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ON FAITHFUL PROJECTIVE REPRESENTATIONS OF FINITE ABELIAN p-GROUPS OVER A FIELD OF CHARACTERISTIC p

BY

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Abstract. Let G be a noncyclic abelian p-group and K be an infinite field of finite characteristic p. For every 2-cocycle $\lambda \in Z^2(G, K^*)$ such that the twisted group algebra $K^\lambda G$ is of infinite representation type, we find natural numbers d for which G has infinitely many faithful absolutely indecomposable λ-representations over K of dimension d.

0. Introduction. Throughout this paper, we use the following notations: $p \geq 2$ is a prime; K is an infinite field of characteristic p; K^* is the multiplicative group of K; $K^p = \{\alpha^p : \alpha \in K\}$; G is a finite p-group of order $|G|$; e is the identity element of G; $|g|$ is the order of $g \in G$; $\operatorname{soc} B$ is the socle of an abelian p-group B and $\exp B$ is the exponent of B. Moreover, we denote by $Z^2(G, K^*)$ the group of all K^*-valued normalized 2-cocycles of the group G, where we assume that G acts trivially on K^* (see [15, Chapter 1]).

Given a cocycle $\lambda : G \times G \to K^*$ in $Z^2(G, K^*)$, we denote by $K^\lambda G$ the twisted group algebra of the group G over the field K with the cocycle λ and by $\operatorname{rad} K^\lambda G$ the radical of $K^\lambda G$. We set $\overline{K^\lambda G} = K^\lambda G / \operatorname{rad} K^\lambda G$. A K-basis $\{u_g : g \in G\}$ of $K^\lambda G$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$ is called natural. All $K^\lambda G$-modules are assumed to be finitely generated left modules. If H is a subgroup of G, we often use the same symbol for an element $\lambda : G \times G \to K^*$ of $Z^2(G, K^*)$ and its restriction to $H \times H$. In this case, $K^\lambda H$ is a subalgebra of $K^\lambda G$.

If M is a $K^\lambda G$-module, then we denote by M_H the $K^\lambda H$-module obtained by restriction of the algebra. If N is a $K^\lambda H$-module then $N^G = K^\lambda G \otimes_{K^\lambda H} N$ is the induced $K^\lambda G$-module.

Let $Z^2(G, K^*)_\infty$ be the set of all cocycles $\lambda \in Z^2(G, K^*)$ such that the algebra $K^\lambda G$ is of infinite representation type, that is, the number of isomorphism classes of finite-dimensional indecomposable $K^\lambda G$-modules is infinite (see [1, p. 25]). Finally, given $\lambda \in Z^2(G, K^*)$, we denote by $\operatorname{Ker}(\lambda)$ the union of all cyclic subgroups $\langle g \rangle$ of G such that the restriction of λ to

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$\langle g\rangle \times \langle g\rangle$ is a coboundary. We recall from Lemma 1 of [2] that $G' \subset \mathrm{Ker}(\lambda)$, $\mathrm{Ker}(\lambda)$ is a normal subgroup of G and the restriction of λ to $\mathrm{Ker}(\lambda)\times\mathrm{Ker}(\lambda)$ is a coboundary. The set $\mathrm{Ker}(\lambda)$ is called the *kernel* of λ.

Let V be a finite-dimensional vector space over K and $\Gamma : G \to \mathrm{GL}(V)$ a projective representation of G with a cocycle $\lambda \in Z^2(G, K^*)$. We refer to Γ as a λ-*representation* of G over the field K (see [15, p. 106]). If we view V as a module over $K^\lambda G$, we say that V is the underlying module of the λ-representation Γ. Let $\mathrm{PGL}(V) = \mathrm{GL}(V)/K^* \cdot 1_V$ and $\pi : \mathrm{GL}(V) \to \mathrm{PGL}(V)$ be the canonical group homomorphism. If $\pi \circ \Gamma : G \to \mathrm{PGL}(V)$ is a monomorphism, the representation Γ is called *faithful*.

We recall from [10, p. 437] that a $K^\lambda G$-module V is defined to be *absolutely indecomposable* if for every field extension L of K, $L \otimes_K V$ is an indecomposable module over $L \otimes_K K^\lambda G$.

In this paper we continue the study of faithful projective representations of finite p-groups over fields of characteristic p as begun in [3]. Our investigations are also motivated by the results of P. M. Gudivok [11] and G. J. Janusz [12, 13]. In particular, they show that a noncyclic abelian p-group A of order $|A| \neq 4$ has infinitely many absolutely indecomposable linear representations in each dimension $d \geq 2$ if the ground field is infinite. This result, together with the result by V. A. Bashev [5], gives a solution of the second Brauer–Thrall conjecture for group algebras of finite groups (see [1, p. 138] for a formulation of the conjecture). Moreover, G. J. Janusz [13] has proved that if $p(d-1) \geq \exp A$, then there exist infinitely many isomorphism classes of absolutely indecomposable faithful linear representations of A of dimension d.

