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THE LJUNGGREN EQUATION REVISITED

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Abstract. We study the Ljunggren equation $Y^2+1=2X^4$ using the "multiplication by 2" method of Chabauty.

1. Introduction. In [5], Ljunggren proved that the only positive integral solutions of the diophantine equation

$$L_2: \quad Y^2+1=2X^4$$

are $(X,Y)=(1,1),(13,239)$. Since the proof was quite complicated, Mordell asked if one could find a simpler proof.

In [8] Tzanakis and Steiner gave a proof using the theory of Baker. Another proof was given by Chen [3], using the Thue–Siegel method combined with Padé approximation of algebraic functions.

In this paper we solve this equation with another method. Our approach is inspired by Chabauty [2] and uses the group structure of an elliptic curve and the multiplication by 2 map. This method was used by Poulakis [6] and later by Bugeaud [1] to obtain an upper bound for the height of integral points. This method eventually also uses Baker's theory since we need to solve a unit equation.

2. The integral solutions of L_2. The proof consists of two parts. The first uses the group structure of the elliptic curve and the second is a reduction to a unit equation in a certain quartic number field.

To solve the equation L_2 it is enough to solve E_2, where

$$E_2: \quad F(X,Y)=Y^2-(X^3-2X)=0.$$

Let $(x,y)\in L_2(\mathbb{Z})$, and set $a=2x^2$, $b=2xy$. Then $P=(a,b)\in E_2(\mathbb{Z})$. We assume that $|a|\geq 2$. Let $R=(s,t)$ be a point of E_2 over the algebraic

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closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ such that $2R = P$. By [7, Chapter 3, p. 59], we have

$$a = \frac{(s^2+2)^2}{4s(s^2-2)} \tag{1}$$

and so s is a root of the polynomial

$$\Theta_a(S) = S^4 - 4aS^3 + 4S^2 + 8aS + 4.$$

The roots of $\Theta_a(S)$ are

$$a \pm \sqrt{a^2-2} \pm \sqrt{2a^2 \pm 2a\sqrt{a^2-2}},$$

where the first $\pm$ coincides with the third. Put $L = \mathbb{Q}(s)$. Since $a = 2x^2$, we have $a^2 - 2 = 4x^4 - 2 = 2y^2$ and so $L = \mathbb{Q}(\sqrt{2x^2 \pm y\sqrt{2}})$. Also, $\mathbb{Q}(\sqrt{2}) \subset L$ and $N_K(2x^2 \pm y\sqrt{2}) = 2$. It follows that the only prime dividing the discriminant of L is 2. So the only prime ramified in L is 2. Furthermore, from [4, Chapter 9, Proposition 9.4.1, p. 461], L is a totally real quartic extension of $\mathbb{Q}$. So from Jones' list ([1]) or the database ([2]) of Jürgen Klüners and Gunter Malle, we conclude that $L = \mathbb{Q}(\sqrt{2+\sqrt{2}})$.

The element $s_\pm = (s \pm \sqrt{2})/2$ is a root of the polynomial with integer coefficients:

$$\begin{aligned}\lambda(S) &= (1/256)\operatorname{res}_W(\Theta_a(2S \mp W), W^2 - 2)\\ &= S^8 - 4aS^7 + \cdots + 1,\end{aligned}$$

where $\operatorname{res}_W(\cdot,\cdot)$ denotes the resultant of two polynomials with respect to W. Thus $s_\pm$ is a unit in L. So $u = (s+\sqrt{2})/2$ and $v = (\sqrt{2}-s)/2$ satisfy the unit equation $u + v = \sqrt{2}$ in L. The algorithm of Wildanger [9], which is implemented in the computer algebra system Magma ([3]) V2.10-22, gives the solutions of this unit equation in L, which are listed in Table 1 where we have put

$$[a_1, a_2, a_3, a_4] = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3,$$

with $\theta = \sqrt{2+\sqrt{2}}$. We substitute to (1) each solution of the unit equation and we check if it gives an integer. Thus, it follows that $a = 2, 338$. So, for $|a| \geq 2$, the solutions of E_2 are $(X, Y) = (2, \pm 2)$, $(338, \pm 6214)$, and for $|a| < 2$, they are $(X, Y) = (0, 0), (-1, \pm 1)$. So $L_2(\mathbb{Z}) = \{(\pm 1, \pm 1), (\pm 13, \pm 239)\}$.

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([1]) Jones, W. J., http://math.la.asu.edu/˜jj/numberfields/. Tables of number fields with prescribed ramification.

([2]) http://www.mathematik.uni-kassel.de/˜klueners/minimum/minimum.html.

([3]) http://magma.maths.usyd.edu.au/magma/.

Table 1. The solutions of the unit equation

$[-1,0,0,0][-1,0,1,0]$	$[1,0,0,0][-30,1,0]$	$[-1,-1,0,0][-1,-1,1,0]$
$[-1,1,0,0][-1,-1,1,0]$	$[-1,-1,1,0][-1,1,0,0]$	$[-3,0,1,0][1,0,0,0]$
$[407,533,-119,-156][-409,-533,120,156]$	$[-1,1,1,0][-1,-1,0,0]$	$[-1,0,1,0][-1,0,0,0]$
$[-409,533,120,-156][407,-533,-119,156]$	$[5,7,-1,-2][-7,-7,2,2]$	$[1,4,0,-1][-3,-4,1,1]$
$[-71,39,120,-65][69,-39,-119,65]$	$[-1,-1,-1,1][-1,1,2,-1]$	$[1,2,-3,-2][-3,-2,4,2]$
$[69,39,-119,-65][-71,-39,120,65]$	$[-7,7,2,-2][5,-7,-1,2]$	$[-3,2,4,-2][1,-2,-3,2]$
$[-71,-39,120,65][69,39,-119,-65]$	$[-1,2,0,-1][-1,-2,1,1]$	$[1,3,0,-1][-3,-3,1,1]$
$[11,14,-3,-4][-13,-14,4,4]$	$[-1,2,1,-1][-1,-2,0,1]$	$[-3,3,1,-1][1,-3,0,1]$
$[-1,1,-1,-1][-1,-1,2,1]$	$[-1,1,2,-1][-1,-1,-1,1]$	$[-3,-4,1,1][1,4,0,-1]$
$[11,-14,-3,4][-13,14,4,-4]$	$[1,-3,0,1][-3,3,1,-1]$	$[-1,-2,0,1][-1,2,1,-1]$
$[-13,14,4,-4][11,-14,-3,4]$	$[-3,-3,1,1][1,3,0,-1]$	$[-1,-2,1,1][-1,2,0,-1]$
$[-409,-533,120,156][407,533,-119,-156]$	$[1,-2,-3,2][-3,2,4,-2]$	$[5,-7,-1,2][-7,7,2,-2]$
$[69,-39,-119,65][-71,39,120,-65]$	$[-1,-1,2,1][-1,1,-1,-1]$	$[1,-4,0,1][-3,4,1,-1]$
$[-13,-14,4,4][11,14,-3,-4]$	$[-3,-2,4,2][1,2,-3,-2]$	$[-3,4,1,-1][1,-4,0,1]$
$[407,-533,-119,156][-409,533,120,-156]$	$[-7,-7,2,2][5,7,-1,-2]$	

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