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ON THE UNIFORM BEHAVIOUR OF THE FROBENIUS CLOSURES OF IDEALS

BY

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Abstract. Let $\mathfrak{a}$ be a proper ideal of a commutative Noetherian ring R of prime characteristic p and let $Q(\mathfrak{a})$ be the smallest positive integer m such that $(\mathfrak{a}^{\mathrm{F}})^{[p^m]} = \mathfrak{a}^{[p^m]}$, where $\mathfrak{a}^{\mathrm{F}}$ is the Frobenius closure of $\mathfrak{a}$. This paper is concerned with the question whether the set $\{Q(\mathfrak{a}^{[p^m]}) : m \in \mathbb{N}_0\}$ is bounded. We give an affirmative answer in the case that the ideal $\mathfrak{a}$ is generated by an u.s.d-sequence $c_1, \dots, c_n$ for R such that

(i) the map $R/\sum_{j=1}^n Rc_j \to R/\sum_{j=1}^n Rc_j^2$ induced by multiplication by $c_1 \dots c_n$ is an R-monomorphism;
(ii) for all $\mathfrak{p} \in \operatorname{ass}(c_1^j, \dots, c_n^j)$, $c_1/1, \dots, c_n/1$ is a $\mathfrak{p}R_{\mathfrak{p}}$-filter regular sequence for $R_{\mathfrak{p}}$ for $j \in \{1, 2\}$.

1. Introduction. Let R be a commutative Noetherian ring of prime characteristic p. The theory of tight closure was introduced by M. Hochster and C. Huneke [3], and many applications of it have been found (see [5]). For a proper ideal $\mathfrak{a}$ of R, the *Frobenius closure* $\mathfrak{a}^{\mathrm{F}}$ of $\mathfrak{a}$, defined as

$$\mathfrak{a}^{\mathrm{F}} := \{r \in R : \text{there exists } n \in \mathbb{N}_0 \text{ such that } r^{p^n} \in \mathfrak{a}^{[p^n]}\},$$

is an ideal relative to the theory of tight closure, where the nth Frobenius power $\mathfrak{a}^{[p^n]}$ of $\mathfrak{a}$ is the ideal of R generated by the p^nth powers of elements of $\mathfrak{a}$ (we use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers). Since $\mathfrak{a}^{\mathrm{F}}$ is finitely generated, $(\mathfrak{a}^{\mathrm{F}})^{[p^{n_0}]} = \mathfrak{a}^{[p^{n_0}]}$ for some $n_0 \in \mathbb{N}_0$. Let $Q(\mathfrak{a})$ be the smallest power of p with this property. An interesting question is whether the set $\{Q(\mathfrak{b}) : \mathfrak{b} \text{ is a proper ideal of } R\}$ of powers of p is bounded. A simpler question is whether, for a given proper ideal $\mathfrak{b}$ of R, the set $\{Q(\mathfrak{b}^{[p^n]}) : n \in \mathbb{N}_0\}$ is bounded.

In [6], M. Katzman and R. Y. Sharp gave an affirmative answer to the latter question in case $\mathfrak{b}$ is generated by a regular sequence. To do this they used the theory of modules of generalized fractions introduced by R. Y. Sharp

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and H. Zakeri in [11]. In fact they constructed a certain module of generalized fractions to which they applied the Hartshorne–Speiser–Lyubeznik theorem (Theorem 2.1). Following some ideas of [6], we are able to establish the following theorem.

THEOREM 1.1. *Suppose that the ideal* $\mathfrak{a}$ *of* R *is generated by an u.s.d-sequence* $c_1, \dots, c_n$ *for* R *such that*

(i) *the map*

$$R/\sum_{j=1}^{n} Rc_j \to R/\sum_{j=1}^{n} Rc_j^2$$

induced by multiplication by $c_1 \dots c_n$ *is an* R*-monomorphism*;

(ii) *for all* $\mathfrak{p} \in \operatorname{ass}(c_1^j, \dots, c_n^j)$, $c_1/1, \dots, c_n/1$ *is a* $\mathfrak{p}R_{\mathfrak{p}}$*-filter regular sequence for* $R_{\mathfrak{p}}$ *for* $j \in \{1, 2\}$.

