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A GENERIC THEOREM IN CARDINAL FUNCTION INEQUALITIES

BY

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Abstract. We establish a general technical result, which provides an algorithm to prove cardinal inequalities and relative versions of cardinal inequalities.

1. Introduction. Among the best known theorems on cardinal functions are those which give an upper bound on the cardinality of a space in terms of other cardinal invariants. In [7] Hodel classified the bounds on $|X|$ in two categories: *easy* and *difficult* to prove. The proofs of several inequalities in the difficult category have a common construction that is inspired by Arkhangel'skiĭ's original proof of the inequality $|X| \leq 2^{L(X)\chi(X)}$ for every Hausdorff space X (for a detailed discussion on this topic, the reader is referred to [5]). This suggests the general problem of finding a result which captures this common core. In [1], Arkhangel'skiĭ established a general result which yields an *algorithm* for proving relative versions of cardinal inequalities and also captures the common construction of several inequalities in the difficult category. However (as Arkhangel'skiĭ commented in [1]), it is not true that all important cardinal inequalities can be proved just following his algorithm. Arkhangel'skiĭ also says he does not know such a proof for Gryzlov's theorem [4]. Other results of this kind are also obtained in [5, Ths. 3.1 and 3.3], [11] and [9].

In this paper, following the ideas of Arkhangel'skiĭ [1] and Hodel [5], we formulate a general technical result (Theorem 3.1), closely related to Theorem 1 in [1], which provides an algorithm for proving a wide range of cardinal inequalities and relative versions of cardinal inequalities. Later we will use Theorem 3.1 to prove three cardinal inequalities, in particular we will prove Gryzlov's inequality: $|X| \leq 2^{\psi(X)}$ *for every compact T_1-space X*, and a relative version of Sun's inequality: $|X| \leq 2^{qL(X)\psi_c(X)t(X)}$ *for every T_2-space X*.

2. Notation and terminology. We refer the reader to [7] and [8] for definitions and terminology on cardinal functions not explicitly given. Let

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L, wL, χ, ψ and ψ_c denote the following standard cardinal functions: Lindelöf degree, weak Lindelöf number, character, pseudocharacter and closed pseudocharacter, respectively.

For any topological space X and any subset A of X, $\mathrm{cl}_X(A)$ is the closure of A in X. For any set X and cardinal κ, $[X]^{\leq\kappa}$ denotes the collection of all subsets of X with cardinality $\leq \kappa$; $[X]^{<\kappa}$ and $[X]^{\kappa}$ are defined analogously.

Definition 2.1 ([9]). Let X be a nonempty set and let τ, κ be infinite cardinals. An operator $c : \mathcal{P}(X) \to \mathcal{P}(X)$ will be called a (τ,κ)-*closure operator* if:

(1) $A \subseteq c(A)$ for every $A \in \mathcal{P}(X)$,
(2) if $A \subseteq B$, then $c(A) \subseteq c(B)$ for every $A, B \in \mathcal{P}(X)$,
(3) if $|A| \leq \tau^{\kappa}$, then $|c(A)| \leq \tau^{\kappa}$.

If the operator $c : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies (1) and (3) only, we say that it is a *quasi-*(τ,κ)*-closure operator*.

Remark 2.2. It is clear that if $\kappa^+ = \tau$, then $\tau^{\kappa} = 2^{\kappa}$; hence in this case, condition (3) in the previous definition states: if $|A| \leq 2^{\kappa}$, then $|c(A)| \leq 2^{\kappa}$.

Clearly every (τ,κ)-closure operator is a quasi-(τ,κ)-closure operator; the following examples show that the converse need not be true.

Example 2.3. Let X be a compact T_1-space and let κ be an infinite cardinal. Define $c : \mathcal{P}(X) \to \mathcal{P}(X)$ by $c(A) = A \cup A'$ where A' is obtained as follows: For every infinite subset B of A with $|B| \leq \kappa$ choose a complete accumulation point of B, and let A' be the set of points chosen in this way (this operator was defined by Stephenson [13]). Then c is a quasi-(κ^+,κ)-closure operator.

