THE LJUNGGREN EQUATION REVISITED

BY

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Abstract. We study the Ljunggren equation $Y^2 + 1 = 2X^4$ using the “multiplication by 2” method of Chabauty.

1. Introduction. In [5], Ljunggren proved that the only positive integral solutions of the diophantine equation

$$L_2 : \quad Y^2 + 1 = 2X^4$$

are $(X, Y) = (1, 1), (13, 239)$. Since the proof was quite complicated, Mordell asked if one could find a simpler proof.

In [8] Tzanakis and Steiner gave a proof using the theory of Baker. Another proof was given by Chen [3], using the Thue–Siegel method combined with Padé approximation of algebraic functions.

In this paper we solve this equation with another method. Our approach is inspired by Chabauty [2] and uses the group structure of an elliptic curve and the multiplication by 2 map. This method was used by Poulakis [6] and later by Bugeaud [1] to obtain an upper bound for the height of integral points. This method eventually also uses Baker’s theory since we need to solve a unit equation.

2. The integral solutions of $L_2$. The proof consists of two parts. The first uses the group structure of the elliptic curve and the second is a reduction to a unit equation in a certain quartic number field.

To solve the equation $L_2$ it is enough to solve $E_2$, where

$$E_2 : \quad F(X, Y) = Y^2 - (X^3 - 2X) = 0.$$ 

Let $(x, y) \in L_2(\mathbb{Z})$, and set $a = 2x^2$, $b = 2xy$. Then $P = (a, b) \in E_2(\mathbb{Z})$. We assume that $|a| \geq 2$. Let $R = (s, t)$ be a point of $E_2$ over the algebraic...
closure \( \bar{\mathbb{Q}} \) of \( \mathbb{Q} \) such that \( 2R = P \). By [7, Chapter 3, p. 59], we have

\[
a = \frac{(s^2 + 2)^2}{4s(s^2 - 2)}
\]

and so \( s \) is a root of the polynomial

\[
\Theta_a(S) = S^4 - 4aS^3 + 4S^2 + 8aS + 4.
\]

The roots of \( \Theta_a(S) \) are

\[
a \pm \sqrt{a^2 - 2} \pm \sqrt{2a^2 \pm 2a\sqrt{a^2 - 2}},
\]

where the first \( \pm \) coincides with the third. Put \( L = \mathbb{Q}(s) \). Since \( a = 2x^2 \), we have \( a^2 - 2 = 4x^4 - 2 = 2y^2 \) and so \( L = \mathbb{Q}(\sqrt{2x^2 \pm y\sqrt{2}}) \). Also, \( \mathbb{Q}(\sqrt{2}) \subset L \) and \( N_K(2x^2 \pm y\sqrt{2}) = 2 \). It follows that the only prime dividing the discriminant of \( L \) is 2. So the only prime ramified in \( L \) is 2. Furthermore, from [4, Chapter 9, Proposition 9.4.1, p. 461], \( L \) is a totally real quartic extension of \( \mathbb{Q} \). So from Jones’ list (1) or the database (2) of Jürgen Klüners and Gunter Malle, we conclude that \( L = \mathbb{Q}(\sqrt{2} + \sqrt{2}) \).

The element \( s_\pm = (s \pm \sqrt{2})/2 \) is a root of the polynomial with integer coefficients:

\[
\lambda(S) = (1/256) \text{res}_W(\Theta_a(2S \mp W), W^2 - 2) = S^8 - 4aS^7 + \cdots + 1,
\]

where \( \text{res}_W(\cdot, \cdot) \) denotes the resultant of two polynomials with respect to \( W \). Thus \( s_\pm \) is a unit in \( L \). So \( u = (s + \sqrt{2})/2 \) and \( v = (\sqrt{2} - s)/2 \) satisfy the unit equation \( u + v = \sqrt{2} \) in \( L \). The algorithm of Wildanger [9], which is implemented in the computer algebra system Magma (3) V2.10-22, gives the solutions of this unit equation in \( L \), which are listed in Table 1 where we have put

\[
[a_1, a_2, a_3, a_4] = a_0 + a_1 \theta + a_2 \theta^2 + a_3 \theta^3,
\]

with \( \theta = \sqrt{2 + \sqrt{2}} \). We substitute to (1) each solution of the unit equation and we check if it gives an integer. Thus, it follows that \( a = 2,338 \). So, for \( |a| \geq 2 \), the solutions of \( E_2 \) are \((X, Y) = (2, \pm 2), (338, \pm 6214)\), and for \( |a| < 2 \), they are \((X, Y) = (0, 0), (-1, \pm 1)\). So \( L_2(\mathbb{Z}) = \{ (\pm 1, \pm 1), (\pm 13, \pm 239) \} \).

Acknowledgments. The author is indebted to Professor D. Poulakis for his valuable remarks. Also the author thanks the referee for his/her suggestions and comments.


(2) http://www.mathematik.uni-kassel.de/~klueners/minimum/minimum.html.

(3) http://magma.maths.usyd.edu.au/magma/.
Table 1. The solutions of the unit equation

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<th>Solution 1</th>
<th>Solution 2</th>
<th>Solution 3</th>
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REFERENCES


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