Then there exists $e \in \mathbb{N}_0$ *such that* $((\mathfrak{a}^{[p^n]})^{\mathrm{F}})^{[p^e]} = (\mathfrak{a}^{[p^n]})^{[p^e]}$ *for all* $n \in \mathbb{N}_0$.

The proof employs the Hartshorne–Speiser–Lyubeznik theorem. The new point of view is the use of Koszul homology with respect to a generalized regular sequence.

Throughout the paper, A will denote a general commutative Noetherian ring and R will denote a commutative Noetherian ring of prime characteristic p. We shall always denote by $f : R \to R$ the Frobenius homomorphism, for which $f(r) = r^p$ for all $r \in R$. Our terminology follows the textbook [1] on local cohomology.

2. An ideal generated by an u.s.d-sequence. In this paper, we shall work with the skew polynomial ring $R[x, f]$ associated to R and f in the indeterminate x over R. Recall that $R[x, f]$ is, as a left R-module, freely generated by $(x^i)_{i \in \mathbb{N}_0}$, and so consists of all polynomials $\sum_{i=0}^{n} r_i x^i$, where $n \in \mathbb{N}_0$ and $r_0, \dots, r_n \in R$; however, its multiplication is subject to the rule

$$xr = f(r)x = r^p x \quad \text{for all } r \in R.$$

Now, let Z be a left $R[x, f]$-module. Then it is easy to see that the x-torsion submodule $\Gamma_x(Z)$ of Z, defined as

$$\Gamma_x(Z) := \{z \in Z : x^j z = 0 \text{ for some } j \in \mathbb{N}\},$$

is an $R[x, f]$-submodule of Z (cf. [6, Lemma and Definition 1.2]).

Crucial to this paper is the following extension, due to G. Lyubeznik, of a result of R. Hartshorne and R. Speiser. It shows that, when R is local, an x-torsion left $R[x, f]$-module which is Artinian (that is, "cofinite" in the terminology of Hartshorne and Speiser) as an R-module exhibits a certain uniformity of behaviour.

THEOREM 2.1 ([9, Proposition 4.4]; cf. [2, Proposition 1.11]). *Suppose that $(R,\mathfrak{m})$ is local, and let G be a left $R[x,f]$-module which is Artinian as an R-module. Then there exists $e \in \mathbb{N}_0$ such that $x^e \Gamma_x(G) = 0$.*

Hartshorne and Speiser first proved this result in the case where R is local and contains its residue field which is perfect. Lyubeznik applied his theory of F-modules to obtain the result for any local ring R of characteristic p.

DEFINITION 2.2 (see [6, Definition 1.5]). Suppose that $(R,\mathfrak{m})$ is local, and let G be a left $R[x,f]$-module which is Artinian as an R-module. By the Hartshorne–Speiser–Lyubeznik Theorem 2.1, there exists $e \in \mathbb{N}_0$ such that $x^e\Gamma_x(G) = 0$; we call the smallest such e the *Hartshorne–Speiser–Lyubeznik number*, or *HSL-number* for short, of G.

Let $a_1, \dots, a_n$ be a sequence of elements of R. Set $\mathfrak{a} := Ra_1 + \cdots + Ra_n$. Then the nth local cohomology module $H^n_{\mathfrak{a}}(R)$ can be interpreted as the direct limit of Koszul homology modules (cf. [1, Theorem 5.2.9]), and in the present situation we have

$$H^n_{\mathfrak{a}}(R) \cong \varinjlim_{k\in\mathbb{N}} R/\sum_{j=1}^{n} Ra_j^k$$

with the map

$$R/\sum_{j=1}^{n} Ra_j^k \to R/\sum_{j=1}^{n} Ra_j^{k+1}$$

induced by multiplication by $a_1 \dots a_n$. For each $t \in \mathbb{N}$, let

$$\Theta_t : R/\sum_{j=1}^{n} Ra_j^t \to \varinjlim_{k\in\mathbb{N}} R/\sum_{j=1}^{n} Ra_j^k$$

be the canonical homomorphism. We will show that the above direct limit has a structure of a left $R[x,f]$-module. To do this we recall the following lemma.