Example 2.4. Let X be an $\aleph_1$-compact space. Define $c : \mathcal{P}(X) \to \mathcal{P}(X)$ by $c(A) = A \cup A'$ where A' is obtained as follows: For each infinite subset B of A with $|B| = \aleph_1$ choose a limit point of B, and let A' be the set of points chosen in this way (this operator was defined by Hodel [5]). It is not difficult to prove that c is a quasi-(κ^+,κ)-closure operator.

We shall use the notation and terminology employed in [1]. For the reader's convenience, we repeat some of the relevant definitions.

Let X be a set and Y be a nonempty subset of X. Here and in what follows, let τ, κ be infinite cardinals such that $\kappa < \mathrm{cf}(\tau)$ and let $\mu = \tau^{\kappa}$.

Let $\mathcal{L}$ be the family of subsets of Y of cardinality not greater than μ, that is, $\mathcal{L} = [Y]^{\leq\mu}$.

A τ*-long increasing sequence in* $\mathcal{L}$ is a transfinite sequence $\{F_\alpha : \alpha < \tau\}$ of elements of $\mathcal{L}$ such that $F_\alpha \subseteq F_\beta$ if $\alpha < \beta < \tau$.

A *sensor* is a pair $(\mathcal{A},\mathcal{F})$, where $\mathcal{A}$ is a family of subsets of Y and $\mathcal{F}$ is a collection of families of subsets of X.

We assume that with each sensor $s = (\mathcal{A}, \mathcal{F})$ a subset $\Theta(s)$ of X is associated, called the *Θ-closure of s*.

DEFINITION 2.5. A sensor $s = (\mathcal{A}, \mathcal{F})$ will be called *small* if:

(1) $|\mathcal{A}| \leq \kappa$ and $|A| \leq \kappa$ for every $A \in \mathcal{A}$,
(2) $|\mathcal{F}| \leq \kappa$ and $|\mathcal{C}| \leq \kappa$ for every $\mathcal{C} \in \mathcal{F}$,
(3) $Y \setminus \Theta(s) \neq \emptyset$.

Let H be a subset of Y and $\mathcal{G}$ a family of subsets of X. A sensor $(\mathcal{A}, \mathcal{F})$ is said to be *generated by the pair* $(H, \mathcal{G})$ if $A \subseteq H$ for each $A \in \mathcal{A}$, and $\mathcal{C} \subseteq \mathcal{G}$ for each $\mathcal{C} \in \mathcal{F}$.

Let $\mathcal{Q}$ be the set of all families $\mathcal{G}$ of subsets of X such that $|\mathcal{G}| \leq \mu$. If g is a mapping of $\mathcal{L}$ into $\mathcal{Q}$ and $\mathcal{E} \subseteq \mathcal{L}$, then $\mathcal{U}_g(\mathcal{E}) = \bigcup\{g(F) : F \in \mathcal{E}\}$.

Let g be a mapping of $\mathcal{L}$ into $\mathcal{Q}$, and let $\mathcal{E}$ be a subfamily of $\mathcal{L}$. A sensor s will be called *good* for $\mathcal{E}$ if it is generated by the pair $(\bigcup \mathcal{E}, \mathcal{U}_g(\mathcal{E}))$ and $\bigcup \mathcal{E} \subseteq \Theta(s)$.

A *propeller* (with respect to (g, Θ)) *in* $\mathcal{L}$ is a τ-long increasing sequence $\mathcal{E}$ in $\mathcal{L}$ such that no small sensor s is good for $\mathcal{E}$.

DEFINITION 2.6. A *quasi-propeller* (with respect to (g, Θ)) *in* $\mathcal{L}$ is a τ-long sequence $\mathcal{E}$ in $\mathcal{L}$ such that no small sensor s is good for $\mathcal{E}$.

Clearly every propeller is a quasi-propeller.

3. The main theorem and some consequences. Now we are ready to state and prove our main result which is a slight generalization of the main result in [9] (see Corollary 3.2 below). The proof of Theorem 3.1 below follows the same pattern as the proof of Theorem 1 in [1], therefore some of the details are omitted.