Now we briefly present the main results of the paper. In Section 1 we prove that if G is a noncyclic abelian p-group, then for any natural $n \geq 2$ and for any cocycle $\lambda \in Z^2(G, K^*)_\infty$ the group G has infinitely many nonequivalent faithful absolutely indecomposable λ-representations over K of dimension $nt|G|$, where $t = 1/p^2$ if $p \neq 2$, and $t = 1/2$ if $p = 2$ (Corollary 1.11).

In Section 2 we study the indecomposable projective representations of a noncyclic abelian p-group G over a nonperfect field K of characteristic p such that the K-algebra

$$K[x]/(x^p - \alpha) \otimes_K K[x]/(x^p - \beta)$$

is not a field for any $\alpha, \beta \in K^*$. Let $\lambda \in Z^2(G, K^*)_\infty$, $d = \dim_K \overline{K^\lambda G}$ and

$$l = \begin{cases} 1 & \text{if } 4d < |G|, \\ 2 & \text{if } 4d = |G|. \end{cases}$$

We show that in this case the group G has infinitely many nonequivalent absolutely indecomposable λ-representations over K of dimension nld for any $n \geq 2$ (Theorem 2.3). If $\mathrm{Ker}(\lambda) = \{e\}$, then $d = \exp G$ and all λ-

representations of G are faithful. Suppose that $G = A\times B$, $\lambda \in Z^2(G, K^*)_\infty$, $H = \mathrm{Ker}(\lambda)$, $\overline{H} = B\cap H$, $|A| > 1$, $|B| > 1$, $\exp B \neq 2$ and $\operatorname{soc} B = \operatorname{soc} H$. We prove that if $\exp A = p^m$, $\exp B = p^s$ and $p^m \geq \exp(B/\overline{H})$, then the group G has infinitely many nonequivalent faithful absolutely indecomposable λ-representations over K of dimension np^m for any $n \geq p^{s-1}+1$ (Theorem 2.5).

The reader is referred to [8], [14] and [15] for basic facts and notation from group representation theory and to [1] and [7] for terminology, notation and introduction to the representation theory of finite-dimensional algebras over a field.

1. Faithful indecomposable projective representations of abelian p-groups over an arbitrary field. In this section, K denotes an infinite field of characteristic p.

LEMMA 1.1 ([13, p. 138]). *Let G be an abelian p-group which is neither cyclic nor of order four. If G has exponent p^s and n is any natural number with $n \geq p^{s-1} + 1$, then G has infinitely many nonequivalent faithful absolutely indecomposable linear K-representations of dimension n.*

Note that it is not shown in [13] that the representations constructed in [13, pp. 139–144] are absolutely indecomposable. However, this follows by an analysis of the construction given in [13]. To convince the reader, we present an outline of the proof.

The general idea of the proof in [13] is to construct a K-algebra A and imbed the group G into the group A^* of all invertible elements of A.

Assume that $p \neq 2$ and $G = \langle g_1\rangle \times \cdots \times \langle g_r\rangle$, where $|g_i| = p^{c_i}$ and $c_1 \geq \cdots \geq c_r$. Let n be a natural number with $p^{c_1-1} + 1 \leq n \leq p^{c_1}$. We set $A = K[X]$, where $X^n = 0$ and $X^{n-1} \neq 0$. Let $\alpha_1, \alpha_2, \ldots$ be a basis for K over the field of p elements. By [12, Theorem 3.1], $A^* = K^* \times U$, where U is a p-primary group. The group U is the direct product of the cyclic groups $\langle w_j(\alpha_i)\rangle$, where $w_j(\alpha_i) = 1 + \alpha_i X^j$ for $j \in \{1, \ldots, n\}$, j is not divisible by p and $i = 1, 2, \ldots$. It follows that there exist infinitely many ways of imbedding G into A^*, so that g_1 is mapped to $1 + X$ in each imbedding. Every such imbedding T gives rise to a faithful indecomposable representation of G of dimension n acting on A. Let $\widetilde{K}$ be a field extension of K and $\widetilde{A} = \widetilde{K} \otimes_K A$. Then T is also a monomorphism of G into $\widetilde{A}^*$ and $T(G)$ generates $\widetilde{A}$ as a $\widetilde{K}$-algebra. Hence T gives rise to an absolutely indecomposable representation of G. Distinct imbeddings of G into A^* give rise to mutually nonequivalent representations of G.

Now suppose that

$$p^{c_1-1} + p^{c_r-1} \leq n \leq p^{c_1+c_r} - 2.$$

Select natural numbers e and f such that

$$p^{c_1-1}+1 \le e \le p^{c_1}, \quad p^{c_r-1}+1 \le f \le p^{c_r}$$

and

$$e+f-2 \le n \le ef-2.$$

Let $A = A_{e,f} = K[X,Y]$ be the K-algebra on two commuting generators X and Y that satisfy

$$\begin{aligned} X^e &= 0, \quad X^{e-1} \ne 0, \\ Y^f &= 0, \quad Y^{f-1} \ne 0. \end{aligned} \tag{1.1}$$

Denote by I_γ the ideal of $A_{e,f}$ generated by the elements

$$X^{a_1-1}Y^{f-1} + \gamma X^{e-1}Y^{b_t-1}, \quad X^{a_i}Y^{b_i},$$

where γ is any nonzero element in K, a_i, b_i are natural numbers for $i = 1,\dots,t$ and

$$\begin{aligned} 1 \le a_1 < a_2 < \cdots < a_t \le e-1, \\ 1 \le b_t < b_{t-1} < \cdots < b_1 \le f-1. \end{aligned} \tag{1.2}$$

Since $A_{e,f}$ is a local algebra, $A_{e,f}/I_\gamma$ is an indecomposable $A_{e,f}$-module. There exists at least one pair of sequences (1.2) such that $\dim_K(A_{e,f}/I_\gamma) = n$. The modules $A_{e,f}/I_\gamma$ and $A_{e,f}/I_\delta$ (both constructed from the same sequences (1.2)) are isomorphic if and only if $\gamma = \delta$.