LEMMA 2.3 ([6, Lemma 1.3]). *Let M be an R-module and $\tau : M \to M$ be a $\mathbb{Z}$-endomorphism of M such that $\tau(rm) = r^p\tau(m)$ for all $r \in R$ and $m \in M$. Then the R-module structure on G can be extended to a structure of a left $R[x,f]$-module in such a way that $xm = \tau(m)$ for all $m \in M$.*

REMARK 2.4. In view of the above lemma, it is routine to check that $H^i_{\mathfrak{a}}(R)$ (and hence $\varinjlim_{k\in\mathbb{N}} R/\sum_{j=1}^n Ra_j^k$) has a natural structure of a left $R[x,f]$-module (see [6, Reminder 2.1]). In the following lemma we determine precisely this structure.

LEMMA 2.5. *Let $a_1, \dots, a_n$ be a sequence of elements of R. Then the module $\varinjlim_{k\in\mathbb{N}} R/\sum_{j=1}^n Ra_j^k$ has a structure of a left $R[x,f]$-module with*

$$x\Big(\Theta_t\Big(r+\sum_{j=1}^{n}Ra_j^t\Big)\Big)=\Theta_{pt}\Big(r^p+\sum_{j=1}^{n}Ra_j^{pt}\Big)$$

for all $r\in R$ and $t\in\mathbb{N}$.

Proof. Suppose that $r,u\in R$ and $t,s\in\mathbb{N}$ are such that

$$\Theta_t\Big(r+\sum_{j=1}^{n}Ra_j^t\Big)=\Theta_s\Big(u+\sum_{j=1}^{n}Ra_j^s\Big)\quad\text{in }\varinjlim_{k\in\mathbb{N}}R/\sum_{j=1}^{n}Ra_j^k.$$

Hence $r(a_1\dots a_n)^{v-t}-u(a_1\dots a_n)^{v-s}\in\sum_{j=1}^{n}Ra_j^v$ for some $v\in\mathbb{N}$ with $v\geq t$ and $v\geq s$. Application of the Frobenius homomorphism yields $r^p(a_1\dots a_n)^{p(v-t)}-u^p(a_1\dots a_n)^{p(v-s)}\in\sum_{j=1}^{n}Ra_j^{pv}$. It follows that $\Theta_{pt}(r^p+\sum_{j=1}^{n}Ra_j^{pt})=\Theta_{ps}(u^p+\sum_{j=1}^{n}Ra_j^{ps})$ and that there is a $\mathbb{Z}$-endomorphism τ of $\varinjlim_{k\in\mathbb{N}}R/\sum_{j=1}^{n}Ra_j^k$ such that $\tau(\Theta_t(r+\sum_{j=1}^{n}Ra_j^t))=\Theta_{pt}(r^p+\sum_{j=1}^{n}Ra_j^{pt})$ for all $r\in R$ and $t\in\mathbb{N}$. Now the claim follows from Lemma 2.3.

Now we recall a certain generalization of the notion of regular sequence. Let M be an A-module. A sequence $b_1,\dots,b_n$ of elements of an ideal $\mathfrak{b}$ of A is said to be a $\mathfrak{b}$-*filter regular sequence* for M if

$$\mathrm{Supp}_A\Big(\frac{(b_1,\dots,b_{i-1})M:_M b_i}{(b_1,\dots,b_{i-1})M}\Big)\subseteq V(\mathfrak{b})$$

for all $i=1,\dots,n$, where $V(\mathfrak{b})$ denotes the set of prime ideals of A containing $\mathfrak{b}$. The concept of a $\mathfrak{b}$-filter regular sequence for M is a generalization of that of a filter regular sequence which has been studied in [10], [12], [7] and has led to some interesting results. Note that both concepts coincide if $\mathfrak{b}$ is the maximal ideal in a local ring. Also note that $b_1,\dots,b_n$ is a weak M-sequence if and only if it is an A-filter regular sequence for M. It is easy to see that the analogue of [12, Appendix 2(ii)] holds true whenever A is Noetherian, M is finitely generated and $\mathfrak{m}$ is replaced by $\mathfrak{a}$; so that, if $b_1,\dots,b_n$ is a $\mathfrak{b}$-filter regular sequence for M, then there is an element $b_{n+1}\in\mathfrak{b}$ such that $b_1,\dots,b_n,b_{n+1}$ is a $\mathfrak{b}$-filter regular sequence for M. Thus, for a positive integer n, there exists a $\mathfrak{b}$-filter regular sequence for M of length n.