THEOREM 3.1. *Let X be a set, Y a nonempty subset of X, and τ and κ infinite cardinals such that $\kappa < \mathrm{cf}(\tau)$. Set $\mu = \tau^\kappa$. If $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is a quasi-(τ, κ)-closure operator, then for every function $g : \mathcal{L} = [Y]^{\leq \mu} \to \mathcal{Q}$, there exists a family $\{E_\alpha : \alpha \in \tau\} \subseteq \mathcal{L}$ such that*:

(1) *for each $0 < \alpha < \tau$, $\bigcup\{c(E_\beta) \cap Y : \beta < \alpha\} \subseteq E_\alpha$,*
(2) *$\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha \in \tau\}$ is a quasi-propeller in $\mathcal{L}$.*

Proof. Let $g : \mathcal{L} \to \mathcal{Q}$ be a function. We construct a sequence $\{E_\alpha : \alpha < \tau\}$ of subsets of Y and a collection $\{U_\alpha : 0 < \alpha < \tau\}$ of families of subsets of X such that:

(a) $|E_\alpha| \leq \mu$, $0 \leq \alpha < \tau$,
(b) $\mathcal{U}_\alpha = \bigcup\{g(c(E_\beta \cap Y)) : \beta < \alpha\}$, $0 < \alpha < \tau$,
(c) if s is a small sensor generated by $(\bigcup\{c(E_\beta) \cap Y : \beta < \alpha\}, \mathcal{U}_\alpha)$, then $E_\alpha \cap (Y \setminus \Theta(s)) \neq \emptyset$.

Fix $0 < \alpha < \tau$ and assume that E_β and $\mathcal{U}_\beta$ are already defined such that (a)–(c) hold for each $\beta \in \alpha$. Note that $\mathcal{U}_\alpha$ has been defined by (2). Put $H_\alpha = \bigcup\{c(E_\beta) \cap Y : \beta < \alpha\}$. Clearly $|H_\alpha| \leq \mu$ and $|\mathcal{U}_\alpha| \leq \mu$. For each small sensor s generated by $(H_\alpha, \mathcal{U}_\alpha)$, choose a point $m(s) \in Y \setminus \Theta(s)$ and let A_α be the set of points chosen in this way. Let $E_\alpha = H_\alpha \cup A_\alpha$. Clearly $E_\alpha \in \mathcal{L}$, $|E_\alpha| \leq \mu$ and E_α satisfies (c). This completes the construction.

Clearly the collection $\{E_\alpha : \alpha < \tau\} \subseteq \mathcal{L}$ satisfies (1). Now, it is sufficient to prove that $\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha < \tau\}$ is a quasi-propeller in $\mathcal{L}$. To see this, let $P = \bigcup \mathcal{E}$ and suppose there is a small sensor $s_0 = (\mathcal{A}, \mathcal{F})$ generated by the pair $(P, \mathcal{U}_g(\mathcal{E}))$ such that $P \subseteq \Theta(s_0)$. Since $\kappa < \mathrm{cf}(\tau)$, there exists $\alpha_0 < \tau$ such that $A \subseteq H_{\alpha_0}$ for each $A \in \mathcal{A}$, and $\mathcal{B} \subseteq \mathcal{U}_{\alpha_0}$ for each $\mathcal{B} \in \mathcal{F}$. Hence the sensor s_0 is generated by the pair $(H_{\alpha_0}, \mathcal{U}_{\alpha_0})$. Hence by (c), there exists $m_{s_0} \in E_{\alpha_0} \cap (Y \setminus \Theta(s_0))$; but then $m_{s_0} \in c(E_{\alpha_0}) \cap Y \subseteq P \subseteq \Theta(s_0)$, which is a contradiction. ∎

Corollary 3.2 ([9]). *If $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is a (τ, κ)-closure operator, then for every function $g : \mathcal{L} \to \mathcal{Q}$, there exists a family $\{E_\alpha : \alpha \in \tau\} \subseteq \mathcal{L}$ such that*:

(1) *for each $0 < \alpha < \tau$, $\bigcup\{c(E_\beta) \cap Y : \beta < \alpha\} \subseteq E_\alpha$,*
(2) *$\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha \in \tau\}$ is a propeller in $\mathcal{L}$.*

Now we will use Theorem 3.1 to prove four cardinal inequalities and one relative version. The first one is the following well-known inequality due to Bell, Ginsburg and Woods [2]:

Corollary 3.3. *Let X be a T_4-space. Then $|X| \leq 2^{wL(X)\chi(X)}$.*

Proof. Let $\kappa = wL(X)\chi(X)$, $\tau = \kappa^+$ and $\mu = 2^\kappa$. For every $x \in X$, let $\mathcal{B}_x$ be a local base of x in X with $|\mathcal{B}_x| \leq \kappa$. For each $F \in \mathcal{L} = [X]^{\leq\mu}$, set $g(F) = \bigcup\{\mathcal{B}_x : x \in \mathrm{cl}_X(F)\}$, and for every sensor $s = (\emptyset, \{\mathcal{F}\})$, put $\Theta(s) = \mathrm{cl}_X(\bigcup \mathcal{F})$. Define $c : \mathcal{P}(X) \to \mathcal{P}(X)$ by $c(A) = \mathrm{cl}_X(A)$. Notice that c is a (κ^+, κ)-closure operator. Thus by Theorem 3.1 there exists a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$ such that $\bigcup\{c(E_\beta) : \beta < \alpha\} \subseteq E_\alpha$ for every $0 < \alpha < \kappa^+$, and $\mathcal{E} = \{c(E_\alpha) : \alpha \in \kappa^+\}$ is a quasi-propeller in $\mathcal{L}$. Let $H = \bigcup \mathcal{E}$ and note that $|H| \leq 2^\kappa$ and $c(H) = H$.

The proof will be complete if we prove that $X \subseteq H$. Suppose not and let $p \in X \setminus H$. Since X is T_4, there is an open subset U of X such that $H \subseteq U$ and $p \notin \mathrm{cl}_X(U)$. Let $\mathcal{F} = \{V : V \in \mathcal{B}_x, x \in H \text{ and } V \subseteq U\}$ and note that $G = \bigcup \mathcal{F}$. Clearly $H \subseteq G \subseteq U$ and $p \notin \mathrm{cl}_X(G)$. Since X is T_4, there exists an open subset W of X such that $H \subseteq W \subseteq \mathrm{cl}_X(W) \subseteq G$. It is not difficult to prove that the collection $\mathcal{F} \cup \{X \setminus \mathrm{cl}_X(W)\}$ is an open cover of X; hence since $wL(X) \leq \kappa$, there exists $\mathcal{F}' \in [\mathcal{F}]^{\leq\kappa}$ such that $X = \mathrm{cl}_X(\bigcup \mathcal{F}') \cup \mathrm{cl}_X(X \setminus \mathrm{cl}_X(W))$.

Let $s = (\emptyset, \{\mathcal{F}'\})$. It is clear that $p \notin \Theta(s)$ while $H \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$, which is a contradiction. ■

We now give a proof of Gryzlov's theorem using Theorem 3.1. First, we need a lemma due to Gryzlov.

LEMMA 3.4 ([4]). *Let X be a compact T_1-space with $\psi(X) \leq \kappa$ and let H be a subset of X such that every infinite subset of H of cardinality $\leq \kappa$ has a complete accumulation point in H. Then H is compact.*

Now we are ready to prove Gryzlov's inequality.