Let $T : G \to A^*_{e,f}$ be a monomorphism such that $T(g_1) = 1+X$, $T(g_i)$ is in $K[X]$ for $i < r$ and $T(g_r) = 1+Y$. Since $T(g)$ generates $A_{e,f}$ as a K-algebra, nonisomorphic $A_{e,f}$-modules give rise to nonequivalent representations of G. Moreover, $A_{e,f}/I_\gamma$ is the underlying module of a faithful indecomposable representation of G over K of dimension n.

Let $V^{(m)}$ be a vector space over K with basis $v_1,\dots,v_m,u_0,u_1,\dots,u_m$. Define

$$Xv_i = u_{i-1}, \quad Yv_i = u_i \quad \text{for } i \in \{1,\dots,m\},$$

and

$$Xu_j = Yu_j \quad \text{for all } j \in \{0,1,\dots,m\}.$$

Then $V^{(m)}$ becomes an $A_{e,f}$-module. Let $e \ge 3$ and $f \ge 3$. We can select sequences (1.2) with $t \ge 2$ such that $\dim_K(A_{e,f}/I_\gamma) = d_0$, where d_0 is any number with $e+f-1 \le d_0 \le ef-3$. Let γ be any nonzero element of K and $M_\gamma = U_\gamma/W_\gamma$, where

$$U_\gamma = V^{(m)} \oplus A_{e,f}/I_\gamma, \quad W_\gamma = K(u_m, X^{a_1}Y^{f-1} + I_\gamma).$$

The $A_{e,f}$-module M_γ is indecomposable and $\dim_K M_\gamma = 2m + d_0$. One can choose m and d_0 in such a way that $2m + d_0 = n$ for any given $n > ef-2$.

We assume again that G acts upon M_γ via $T(G)$ from the previous case. Then M_γ is the underlying KG-module of a faithful indecomposable representation of G.

Let $\widetilde{K}$ be a field extension of K, $\widetilde{A}_{e,f} = \widetilde{K}[X, Y]$ the $\widetilde{K}$-algebra on two commuting generators X and Y that satisfy relations (1.1), and $\widetilde{A}_{e,f}/\widetilde{I}_\gamma$, $\widetilde{M}_\gamma$ be indecomposable $\widetilde{A}_{e,f}$-modules constructed by the same rules as the $A_{e,f}$-modules $A_{e,f}/I_\gamma$, M_γ. Then we can identify $\widetilde{A}_{e,f}$, $\widetilde{I}_\gamma$, $\widetilde{A}_{e,f}/\widetilde{I}_\gamma$, $\widetilde{M}_\gamma$ with

$$\widetilde{K} \otimes_K A_{e,f}, \quad \widetilde{K} \otimes_K I_\gamma, \quad \widetilde{K} \otimes_K A_{e,f}/I_\gamma, \quad \widetilde{K} \otimes_K M_\gamma,$$

respectively. It follows that the $A_{e,f}$-modules $A_{e,f}/I_\gamma$ and M_γ are absolutely indecomposable.

Lemma 1.2 ([5]). *Let G be an abelian group of type $(2,2)$ and K an infinite field of characteristic 2. Then G has infinitely many nonequivalent faithful absolutely indecomposable linear K-representations of dimension $2n$ for any natural number n.*

Lemma 1.3 ([11], [13]). *Let G be an abelian p-group which is neither cyclic nor of order four. Then G has infinitely many nonequivalent absolutely indecomposable linear K-representations of any dimension $n \geq 2$.*

By Theorem 1.1 in [4], an algebra $K^\lambda G$ is of finite representation type if and only if $K^\lambda G$ is a uniserial algebra. It is well known (see [15, p. 74]) that for any $\lambda \in Z^2(G, K^*)$, $\overline{K^\lambda G}$ is a finite purely inseparable field extension of K. Hence, $\dim_K \overline{K^\lambda G}$ divides $|G|$. If G is an abelian group, then $K^\lambda G$ is a commutative algebra for any λ.

Set $i_K = \sup\{0, m\}$, where m is a natural number such that the K-algebra

$$K[x]/(x^p - \gamma_1) \otimes_K \cdots \otimes_K K[x]/(x^p - \gamma_m)$$

is a field for some $\gamma_1, \dots, \gamma_m \in K^*$. By Proposition 1.1 of [4], for any natural t, there exists a field K such that $i_K = t$.

