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ASYMPTOTIC DIMENSION OF ONE RELATOR GROUPS

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Abstract. We show that one relator groups viewed as metric spaces with respect to the word-length metric have finite asymptotic dimension in the sense of Gromov, and we give an improved estimate of that dimension in terms of the relator length. The construction is similar to one of Bell and Dranishnikov, but we produce a sharper estimate.

1. Introduction. A finitely generated group has asymptotic dimension not more than n if its underlying metric space (with respect to the length function corresponding to a given set of generators) has the following property: for any $R > 0$ there exists a uniformly bounded cover of the group such that any R-ball meets not more than $n + 1$ elements of the cover (see [Gr]). There is a way to extend this notion to all countable groups, not necessarily finitely generated.

Not all groups have finite asymptotic dimension ($\mathbb{Z} \wr \mathbb{Z}$ is the standard example of a group which does not have the property above for any n), but for some classes of groups it is known that their asymptotic dimension is finite: for instance, Gromov [Gr] noticed this for the case of hyperbolic groups (for a precise proof, see [R]), Ji (see [Ji]) proved this for arithmetic groups, and of course there are more examples.

The question whether a given group has finite asymptotic dimension has drawn more attention after the results of Yu (see [Yu]), who showed that such a group satisfies the Novikov conjecture, and Higson and Roe [HR], who deduced the exactness of such groups.

The one relator groups in the title are the ones which admit a presentation $\langle S \mid r \rangle$, where the generating set S is at most countable, and the only relator r is a word in S. Historically, one relator groups appeared as the fundamental groups of 2-manifolds, and one may regard them as a simplest class of groups which are close to free groups.

Within the scope of this work, the main motivation for the study of the asymptotic dimension of one relator groups were the results of Guentner

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(see [Gu]), showing that one relator groups are exact, and those of Beguin, Bettaieb, and Valette [BBV], showing that the Baum–Connes conjecture (and therefore the Novikov conjecture as well) holds for such groups. A natural question which arises immediately after comparing all the results mentioned above is: do one relator groups have finite asymptotic dimension? In this paper we give an affirmative answer to this question and estimate the asymptotic dimension in terms of the relator length.

During the preparation of this text the preprint [BD2] was brought to the author's attention; the author is grateful to an anonymous referee for that. The main result of the present paper is similar to Corollary 27 in [BD2], but our estimate is sharper.

2. Asymptotic dimension of groups. First we outline a few basic definitions and some theorems on asymptotic dimension which will be used later. For a more detailed discussion and precise proofs, consult [R].

DEFINITION 2.1. Given a metric space G, we say that its *asymptotic dimension* does not exceed n and write $\operatorname{asdim} G \leq n$ if for any $R > 0$ there exists a uniformly bounded cover of G such that any R-ball in G meets not more than $n+1$ elements of the cover. The *asymptotic dimension* of G, $\operatorname{asdim} G$, is then the minimal n satisfying this condition.

Any finitely generated group Γ with a generating set S can be endowed with a word-length metric

$$\operatorname{dist}(\gamma_1, \gamma_2) = \text{length of a shortest word in } S \cup S^{-1} \text{ representing } \gamma_1^{-1}\gamma_2.$$

Since any two metrics for the same group, arising from different generating sets, are bi-Lipschitz equivalent, the notion of asymptotic dimension of a finitely generated group, viewed as a discrete metric space as stated above, is independent of the particular generating set.

DEFINITION 2.2. For a (not necessarily finitely generated) group G, its *asymptotic dimension*, $\operatorname{asdim} G$, is defined to be a supremum of $\operatorname{asdim} \Gamma$ over all finitely generated subgroups Γ of G.

This definition is consistent, for any inclusion of a subgroup into a group is a coarse embedding.

Now we collect some facts on the asymptotic dimension of groups, to be used in our discussion later on.

THEOREM 2.3. *For a subgroup H of a group G, $\operatorname{asdim} H \leq \operatorname{asdim} G$.*

THEOREM 2.4 (Bell and Dranishnikov, [BD1]). *For an HNN extension $*_A G$ of a group G, $\operatorname{asdim} *_A G \leq 1 + \operatorname{asdim} G$.*

The original theorem in [BD1] was formulated for a finitely generated base group G, but since any finitely generated subgroup of an HNN extension of G is a subgroup of an HNN extension of some finitely generated subgroup of G, one can run the original argument of Bell and Dranishnikov for every finitely generated subgroup of $*_A G$ to obtain the theorem in the form we formulated here.

THEOREM 2.5 (Bell and Dranishnikov, [BD1]). *For a free product $G * H$ of two groups G and H*, $\operatorname{asdim} G * H \leq \max\{\operatorname{asdim} G, \operatorname{asdim} H, 1\}$.

Again, the assumption that both G and H are finitely generated is not crucial for the proof.

Finally, as a base for our inductive argument in the next section, we state that the asymptotic dimension of a finite group is 0, and the asymptotic dimension of a free group is 1; see [R] for formal proofs.

3. One relator groups. Throughout this section, let G be a one relator group with a (possibly infinite) generating set S and relator r, that is, $G = \langle S \mid r \rangle$ is the quotient of a free group on S by the minimal normal subgroup generated by r. We assume that r, a finite word in $S \cup S^{-1}$, is cyclically reduced as a word in the free group on S, and use $|r|$ to denote its length in this free group. To omit the trivial cases, we assume that S contains at least two elements and $|r| > 0$.