COROLLARY 3.5 ([4]). *Let X be a compact T_1-space. Then $|X| \leq 2^{\psi(X)}$.*

Proof. Let $\kappa = \psi(X)$, $\tau = \kappa^+$ and $\mu = 2^\kappa$. For every $x \in X$, let $\mathcal{B}_x$ be a local pseudobase of x in X with $|\mathcal{B}_x| \leq \kappa$. For each $F \in \mathcal{L} = [X]^{\leq\mu}$, set $g(F) = \bigcup\{\mathcal{B}_x : x \in \mathrm{cl}_X(F)\}$, and for every sensor $s = (\emptyset, \{\mathcal{F}\})$, put $\Theta(s) = \bigcup \mathcal{F}$. Let $c : \mathcal{P}(X) \to \mathcal{P}(X)$ be defined as in Example 2.3 (i.e., $c(A) = A \cup A'$). Since c is a quasi-(κ^+, κ)-closure operator, by Theorem 3.1 there exists a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$ such that $\bigcup\{c(E_\beta) : \beta < \alpha\} \subseteq E_\alpha$ for every $0 < \alpha < \kappa^+$, and $\mathcal{E} = \{c(E_\alpha) : \alpha \in \kappa^+\}$ is a quasi-propeller in $\mathcal{L}$. Let $H = \bigcup \mathcal{E}$ and note that $|H| \leq 2^\kappa$ and $H = c(H)$, hence by Lemma 3.4, H is compact.

Now, it is enough to prove that $X \subseteq H$. Suppose not and let $p \in X \setminus H$. For each $x \in H$, let $V_x \in \mathcal{B}_x$ be such that $p \notin V_x$. It is clear that the collection $\{V_x : x \in H\}$ covers H, hence there exist $x_1, \ldots, x_n \in H$ such that $H \subseteq \bigcup\{V_{x_i} : i \in \{1, \ldots, n\}\}$. Let $\mathcal{F} = \{V_{x_i} : i \in \{1, \ldots, n\}\}$ and $s = (\emptyset, \{\mathcal{F}\})$. It is clear that $p \notin \Theta(s)$ while $H \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$, which is a contradiction. ■

The following result was proved in [5]. We will use Theorem 3.1 to prove it.

COROLLARY 3.6 ([5]). *Let X be an $\aleph_1$-compact space*, *and assume that*

(1) $\psi(X) \leq 2^{\aleph_1}$,
(2) *if $Y \subseteq X$ and $|Y| \leq 2^{\aleph_1}$, then Y is meta-Lindelöf.*

Then $|X| \leq 2^{\aleph_1}$.

Proof. Let $\tau = \aleph_2$ and let $\mu = \aleph_2^{\aleph_1} = 2^{\aleph_1}$. For every $x \in X$, let $\mathcal{B}_x$ be a local pseudobase of x in X with $|\mathcal{B}_x| \leq 2^{\aleph_1}$. For each $F \in \mathcal{L} = [X]^{\leq\mu}$, set $g(F) = \bigcup\{\mathcal{B}_x : x \in \mathrm{cl}_X(F)\}$, and for every sensor $s = (\emptyset, \{\mathcal{F}\})$, put $\Theta(s) = \bigcup \mathcal{F}$. Let $c : \mathcal{P}(X) \to \mathcal{P}(X)$ be defined as in Example 2.4. Since c is a quasi-$(\aleph_2, \aleph_1)$-closure operator, by Theorem 3.1 there exists a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$ such that $\bigcup\{c(E_\beta) : \beta < \alpha\} \subseteq E_\alpha$ for every $0 < \alpha < \aleph_2$, and $\mathcal{E} = \{c(E_\alpha) : \alpha \in \aleph_2\}$ is a quasi-propeller in $\mathcal{L}$. Let $H = \bigcup \mathcal{E}$ and note that $|H| \leq 2^{\aleph_1}$ and $H = c(H)$; hence H is $\aleph_1$-compact and, by (2), meta-Lindelöf; therefore H is Lindelöf.

The proof will be complete if we prove that $X \subseteq H$. Suppose not and let $p \in X \setminus H$. For each $x \in H$, let $V_x \in \mathcal{B}_x$ be such that $p \notin V_x$. It is clear that the collection $\{V_x : x \in H\}$ covers H, hence there exist $x_1, \ldots, x_n \in H$ such that $H \subseteq \bigcup\{V_{x_i} : i \in \{1, \ldots, n\}\}$. Let $\mathcal{F} = \{V_{x_i} : i \in \{1, \ldots, n\}\}$ and let $s = (\emptyset, \{\mathcal{F}\})$. It is clear that $p \notin \Theta(s)$ while $H \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$, which is a contradiction. ∎

Before presenting our next result (Corollary 3.9), we need some notations and results.