Let $G = \langle a_1 \rangle \times \cdots \times \langle a_s \rangle$ be an abelian p-group. We recall that from Proposition 1.3 in [4], the following statements hold:

(i) If $s \geq i_K + 2$, then $K^\lambda G$ is of infinite representation type for every $\lambda \in Z^2(G, K^*)$.

(ii) If $2 \leq s \leq i_K + 1$, then the group algebra KG is of infinite representation type and there exists an algebra $K^\lambda G$ that is of finite representation type.

(iii) If $s = 1$, then $K^\lambda G$ is of finite representation type for any $\lambda \in Z^2(G, K^*)$.

LEMMA 1.4. *Let G be an abelian p-group and $\lambda \in Z^2(G, K^*)$. The group G has a faithful irreducible λ-representation over K if and only if* $\mathrm{Ker}(\lambda) = \{e\}$.

Proof. Apply [3, Proposition 9]. ∎

Note that if K is not a perfect field, then the factor group $K^*/(K^p)^*$ is infinite [6]. In this case there exist infinitely many pairwise noncohomologous cocycles $\lambda \in Z^2(G, K^*)$ such that $\mathrm{Ker}(\lambda) = \{e\}$.

LEMMA 1.5 ([9, p. 119]). *Let G be an abelian p-group, and T a subgroup of* $\operatorname{soc} G$. *Then there exists a decomposition $G = A \times B$ such that* $\operatorname{soc} B = T$.

PROPOSITION 1.6. *Let G be an abelian p-group, $K^\lambda G$ a uniserial algebra, p^r the nilpotency index of* $\operatorname{rad} K^\lambda G$ *and* $H = \mathrm{Ker}(\lambda)$.

(i) *Every indecomposable $K^\lambda G$-module is isomorphic to one of $V_j = K^\lambda G/(\operatorname{rad} K^\lambda G)^j$, where $j \in \{1, \dots, p^r\}$. If $d = \dim_K V_1$, then K-dimension of V_j is equal to dj.*

(ii) *If $H = \{e\}$, then every V_j is the underlying $K^\lambda G$-module of a faithful indecomposable λ-representation of the group G over K.*

(iii) *If $H \neq \{e\}$, then V_j is the underlying $K^\lambda G$-module of a faithful indecomposable λ-representation of G over K if and only if $j \geq p^{r-1} + 1$.*

Proof. By Proposition 1.3 in [4], there exists a decomposition of G into a direct product $G = A \times B$ such that $K^\lambda A$ is a field and $B = \langle b \rangle$. Let $L = K^\lambda A$ and $|B| = p^n$. Then

$$K^\lambda G = L^\mu B = \bigoplus_{i=0}^{p^n-1} L u_b^i, \qquad u_b^{p^n} = \gamma^{p^r},$$

where $r \leq n$ and $\gamma \in L^*$. Moreover $\gamma \notin L^p$ if $r < n$. Let $m = n - r$. Now we have $\operatorname{rad} L^\mu B = (u_b^{p^m} - \gamma) L^\mu B$. Up to a $K^\lambda G$-isomorphism, the indecomposable $K^\lambda G$-modules are exhausted by the modules $V_j = L^\mu B/(\operatorname{rad} L^\mu B)^j$, where $j = 1, \dots, p^r$. If $H = \{e\}$, then, by Lemma 1.4, every V_j is the underlying $K^\lambda G$-module of a faithful indecomposable λ-representation of the group G.

Assume that $H \neq \{e\}$. Since KH is of finite representation type, H is a cyclic group. In view of Lemma 1.5, we may assume that $\operatorname{soc} H = \operatorname{soc} B$. Let

$$c = b^{p^{n-1}} \quad \text{and} \quad u_c = u_b^{p^{n-1}}.$$

Then $c \in H$ and $|c| = p$. The $K^\lambda G$-module V_j is not the underlying module of a faithful λ-representation of G if and only if $(u_c - \varrho u_e) L^\mu B \subset (\operatorname{rad} L^\mu B)^j$ for some $\varrho \in K^*$. Then $u_c^p - \varrho^p u_e = 0$, which yields $\varrho u_e = \gamma^{p^{r-1}}$. Since

$$u_c - \varrho u_e = (u_b^{p^m} - \gamma)^{p^{r-1}},$$

it follows that V_j is not the underlying module of a faithful λ-representation of G over K if and only if $j \le p^{r-1}$. ∎

LEMMA 1.7. *Let H be a subgroup of a p-group G and $\lambda \in Z^2(G, K^*)$. If V is an absolutely indecomposable $K^\lambda H$-module, then the induced module V^G is also absolutely indecomposable.*