PROPOSITION 2.6 (see [7, Proposition 1.2]). *Let $b_1,\dots,b_n$ $(n>1)$ be a $\mathfrak{b}$-filter regular sequence for M. Then*

$$H_{\mathfrak{b}}^i(M)\cong\begin{cases}H^i_{(b_1,\dots,b_n)}(M) & \text{for } 0\leq i<n,\\ H_{\mathfrak{b}}^{i-n}(H^n_{(b_1,\dots,b_n)}(M)) & \text{for } n\leq i.\end{cases}$$

PROPOSITION 2.7. *Suppose that $(R,\mathfrak{m})$ is a local ring and $t\in\mathbb{N}$. Then $H^t_{\mathfrak{m}}(R)$ is an $R[x,f]$-submodule of $H^t_{(r_1,\dots,r_t)}(R)$ for every $\mathfrak{m}$-filter regular sequence $r_1,\dots,r_t\in\mathfrak{m}$ for R.*

Proof. Let $r_1, \ldots, r_t \in \mathfrak{m}$ be an $\mathfrak{m}$-filter regular sequence for R. If $\alpha \in H^t_{(r_1,\ldots,r_t)}(R)$ is annihilated by $\mathfrak{m}^h$ for a positive integer h, then $(\mathfrak{m}^h)^{[p]}x\alpha = 0$, and so, in view of Lemma 2.3 and Proposition 2.6, $\Gamma_{\mathfrak{m}}(H^t_{(r_1,\ldots,r_t)}(R)) \cong H^t_{\mathfrak{m}}(R)$ is an $R[x,f]$-submodule of $H^t_{(r_1,\ldots,r_t)}(R)$.

The theory of d-sequences was introduced by Huneke in [4]. Let M be an A-module. The sequence $a_1, \ldots, a_n$ in A is called a *d-sequence* for M if, for each $i = 1, \ldots, n-1$, the equality

$$\Big(\sum_{j=1}^{i} Aa_j\Big)M :_M a_{i+1}a_k = \Big(\sum_{j=1}^{i} Aa_j\Big)M :_M a_k$$

holds for all $k \geq i+1$ (this is actually a slight weakening of Huneke's definition); it is an *unconditioned strong d-sequence* (u.s.d-sequence) for M if $a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}$ is a d-sequence in any order for all positive integers $\alpha_1, \ldots, \alpha_n$. d-sequences are closely related to filter regular sequences. It is easy to see that if $a_1, \ldots, a_n$ is a d-sequence on M, then it is a $\sum_{i=1}^n Aa_i$-filter regular sequence for M.

REMARK 2.8. Let $\mathfrak{a}$ be an ideal of R and $i \in \mathbb{N}_0$. For any prime ideal $\mathfrak{p}$ of R we have

(i) $(\mathfrak{a}R_{\mathfrak{p}})^{\mathrm{F}} = (\mathfrak{a}^{\mathrm{F}})R_{\mathfrak{p}}$,
(ii) $(\mathfrak{a}R_{\mathfrak{p}})^{[p^i]} = (\mathfrak{a}^{[p^i]})R_{\mathfrak{p}}$,
(iii) if $(\mathfrak{a}^{\mathrm{F}})^{[p^i]} = \mathfrak{a}^{[p^i]}$, then $(\mathfrak{a}^{\mathrm{F}})^{[p^{i+j}]} = \mathfrak{a}^{[p^{i+j}]}$ for all $j \in \mathbb{N}$.

The proof of the following theorem relies heavily on ideas in M. Katzman and R. Y. Sharp's proof of [6, Theorem 4.2].