Definition 3.7 ([10]). The *Urysohn pseudocharacter* of X, denoted by $U\psi(X)$, is the smallest infinite cardinal κ such that for each $x \in X$, there is a collection $\mathcal{B}_x$ of open neighborhoods of X such that:

(1) $|\mathcal{B}_x| \leq \kappa$,
(2) if $x \neq y$, then there exist $V_x \in \mathcal{B}_x$ and $V_y \in \mathcal{B}_y$ such that $\mathrm{cl}_X(V_x) \cap \mathrm{cl}_X(V_y) = \emptyset$.

We have the inequalities $\psi_c(X) \leq U\psi(X) \leq \chi(X)$ for every Urysohn space X.

Let κ be an infinite cardinal, and let X be a set. Suppose that for each $x \in X$, $\mathcal{V}_x$ is a family of subsets of X which contains x. For every $L \subseteq X$, let $L^* = \{x \in X : V \cap L \neq \emptyset \text{ for all } V \in \mathcal{V}_x\}$. This operator was defined by Hodel in [6].

We shall use the following result proved in [6].

Theorem 3.8. *Let κ be an infinite cardinal and X a set. For each $x \in X$, let $\mathcal{V}_x = \{V_\gamma(x) : \gamma < \kappa\}$ be a family of subsets of X which contains x such that if $x \neq y$, then there exists $\gamma \in \kappa$ with $V_\gamma(x) \cap V_\gamma(y) = \emptyset$. Then*

(1) $|L^*| \leq |L|^\kappa$.
(2) *If $L = \bigcup_{\alpha<\kappa^+} E^*_\alpha$, where $\{E_\alpha : 0 \leq \alpha < \kappa^+\}$ is a sequence of subsets of X with $\bigcup_{\beta<\alpha} E^*_\beta \subseteq E_\alpha$ for all $\alpha < \kappa^+$, then $L^* = L$.*

Finally, we recall that a subset Y of a space X is called an *H-set* in X if for every open family $\mathcal{U}$ in X such that $Y \subseteq \bigcup \mathcal{U}$ there exists $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ satisfying $Y \subseteq \mathrm{cl}_X(\bigcup \mathcal{V})$.

The next result is another consequence of Theorem 3.1.

Corollary 3.9. *If Y is an H-set in the Urysohn space X, then $|Y| \leq 2^{U\psi(X)}$.*

Proof. Let $\kappa = U\psi(X)$, $\tau = \kappa^+$ and $\mu = 2^\kappa$. For every $x \in X$, let $\mathcal{B}_x$ be a collection of open neighborhoods of x in X with $|\mathcal{B}_x| \leq \kappa$, closed under finite intersections, and satisfying (2) of Definition 3.7. For each $F \in \mathcal{L} = [Y]^{\leq\mu}$, set $g(F) = \bigcup\{\mathcal{B}_x : x \in F\}$, and for every sensor $s = (\emptyset, \{\mathcal{F}\})$ put $\Theta(s) = \mathrm{cl}_X(\bigcup \mathcal{F})$. Define $c : \mathcal{P}(X) \to \mathcal{P}(X)$ by $c(L) = L^* = \{x \in X : \mathrm{cl}_X(V) \cap L \neq \emptyset$ for all $V \in \mathcal{B}_x\}$. By Theorem 3.8, c is a (κ^+, κ)-closure operator, hence by Theorem 3.1 there exists a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$ such that $\bigcup\{c(E_\beta) \cap Y : \beta < \alpha\} \subseteq E_\alpha$ for every $0 < \alpha < \kappa^+$, and $\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha \in \kappa^+\}$ is a quasi-propeller in $\mathcal{L}$. Let $H = \bigcup \mathcal{E}$ and note that $|H| \leq 2^\kappa$, hence $|c(H)| \leq 2^\kappa$. Moreover, it is not difficult to prove that if $x \in c(H)$, then there exists $\alpha < \kappa^+$ such that $x \in c(E_\alpha)$.