Proof. Let $\widetilde{K}$ be the algebraic closure of K, $\widetilde{K}^\lambda H = \widetilde{K} \otimes_K K^\lambda H$, $\widetilde{K}^\lambda G = \widetilde{K} \otimes_K K^\lambda G$ and $\widetilde{V} = \widetilde{K} \otimes_K V$. We may consider $\widetilde{K}^\lambda H$ to be a subalgebra of $\widetilde{K}^\lambda G$. Every cocycle from $Z^2(G, \widetilde{K})$ is a coboundary (see [15, p. 43]). Hence $\widetilde{K}^\lambda G$ is the group algebra of G over $\widetilde{K}$. By Green's theorem (see [10, p. 438]), the induced module

$$\widetilde{V}^G = \widetilde{K}^\lambda G \otimes_{\widetilde{K}^\lambda H} \widetilde{V}$$

is indecomposable. Since

$$\widetilde{K} \otimes_K (K^\lambda G \otimes_{K^\lambda H} V) \cong \widetilde{K}^\lambda G \otimes_{\widetilde{K}^\lambda H} (\widetilde{K} \otimes_K V)$$

as $\widetilde{K}^\lambda G$-modules (see [14, p. 209]), the $\widetilde{K}^\lambda G$-module $\widetilde{K} \otimes_K V^G$ is indecomposable. Consequently, the $K^\lambda G$-module V^G is absolutely indecomposable. ∎

Denote by $[M]$ the isomorphism class of $K^\lambda G$-modules that contains M. Let $\mathrm{AInd}(K^\lambda G, s)$ be the set of all $[V]$ where V is an absolutely indecomposable $K^\lambda G$-module of K-dimension s. We denote by $\mathrm{FAInd}(K^\lambda G, s)$ the set of all $[W]$ where W is the underlying $K^\lambda G$-module of a faithful absolutely indecomposable λ-representation of G over K of dimension s.

LEMMA 1.8. *Let G be an abelian p-group, $\lambda, \mu \in Z^2(G, K^*)$, $K^\lambda G = K^\mu G$, $\{u_g : g \in G\}$ a natural K-basis of $K^\lambda G$ corresponding to λ and $\{v_g : g \in G\}$ a natural K-basis of $K^\lambda G$ corresponding to μ. Assume that C is the socle of* $\mathrm{Ker}(\lambda)$ *and $u_x = \alpha_x v_x$ for every $x \in C$, where $\alpha_x \in K^*$. Let D be a subgroup of G, $C \subset D$, V an absolutely indecomposable $K^\mu D$-module and let V_C be the underlying $K^\lambda C$-module of a faithful λ-representation of C. Then the induced module $V^G = K^\mu G \otimes_{K^\mu D} V$ is the underlying $K^\lambda G$-module of a faithful absolutely indecomposable λ-representation of G. Moreover, if $[V_1^G] = [V_2^G]$ then $[V_1] = [V_2]$.*

Proof. In view of Lemma 1.7, V^G is an absolutely indecomposable $K^\lambda G$-module. Suppose that $(u_g - \alpha u_e)V^G = 0$ for some $g \in \operatorname{soc} G$ and some $\alpha \in K^*$. Since $(u_g - \alpha u_e)^p V^G = 0$, we have $u_g^p = \alpha^p u_e$, which yields $g \in C$. Therefore, $(u_g - \alpha u_e)V = 0$. It follows that $g = e$. Consequently, V^G is the underlying $K^\lambda G$-module of a faithful absolutely indecomposable λ-representation of G. If $V_1^G \cong V_2^G$ then $(V_1^G)_D \cong (V_2^G)_D$. Since $(V_j^G)_D \cong V_j \oplus \cdots \oplus V_j$ for $j = 1, 2$, we have $V_1 \cong V_2$. ∎

Proposition 1.9. *Let G be an abelian p-group, $\lambda \in Z^2(G, K^*)$, $H = \mathrm{Ker}(\lambda)$ and $p^s = \exp H$. Assume that H is noncyclic. Let*

$$l = \begin{cases} 1 & \text{if } |H| > 4, \\ 2 & \text{if } |H| = 4. \end{cases}$$

Then the set $\mathrm{FAInd}(K^\lambda G, nl|G:H|)$ *is infinite for any* $n \geq p^{s-1} + 1$.

Proof. In view of Lemmas 1.1 and 1.2, $\mathrm{FAInd}(K^\lambda H, nl)$ is infinite for $n \geq p^{s-1} + 1$. By Lemma 1.8, the formula $f([V]) = [V^G]$ defines an injective map $f : \mathrm{FAInd}(K^\lambda H, nl) \to \mathrm{FAInd}(K^\lambda G, nl|G:H|)$. ∎

Theorem 1.10. *Let G be a noncyclic abelian p-group, $G_0 = \mathrm{soc}\, G$, $\lambda \in Z^2(G, K^*)_\infty$, $d = \dim_K \overline{K^\lambda G_0}$ and*

$$l = \begin{cases} 1 & \text{if } 4d < |G_0|, \\ 2 & \text{if } 4d = |G_0|. \end{cases}$$

Then the set $\mathrm{FAInd}(K^\lambda G, nld|G:G_0|)$ *is infinite for all* $n \geq 2$.

