals, and their Applications in Theoretical Physics (Leipzig, 1983), Teubner-Texte Math. 67, Teubner, Leipzig, 1983, 185–199.

[Pis96] G. Pisier, *The operator Hilbert space* OH*, complex interpolation and tensor norms*, Mem. Amer. Math. Soc. 122 (1996), no. 585.

[Pis98] —, *Non-commutative vector valued* L_p*-spaces and completely p-summing maps*, Astérisque 247 (1998), 1–131.

[Sch91] B. Schneider, *On absolutely p-summing and related multilinear mappings*, Brandenburgische Landeshochschule Wiss. Z. 35 (1991), 105–117.

Laboratoire de Mathématiques Pures et Appliquées
Université de M'sila
BP 166 Ichbilia, 28003 M'sila, Algérie
E-mail: lmezrag@yahoo.fr

Received 21 December 2006 (4843)