To finish the proof let us show that $Y \subseteq c(H)$. Suppose not and let $p \in Y \setminus c(H)$. For each $y \in Y \cap c(H)$, fix $V_y \in \mathcal{B}_y$ and let $V(y, p) \in \mathcal{B}_p$ be such that $\mathrm{cl}_X(V_y) \cap \mathrm{cl}_X(V(y, p)) = \emptyset$, and for every $y \in Y \setminus c(H)$, let $V_y \in \mathcal{B}_y$ be such that $\mathrm{cl}_X(V_y) \cap H = \emptyset$. It is clear that the collection $\{V_x : x \in Y\}$ covers Y, hence since Y is an H-set in X, there exist $x_1, \ldots, x_n \in Y$ such that $Y \subseteq \mathrm{cl}_X(\bigcup\{V_{x_i} : i \in \{1, \ldots, n\}\})$. Let $C = \{x_1, \ldots, x_n\} \cap c(H)$, $\mathcal{F} = \{V_x : x \in C\}$ and $s = (\emptyset, \{\mathcal{F}\})$. It is clear that $p \notin \Theta(s)$ while $H \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$, which is a contradiction. ∎

As a consequence of Corollary 3.9, we have:

COROLLARY 3.10 ([3]). *If Y is an H-set in the Urysohn space X, then* $|Y| \leq 2^{\chi(X)}$.

At the moment the author does not know the answer to the following question:

QUESTION 3.11. Let Y be an H-set in the Hausdorff space X. Is it true that $|Y| \leq 2^{H\psi(X)}$? (The definition of $H\psi(X)$ can be found in [6].)

Now we turn to the final result of this paper. The following inequality was proved in [12]: $|X| \leq 2^{qL(X)\psi_c(X)t(X)}$ *for every Hausdorff space X*. We will prove a relative version of this result. To formulate it, we have to introduce a relative version of qL.

DEFINITION 3.12. Let X be a topological space, $Y \subseteq X$, and κ an infinite cardinal.

(1) We say that $A \in [Y]^{\leq 2^\kappa}$ is *κ-quasi-dense* in Y with respect to X if for every open cover $\mathcal{U}$ of X, there exist $B \in [A]^{\leq\kappa}$ and $\mathcal{V} \in [\mathcal{U}]^\kappa$ such that $Y \subseteq \mathrm{cl}_X(B) \cup \bigcup \mathcal{V}$.
(2) We define $qL(Y, X)$ as the smallest infinite cardinal κ such that there is a subset κ-quasi-dense in Y with respect to X.

It follows from Definition 3.12 that if $Y = X$, then $qL(Y, X) = qL(X)$.

COROLLARY 3.13. *Let X be a T_1-space and let Y be a subspace of X. Suppose that*:

(1) $t(X) \leq \kappa$,
(2) $|\mathrm{cl}_X(B)| \leq 2^\kappa$ *for every* $B \in [X]^{\leq 2^\kappa}$,
(3) $\psi(X) \leq 2^\kappa$.

If $qL(Y,X) \leq \kappa$, *then* $|Y| \leq 2^\kappa$.

Proof. Let $\tau = \kappa^+$, $\mu = 2^\kappa$ and for each $x \in X$ let $\mathcal{B}_x$ be a local pseudobase of x in X with $|\mathcal{B}_x| \leq \kappa$. Since $qL(Y,X) \leq \kappa$, there exists $A \in [Y]^{\leq\mu}$ which is κ-quasi-dense in Y with respect to X. For every $F \in \mathcal{L} = [Y]^{\leq\mu}$ we put $g(F) = \bigcup\{\mathcal{B}_x : x \in \mathrm{cl}_X(F)\}$. For every sensor $s = (\mathcal{A},\mathcal{F})$ we put $\Theta(s) = \mathrm{cl}_X(\bigcup\mathcal{A}) \cup \bigcup\{\bigcup\mathcal{C} : \mathcal{C} \in \mathcal{F}\}$. Since $c(D) = \mathrm{cl}_X(D)$ is a (τ,κ)-closure operator, by Theorem 3.1 there is $\{E_\alpha : \alpha < \tau\} \subseteq \mathcal{L}$ such that $\bigcup\{c(E_\beta) \cap Y : \beta < \alpha\} \subseteq E_\alpha$ for every $\alpha < \kappa$ and $\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha < \tau\}$ is a quasi-propeller in $\mathcal{L}$. We suppose, without loss of generality, that $E_0 = A$. Let $P = \bigcup\mathcal{E}$ and note that $|P| \leq \mu$; hence $|\mathrm{cl}_X(P)| \leq \mu$.