Proof. Let $H = G_0 \cap \mathrm{Ker}(\lambda)$ and B be a maximal subgroup of G_0 with $K^\lambda B$ a field. Then $G_0 = B \times C \times H$ and $K^\lambda G_0 = K^\mu G_0$, where

$$\mu_{bch, b'c'h'} = \lambda_{b,b'}$$

for all $b, b' \in B$, $c, c' \in C$ and $h, h' \in H$. Obviously, $d = \dim_K K^\lambda B = |B|$. Let $D = C \times H$. Since $\lambda \in Z^2(G, K^*)_\infty$, the group D is noncyclic [4, p. 176]. By Lemmas 1.1 and 1.2, $\mathrm{FAInd}(KD, nl)$ is infinite for every $n \geq 2$. Hence, by Lemma 1.8, $\mathrm{FAInd}(K^\lambda G_0, nld)$ is infinite. Applying again Lemma 1.8, we conclude that $\mathrm{FAInd}(K^\lambda G, nld|G:G_0|)$ is infinite for any $n \geq 2$. ∎

Corollary 1.11. *Let G be a noncyclic abelian p-group and*

$$t = \begin{cases} 1/p^2 & \text{if } p \neq 2, \\ 1/2 & \text{if } p = 2. \end{cases}$$

Then $\mathrm{FAInd}(K^\lambda G, nt|G|)$ *is infinite for any $n \geq 2$ and any cocycle $\lambda \in Z^2(G, K^*)_\infty$.*

Let G be a noncyclic abelian p-group with at most i_K invariants. By Proposition 1 of [2], there exists a cocycle $\lambda \in Z^2(G, K^*)_\infty$ such that $\dim_K \overline{K^\lambda G} = |G| \cdot p^{-2}$. Hence, in this case, Corollary 1.11 gives all dimensions for which the group G has infinitely many faithful absolutely indecomposable λ-representations.

2. Faithful indecomposable projective representations of abelian p-groups over a field K with $i_K = 1$. In this section we assume that K is a field of characteristic p with $i_K = 1$. That is, there exists $\alpha \in$

K^* such that the K-algebra $K[x]/(x^p - \alpha)$ is a field, and the K-algebra $K[x]/(x^p - \beta) \otimes_K K[x]/(x^p - \gamma)$ is not a field for any $\beta, \gamma \in K^*$. Since K is not perfect, K is an infinite field. For example, if F is a perfect field of characteristic p and $L = F(x)$ is the quotient field of the polynomial ring $F[x]$, then $i_L = 1$ (see [4, p. 174]).

LEMMA 2.1. *Let θ be a root of an irreducible polynomial $x^{p^m} - \alpha \in K[x]$ in some extension of K. Then for every $\beta \in K^*$ there exists $\gamma \in K(\theta)^*$ such that $\beta = \gamma^{p^m}$.*

Proof. Because $i_K = 1$, we have

$$\beta = \Big(\sum_{r=0}^{p-1} \mu_r \theta^{rp^{m-1}}\Big)^p \tag{2.1}$$

for some $\mu_r \in K$. Let $m \geq 2$. We have

$$\mu_r = \Big(\sum_{s=0}^{p-1} \nu_{rs} \theta^{sp^{m-1}}\Big)^p, \tag{2.2}$$

where $\nu_{rs} \in K$. It follows from (2.1) and (2.2) that

$$\beta = \Big(\sum_{i=0}^{p^2-1} \varrho_i \theta^{ip^{m-2}}\Big)^{p^2}, \quad \varrho_i \in K.$$

If $m > 2$, we inductively continue the above construction. ∎

LEMMA 2.2. *Let $G = \langle a \rangle$, $|a| = p^n$ and*

$$K^\lambda G = \bigoplus_{i=0}^{p^n-1} K u_a^i, \qquad u_a^{p^n} = \gamma^{p^m} u_e,$$

where $\gamma \in K^*$, $\gamma \notin K^p$ and $m < n$. Then for every $\beta \in K^*$ there exists an invertible element z in $K^\lambda G$ such that

$$z^{p^n} = \beta^{p^m} u_e.$$

Proof. Let θ be a root of the polynomial $x^{p^r} - \gamma$, where $r = n - m$. By Lemma 2.1,

$$\beta = \Big(\sum_{j=0}^{p^r-1} \delta_j \theta^j\Big)^{p^r}, \quad \delta_j \in K.$$

It follows that

$$\Big(\sum_{j=0}^{p^r-1} \delta_j u_a^j\Big)^{p^n} = \beta^{p^m} u_e. \ ∎$$

THEOREM 2.3. *Let G be a noncyclic abelian p-group, $\lambda \in Z^2(G, K^*)_\infty$, $d = \dim_K \overline{K^\lambda G}$ and*

$$l = \begin{cases} 1 & \text{if } 4d < |G|, \\ 2 & \text{if } 4d = |G|. \end{cases}$$

Then the set $\mathrm{AInd}(K^\lambda G, nld)$ *is infinite for any* $n \geq 2$.

Proof. By Lemmas 1.2 and 1.3, it is sufficient to consider the case $d \neq 1$. Let $\{u_g : g \in G\}$ be a natural K-basis of $K^\lambda G$. There exists a decomposition $G = \langle a \rangle \times B$ such that if $|a| = p^r$ and H is the kernel of the restriction of λ to $B \times B$, then $u_a^{p^r} = \gamma^{p^s} u_e$, where $s < r$, $\gamma \in K^*$, $\gamma \notin K^p$, and $p^{r-s} \geq \exp(B/H)$. Let $C = \langle c \rangle$ be a group of order p^{r-s} and $D = C \times B$. There exists an algebra homomorphism of $K^\lambda G$ onto $K^\mu D = K^\nu C \otimes_K K^\lambda B$, where

$$K^\nu C = \bigoplus_i K v_c^i, \quad v_c^{p^{r-s}} = \gamma v_e.$$

By Lemma 2.1, $K^\mu D \cong K^\nu C \otimes_K KB$. Evidently $d = p^{r-s}$. If B is not cyclic and $|B| > 4$ then, in view of Lemmas 1.3 and 1.7, the set $\mathrm{AInd}(K^\mu D, n|C|)$ is infinite for every $n \geq 2$.