THEOREM 2.9. *Suppose that the ideal $\mathfrak{a}$ of R is generated by an u.s.d-sequence $c_1, \ldots, c_n$ for R such that*

(i) *the map*

$$R/\sum_{j=1}^{n} Rc_j \to R/\sum_{j=1}^{n} Rc_j^2$$

induced by multiplication by $c_1 \ldots c_n$ is an R-monomorphism;
(ii) *for all $\mathfrak{p} \in \operatorname{ass}(c_1^j, \ldots, c_n^j)$, $c_1/1, \ldots, c_n/1$ is a $\mathfrak{p}R_{\mathfrak{p}}$-filter regular sequence for $R_{\mathfrak{p}}$ for $j \in \{1,2\}$.*

Then there exists $e \in \mathbb{N}_0$ such that $((\mathfrak{a}^{[p^n]})^{\mathrm{F}})^{[p^e]} = (\mathfrak{a}^{[p^n]})^{[p^e]}$ for all $n \in \mathbb{N}_0$.

Proof. First of all note that we can assume $\mathfrak{a} \neq 0$. Let $\mathfrak{p} \in \operatorname{ass}(c_1, \ldots, c_n) \cup \operatorname{ass}(c_1^2, \ldots, c_n^2)$. It is well known that the local cohomology module $H^n_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is an Artinian $R_{\mathfrak{p}}$-module (cf. [1, Theorem 7.1.3]). Now, in view of Proposition 2.7, let $e_{\mathfrak{p}}$ be the HSL-number of $H^n_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$. Let $\{\operatorname{ht}\mathfrak{p} : \mathfrak{p} \in \operatorname{ass}(c_1, \ldots, c_n) \cup \operatorname{ass}(c_1^2, \ldots, c_n^2)\} = \{h_1, \ldots, h_w\}$, where $h_1 < \cdots < h_w$. We define (for each $i = 1, \ldots, w$)

$$e_i = \max\{e_{\mathfrak{p}} : \mathfrak{p} \in \operatorname{ass}(c_1, \dots, c_n) \cup \operatorname{ass}(c_1^2, \dots, c_n^2) \text{ and } \operatorname{ht}\mathfrak{p} = h_i\},$$

and we claim that $e = \sum_{i=1}^{w} e_i$ has the desired property.

Now, let m be a positive integer such that $((\mathfrak{a}^{[p^m]})^{\mathrm{F}})^{[p^e]} \neq (\mathfrak{a}^{[p^m]})^{[p^e]}$. Then choose $\mathfrak{q} \in \operatorname{Ass}((\mathfrak{a}^{[p^m]})^{\mathrm{F}})^{[p^e]}/(\mathfrak{a}^{[p^m]})^{[p^e]}$. Hence by Remark 2.8, $((\mathfrak{a}^{[p^m]}R_{\mathfrak{q}})^{\mathrm{F}})^{[p^e]} \neq (\mathfrak{a}^{[p^m]}R_{\mathfrak{q}})^{[p^e]}$. Note that such a $\mathfrak{q}$ has to be an associated prime of $(\mathfrak{a}^{[p^m]})^{[p^e]}$. Since $\mathfrak{a}$ can be generated by a u.s.d-sequence $c_1, \dots, c_n$, in view of [8, Lemma 3], we have $\mathfrak{q} \in \operatorname{ass}(c_1, \dots, c_n) \cup \operatorname{ass}(c_1^2, \dots, c_n^2)$. By using this observation, in conjunction with Remark 2.8, in order to establish our claim that e has the desired property, it is enough to show that, for each $\mathfrak{p} \in \operatorname{ass}(c_1, \dots, c_n) \cup \operatorname{ass}(c_1^2, \dots, c_n^2)$,

$$((\mathfrak{a}^{[p^n]}R_{\mathfrak{q}})^{\mathrm{F}})^{[p^{e_1+\cdots+e_i}]}/(\mathfrak{a}^{[p^n]}R_{\mathfrak{q}})^{[p^{e_1+\cdots+e_i}]} = 0$$