Now, let us show that $Y \subseteq \mathrm{cl}_X(P)$. Suppose not and fix $p \in Y \setminus \mathrm{cl}_X(P)$. For every $x \in \mathrm{cl}_X(P)$, let $V_x \in \mathcal{B}_x$ be such that $p \notin V_x$. It is clear that the collection $\{V_x : x \in \mathrm{cl}_X(P)\} \cup \{X \setminus \mathrm{cl}_X(P)\}$ covers X. Since $qL(Y,X) \leq \kappa$, there exist $D \in [A]^{\leq\kappa}$ and $B \in [\mathrm{cl}_X(P)]^{\leq\kappa}$ such that $Y \subseteq \mathrm{cl}_X(D) \cup \bigcup\{V_x : x \in B\} \cup X \setminus \mathrm{cl}_X(P)$. Let $\mathcal{A} = \{D\}$, $\mathcal{F} = \{V_x : x \in B\}$ and $s = (\mathcal{A},\mathcal{F})$. Clearly $p \notin \Theta(s)$ and $P \subseteq \Theta(s)$. Thus s is a small sensor good for $\mathcal{E}$, a contradiction. ∎

COROLLARY 3.14. *If* X *is a* T_2*-space, then* $|Y| \leq 2^{qL(Y,X)\psi_c(X)t(X)}$.

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REFERENCES

[1] A. V. Arhangel'skij [A. V. Arkhangel'skiĭ], *A generic theorem in the theory of cardinal invariants of topological spaces*, Comment. Math. Univ. Carolin. 36 (1995), 303–325.

[2] M. Bell, J. Ginsburg and G. Woods, *Cardinal inequalities for topological spaces involving the weak Lindelöf number*, Pacific J. Math. 79 (1978), 37–45.

[3] A. Bella, *A couple of open questions concerning cardinal invariants*, Questions Answers Gen. Topology 14 (1996), 139–143 (http://at.yorku.ca/e/a/p/p/12.htm).

[4] A. A. Gryzlov, *Two theorems on the cardinality of topological spaces*, Soviet Math. Dokl. 21 (1980), 506–509.

[5] R. E. Hodel, *Arhangel'skiĭ's solution to Alexandroff's problem: A survey*, Topology Appl. 153 (2006), 2199–2217.

[6] —, *Combinatorial set theory and cardinal function inequalities*, Proc. Amer. Math. Soc. 111 (1991) 567–575.

[7] —, *Cardinal functions I*, in: K. Kunen and J. Vaughan (eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 1–61.

[8] I. Juhász, *Cardinal Functions in Topology—Ten Years Later*, Math. Centrum, Amsterdam, 1980.
[9] A. Ramírez-Páramo and N. T. Tapia-Bonilla, *Other generic theorem in the theory of cardinal invariants of topological spaces*, Comment. Math. Univ. Carolin. 48 (2007), 177–187.
[10] D. Stavrova, *Separation pseudocharacter and the cardinality of topological spaces*, Topology Proc. 25 (2000), 333–343.
[11] —, *Unified approach to the theory of topological cardinal invariants*, C. R. Acad. Bulgare Sci. 50 (1997), 5–8.
[12] S. H. Sun, *Two new topological cardinal inequalities*, Proc. Amer. Math. Soc. 104 (1988), 313–316.
[13] R. M. Stephenson, *Initially κ-compact and related spaces*, in: Handbook of Set-Theoretic Topology, K. Kunen and J. Vaughan (eds.), North-Holland, Amsterdam, 1984, 63–109.

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