Now let B be noncyclic and $|B| = 4$. If $s = 0$ then $d = 2^r$ and $|G| = 4d$. By Lemmas 1.2 and 1.7, $\mathrm{AInd}(K^\lambda G, 2nd)$ is infinite for any n. Assume that $s \neq 0$. We have

$$K^\lambda G = \bigoplus_{i,j_1,j_2} K u_a^i u_{b_1}^{j_1} u_{b_2}^{j_2}, \quad u_a^{2^r} = \gamma^{2^s} u_e, \quad u_{b_1}^2 = \delta_1 u_e, \quad u_{b_2}^2 = \delta_2 u_e,$$

where $\delta_1, \delta_2 \in K^*$. Let $\delta_1 \notin K^2$. Then we may suppose that $\delta_2 = 1$. Let $\varrho \in K[u_{b_1}]$ and $\varrho^2 = \gamma^{-1} u_e$. Then

$$(\varrho u_a^{2^{r-s-1}})^{2^{s+1}} = u_e.$$

The order of the subgroup of G generated by $a^{2^{r-s-1}}$ and b_2 is equal to $2^{s+2} \geq 8$. It follows from this and Lemmas 1.3 and 1.7 that $\mathrm{AInd}(K^\lambda G, nd)$ is infinite for every $n \geq 2$.

Assume that $B = \langle b \rangle$ and $|B| = p^t$. Since $K^\lambda G$ is not a uniserial algebra, we have

$$K^\lambda G = \bigoplus_{i,j} K u_a^i u_b^j, \quad u_a^{p^r} = \gamma^{p^s} u_e, \quad u_b^{p^t} = \delta^{p^m} u_e,$$

where $s > 0$, $m \leq t$, moreover, if $m < t$ then $\delta \notin K^p$ and if $m = t$ then $\delta = 1$. Let $\delta \notin K^p$. There exists an algebra homomorphism of $K^\lambda G$ onto

$$K^\mu \overline{G} = \bigoplus_{i,j} K v_{\overline{a}}^i v_{\overline{b}}^j, \quad v_{\overline{a}}^{p^{r-s+1}} = \gamma^p v_{\overline{e}}, \quad v_{\overline{b}}^{p^{t-m+1}} = \delta^p v_{\overline{e}}.$$

By Lemma 2.2, we have

$$K^{\mu}\overline{G}=\bigoplus_{i,j}Kv_{\overline{a}}^{i}w_{\overline{b}}^{j},\quad w_{\overline{b}}^{p^{t-m+1}}=v_{\overline{e}}.$$

Because $p^{t-m+1}>2$, $\mathrm{AInd}(K^{\mu}\overline{G},nd)$ is infinite for any $n\geq 2$ by Lemmas 1.3 and 1.7. Let $\delta=1$. If $p^t>2$ or $p^s>2$ then $\mathrm{AInd}(K^{\lambda}G,nd)$ is infinite for any $n\geq 2$. If $p=2$, $s=1$, $t=1$, then $4d=|G|$. In view of Lemmas 1.2 and 1.7, $\mathrm{AInd}(K^{\lambda}G,2nd)$ is infinite for all n. ∎

Corollary 2.4. *Let G be a noncyclic abelian p-group of exponent p^m, $\lambda\in Z^2(G,K^*)_{\infty}$, $\mathrm{Ker}(\lambda)=\{e\}$, $d=\dim_K\overline{K^{\lambda}G}$ and*

$$l=\begin{cases}1 & \text{if } 4d<|G|,\\ 2 & \text{if } 4d=|G|.\end{cases}$$

Then $d=p^m$ and $\mathrm{FAInd}(K^{\lambda}G,nld)$ is infinite for any $n\geq 2$.

Proof. Apply Lemma 1.4. ∎

Let us remark that K. Sobolewska in [16] has found some infinite subsets of the set of all natural numbers m for which an abelian p-group G has infinitely many indecomposable λ-representations over K of dimension m, where K is an arbitrary field and $\lambda\in Z^2(G,K^*)_{\infty}$.

Theorem 2.5. *Let $G=A\times B$ be an abelian p-group, $\lambda\in Z^2(G,K^*)_{\infty}$, $H=\mathrm{Ker}(\lambda)$, $\overline{H}=B\cap H$, $p^m=\exp A$ and $p^r=\exp(B/\overline{H})$. Assume that $|A|>1$, $|B|>1$ and $\mathrm{soc}\,B=\mathrm{soc}\,H$.*

(i) *Let $m\geq r$ and*

$$l=\begin{cases}1 & \text{if } \exp B\neq 2 \text{ or if } \exp B=2 \text{ and } |\mathrm{soc}\,G|>8,\\ 2 & \text{if } \exp B=2 \text{ and } |\mathrm{soc}\,G|=8.\end{cases}$$

Then $p^m=\dim_K\overline{K^{\lambda}G}$. If $p^s=\exp B$ then $\mathrm{FAInd}(K^{\lambda}G,nlp^m)$ is infinite for all $n\geq p^{s-1}+1$. Moreover, the smallest dimension of a faithful indecomposable λ-representation of G over K equals $p^m(p^{s-1}+1)$.