For any real number x, we denote by $\lceil x \rceil$ the minimal integer greater than or equal to x.

THEOREM 3.1. *In the notations above*, $\operatorname{asdim} G \leq \lceil |r|/2 \rceil$.

The rest of this section is devoted to the proof of this theorem.

First note that we can assume that G is finitely generated and every letter of S appears in r. Indeed, G is isomorphic to a free product of a finitely generated one relator group Γ with relator r and generating set consisting of the letters which appear in r, and the free group on all other letters. According to Theorem 2.5, $\operatorname{asdim} G \leq \max\{\operatorname{asdim} \Gamma, 1\}$. If we can prove that $\operatorname{asdim} \Gamma \leq \lceil |r|/2 \rceil$, the statement of the theorem for G would follow.

The argument is by induction on the length of r. For $|r| = 1$ the group G is isomorphic to a free group on all letters in S except the one which appears in r. Thus $\operatorname{asdim} G = 1 \leq 1 = \lceil |r|/2 \rceil$.

For the inductive step suppose that the statement has been proven for all one relator groups with relator length strictly less than $|r|$. Following the standard arguments of Magnus and Molchanovskii (see [LS]), which we shall briefly describe in what follows, consider two cases:

Case 1: There exists a letter $t \in S$ whose exponent sum in r is 0. To fix notation, let $S = \{t, b, c, d, \ldots\}$, and, by means of a cyclic permutation of r, one may assume that the latter word begins with b or b^{-1}.

Let b_i denote $t^i b t^{-i}$ for $i \in \mathbb{Z}$, c_i denote $t^i c t^{-i}$ for $i \in \mathbb{Z}$, and so on. Rewrite r scanning it from left to right and changing any occurrence of $t^i x$ into $t^i x t^{-i} t^i = x_i t^i$ (here x represents any letter among $b, c, d, \ldots$, or its inverse), collecting the powers of adjacent t-letters together, and continuing with the leftmost occurrence of t or its inverse in the modified word. This way we make at least one cancellation of t and its inverse which happen to be next to each other, and the resulting word s, which represents r in terms of t, b_i, c_i, $\ldots$, and their inverses, has length not more than $|r| - 2$.

Let m and M be the minimal and maximal subscripts of b_i occurring in s. Then

$$G \cong \langle t, b_m, \ldots, b_M, c_i, d_i, \ldots (i \in \mathbb{Z}) \mid s, t b_i t^{-1} b_{i+1}^{-1}\ (i = m, \ldots, M-1), \\ t c_i t^{-1} c_{i+1}^{-1}, t d_i t^{-1} d_{i+1}^{-1}, \ldots\ (i \in \mathbb{Z}) \rangle.$$

Consider

$$H = \langle b_m, \ldots, b_M, c_i, d_i, \ldots\ (i \in \mathbb{Z}) \mid s \rangle.$$

According to our inductive assumption, $\operatorname{asdim} H \le \lceil |s|/2 \rceil \le \lceil |r|/2 \rceil - 1$. Now $G \cong *_{F\langle b_m, \ldots, b_M, c_i, d_i, \ldots (i \in \mathbb{Z}) \rangle} H$, and, via Theorem 2.4, $\operatorname{asdim} G \le 1 + \operatorname{asdim} H \le \lceil |r|/2 \rceil$.

Case 2: For *all* letters in S, their exponent sums in r are nonzero. Let $S = \{u, v, c, d, \ldots\}$, and assume that the exponent sums of u and v in r are α and β respectively. Define the following homomorphism:

$$\Psi : u \mapsto b t^{-\beta},\ v \mapsto t^{\alpha},\ c \mapsto c,\ d \mapsto d,\ \ldots$$

Our group G embeds via Ψ into

$$C = \langle t, b, c, d, \ldots \mid r(b t^{-\beta}, t^{\alpha}, c, d, \ldots) \rangle,$$

and if p is the cyclically reduced $r(b t^{-\beta}, t^{\alpha}, c, d, \ldots)$, the exponent sum of t in p is 0, and b occurs in p.

Since G can be thought of as a subgroup of C, it is enough to show that $\operatorname{asdim} C \le \lceil |r|/2 \rceil$.

Now C is an HNN extension of some group H as in Case 1; assuming that p starts with b or b^{-1}, we introduce new variables $b_i = t^i b t^{-i}$, $c_i = t^i c t^{-i}$, and so on for $i \in \mathbb{Z}$. Using these variables, we rewrite p (and therefore r as well) as a word s, eliminating all occurrences of t and its inverse and substituting appropriate x_i for any other letter x among $b, c, d, \ldots$ and their inverses. If p had at least two occurrences of t or t^{-1}, then $|s| \le |r| - 2$, and, using our inductive assumption for s, $\operatorname{asdim} H \le \lceil (|r| - 2)/2 \rceil$ as before. Invoking Theorem 2.4, we obtain the desired inequality $\operatorname{asdim} C \le \lceil |r|/2 \rceil$.

If, however, p contains t or its inverse only at one place, then p expresses t in terms of other generators, so that we can eliminate t from the generating set, and C is indeed a free group on the remaining generators with $\operatorname{asdim} C = 1 \leq \lceil |r|/2 \rceil$.

Now the theorem is proven completely.

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