for all $n \in \mathbb{N}$, where $\operatorname{ht}\mathfrak{p} = h_i$. Suppose that this is not the case, and let $\mathfrak{p}$ be a minimal counterexample. Set $\operatorname{ht}\mathfrak{p} = h_l$; there must exist a positive integer m such that $((\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{\mathrm{F}})^{[p^{e_1+\cdots+e_l}]}/(\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{[p^{e_1+\cdots+e_l}]} \neq 0$. Set $e' := \sum_{\gamma=1}^{l-1} e_\gamma$ (interpreted as 0 if $l = 1$). By choice of $\mathfrak{p}$, each of the $R_{\mathfrak{p}}$-modules $((\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{\mathrm{F}})^{[p^{e'}]}/(\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{[p^{e'}]}$ and $((\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{\mathrm{F}})^{[p^{e'+e_l}]}/(\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{[p^{e'+e_l}]}$ has $\mathfrak{p}R_{\mathfrak{p}}$ as its only possible associated prime, because a smaller associated prime would contradict the minimality of $\mathfrak{p}$. Therefore, both $R_{\mathfrak{p}}$-modules in the last display have finite length.

There exists $\varrho \in (\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{\mathrm{F}}$ such that $\varrho^{p^{e'+e_l}} \notin (\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{[p^{e'+e_l}]}$. Consider the element

$$\begin{aligned} \alpha &:= \Theta_{p^m}(\varrho + (c_1^{p^m}/1, \dots, c_n^{p^m}/1)) \\ &\in \varinjlim_{k\in\mathbb{N}} R_{\mathfrak{p}}/(c_1^k/1, \dots, c_n^k/1) \cong H^n_{(c_1/1,\dots,c_n/1)}(R_{\mathfrak{p}}). \end{aligned}$$

By using Lemma 2.5, we have

$$\begin{aligned} x^{e'}\alpha &= x^{e'}\Theta_{p^m}(\varrho + (c_1^{p^m}/1, \dots, c_n^{p^m}/1)) \\ &= \Theta_{p^{e'+m}}(\varrho^{p^{e'}} + (c_1^{p^{e'+m}}/1, \dots, c_n^{p^{e'+m}}/1)). \end{aligned}$$

Since $\varrho^{p^{e'}} \in ((\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{\mathrm{F}})^{[p^{e'}]}$ and the R-module

$$((\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{\mathrm{F}})^{[p^{e'}]}/(\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{[p^{e'}]}$$

has finite length, it is routine to check that

$$x^{e'}\alpha \in \Gamma_{\mathfrak{p}R_{\mathfrak{p}}}(H^n_{(c_1/1,\dots,c_n/1)}(R_{\mathfrak{p}})).$$

Also, by assumption (ii) in conjunction with Proposition 2.6, we have the isomorphism

$$\Gamma_{\mathfrak{p}R_{\mathfrak{p}}}(H^n_{(c_1/1,\dots,c_n/1)}(R_{\mathfrak{p}})) \cong H^n_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}).$$

Moreover $\alpha \in \Gamma_x(H^n_{(c_1/1,\dots,c_n/1)}(R_{\mathfrak{p}}))$ because $\varrho \in (\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{\mathrm{F}}$. Now, since by assumption (i), the map

$$R/\sum_{j=1}^{n} Rc_j \to R/\sum_{j=1}^{n} Rc_j^2$$

induced by multiplication by $c_1 \ldots c_n$ is an R-monomorphism and $\varrho^{p^{e'+e_l}} \notin (\mathfrak{a}^{[p^m]}R_{\mathfrak{p}})^{[p^{e'+e_l}]}$, in view of Lemma 2.5, it is routine to check that $x^{e_j}(x^{e'}\alpha) \neq 0$, which is the required contradiction. So the theorem is proved.

COROLLARY 2.10 ([6, Corollary 4.3]). *Suppose that the ideal* $\mathfrak{a}$ *of* R *can be generated by a regular sequence. Then there exists* $e \in \mathbb{N}_0$ *such that* $((\mathfrak{a}^{[p^n]})^{\mathrm{F}})^{[p^e]} = (\mathfrak{a}^{[p^n]})^{[p^e]}$ *for all* $n \in \mathbb{N}_0$.

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