(ii) *Let $m<r$. Denote by D a maximal subgroup of B with $\overline{H}\subset D$ and $\exp(D/\overline{H})=p^m$. If $p^s=\exp D$ then $\mathrm{FAInd}(K^{\lambda}G,np^m|B:D|)$ is infinite for all $n\geq p^{s-1}+1$.*

Proof. Let $A=A_1\times A_2$, where A_1 is a cyclic group and $|A_1|=\exp A$. Since $A\cap H=\{e\}$ and $\lambda\in Z^2(G,K^*)_{\infty}$, it follows that $K^{\lambda}A_1$ is a field and $A_2\times B$ is not a cyclic group.

(i) Assume that $m\geq r$. Denote by $\{u_g : g\in G\}$ a natural K-basis of $K^{\lambda}G$ corresponding to λ. Let $C=A_2\times B$. Up to cohomology $u_h^{|h|}=u_e$ for every $h\in\mathrm{soc}\,B$, and if $g=a_1c$, where $a_1\in A_1$, $c\in C$, then $u_g=u_{a_1}u_c$. We can view $K^{\lambda}G$ as the twisted group algebra $L^{\lambda}C$ of the group C over the field $L = K^\lambda A_1$ with the cocycle λ. By Lemma 2.1, the algebra $L^\lambda C$ has a group L-basis $\{v_c : c \in C\}$, that is, $v_c v_{c'} = v_{cc'}$ for all $c, c' \in C$. We choose this basis in such a way that $v_h = u_h$ for every $h \in \operatorname{soc} B$. We set $v_g = u_{a_1} v_c$ for every $g = a_1 c$, where $a_1 \in A_1$, $c \in C$. If $g' = a_1' c'$, where $a_1' \in A_1$, $c' \in C$, then $v_g v_{g'} = \lambda_{a_1,a_1'} u_{a_1 a_1'} v_{cc'} = \lambda_{a_1,a_1'} v_{gg'}$. Let $\mu_{g,g'} = \lambda_{a_1,a_1'}$ for any $g, g' \in G$. Then $\mu \in Z^2(G, K^*)$, $K^\lambda G = K^\mu G$ and $\{v_g : g \in G\}$ is a natural K-basis of $K^\lambda G$ corresponding to μ.

Let $\widetilde{A}_2$ be an elementary abelian p-group of order $|\operatorname{soc} A_2|$ and $\widetilde{C} = \widetilde{A}_2 \times B$. In view of Lemmas 1.1 and 1.2, $\operatorname{FAInd}(K\widetilde{C}, nl)$ is infinite for all $n \ge p^{s-1} + 1$. It follows that $\operatorname{AInd}(KC, nl)$ has infinitely many elements $[W]$ such that W_B is the underlying KB-module of a faithful linear representation of B. Hence, by Lemma 1.8, $\operatorname{FAInd}(K^\lambda G, nlp^m)$ is infinite for all $n \ge p^{s-1}+1$.

Let $G_1 = A_1 \times B_1$, where B_1 is a cyclic subgroup of B and $|B_1| = p^s$. By [7, p. 170], the algebra $K^\lambda G_1$ is uniserial. The nilpotency index of $\operatorname{rad} K^\lambda G_1$ is equal to p^s. Since $\operatorname{soc} B_1 \subset H$, by Proposition 1.6, the smallest dimension of a faithful λ-representation of G_1 over K equals $p^m(p^{s-1} + 1)$. It follows that the smallest dimension of a faithful indecomposable λ-representation of G over K also equals $p^m(p^{s-1} + 1)$.

(ii) Let $m < r$ and $T = A \times D$. Since $\exp D > 2$, by case (i), $\operatorname{FAInd}(K^\lambda T, np^m)$ is infinite for all $n \ge p^{s-1} + 1$, where $p^s = \exp D$. Hence, in view of Lemma 1.8, $\operatorname{FAInd}(K^\lambda G, np^m \cdot |G : T|)$ is also infinite. Since $|G : T| = |B : D|$, the theorem is proved. ∎

Corollary 2.6. *Let G be an elementary abelian p-group of order p^m, where $m \ge 3$, $\lambda \in Z^2(G, K^*)$, $\operatorname{Ker}(\lambda) \ne G$ and*

$$l = \begin{cases} 1 & \text{if } p \ne 2 \text{ or if } p = 2 \text{ and } m \ge 4, \\ 2 & \text{if } p = 2 \text{ and } m = 3. \end{cases}$$

Then $\dim_K \overline{K^\lambda G} = p$ and $\operatorname{FAInd}(K^\lambda G, nlp)$ is infinite for all $n \ge 2$.

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