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POLYNOMIALLY GROWING PLURIHARMONIC FUNCTIONS ON SIEGEL DOMAINS

BY

MONIKA GILŻYŃSKA (Oława)

Abstract. Let $\mathcal{D}$ be a symmetric type two Siegel domain over the cone of positive definite Hermitian matrices and let $\mathbf{N}(\Phi)\mathbf{S}$ be a solvable Lie group acting simply transitively on $\mathcal{D}$. We characterize polynomially growing pluriharmonic functions on $\mathcal{D}$ by means of three $\mathbf{N}(\Phi)\mathbf{S}$-invariant second order elliptic degenerate operators.

Introduction. Let $\mathcal{D}$ be a type two symmetric Siegel domain and let G be a solvable Lie group that acts simply transitively on $\mathcal{D}$. By means of G-invariant operators we study polynomially growing pluriharmonic functions on $\mathcal{D}$. More precisely, we consider second order real elliptic degenerate G-invariant operators which annihilate holomorphic functions and so their real and imaginary parts. Such operators are called *admissible* and they have already been used to characterize pluriharmonic functions by a number of people: [B], [BBDHPT], [BBDR], [BDH], [BBDHJ], [DDHT], [DH2], [DHMP], [DHP], [T1], [T2]. All the results, except those in [B], have concerned only bounded functions. Only in the specific case of Hua operators invariant under the full group of isometries of the domain $\mathcal{D}$, no growth conditions have been imposed.

In this paper we go a step further—we obtain an analogous characterization of polynomially growing functions on type two Siegel domains $\mathcal{D}$ over cones of positive hermitian matrices. In the case when the cone is the half-line, the domain is the Siegel half-space biholomorphically equivalent to the complex ball.

The main result of the paper is:

There are three admissible operators $H, L, \mathcal{L}$ on $\mathcal{D}$ such that if a function F polynomially growing in the sense of (3.4) *satisfies $HF = LF = \mathcal{L}F = 0$, then F is pluriharmonic modulo a polynomial. $H+L+\mathcal{L}$ is an elliptic operator.*

When the cone is the half-line, *the growth condition* (*see* (1.18)) *is considerably weaker and the system of operators reduces to $\mathcal{L}$, H.*

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The characterization is considerably stronger than the one in [BDH], where boundedness has been assumed. Moreover, while in [BDH] the group picture was dominant, here we rather concentrate on the domain and the action of G on it, which proves to be very fruitful. Unfortunately we have not been able to obtain the result for all type two symmetric Siegel domains.

The paper consists of three sections. In the first one we consider the Siegel half-space, which is identified with a solvable Lie group $\mathcal{H}\times\mathbb{R}^+$, a semidirect product of the Heisenberg group $\mathcal{H}$ and $\mathbb{R}^+$. Writing any point in $\mathcal{H}\times\mathbb{R}^+$ as $(\xi,x)a$, where $(\xi,x)\in\mathcal{H}$, $a\in\mathbb{R}^+$, we consider a function F that satisfies

$$|F((\xi,x)a)|\le C\eta(a)(1+|\xi|+|x|^{1/2})^N$$

for some N and a locally integrable function η. Using a relative fundamental solution for the canonical sublaplacian on $\mathcal{H}$, we obtain additional differential equations on F which make it possible to remove step by step terms of type $w^\alpha\overline{w}^\beta$ for sufficiently large $|\alpha|+|\beta|$ from the Taylor expansion of F, which gives the conclusion. The arguments are different from the ones from [BDH] and based on an idea communicated to us by Aline Bonami, with whom we have discussed the case of $\dim\mathcal{H}=3$, i.e. the complex ball in $\mathbb{C}^2$.

The existence of nonzero polynomials which are annihilated by our system of operators $\mathcal{L},H$ is not surprising. Indeed, it is easy to construct examples of such.

In the third section the domain $\mathcal{D}$ over the cone of positive definite hermitian matrices is considered. We identify $\mathcal{D}$ with the group $G=\mathbf{N}(\Phi)\mathbf{S}$, a semidirect product of a step two nilpotent Lie group $\mathbf{N}(\Phi)$ and a solvable Lie group $\mathbf{S}$ of lower triangular matrices (see Section 2.1). $\mathbf{S}$ acts on the center $\mathcal{V}$ of $\mathbf{N}(\Phi)$ and the group $\mathcal{V}\mathbf{S}$ acts simply transitively on a type one domain immersed in $\mathcal{D}$.

The proof in [BDH], based on the spectral synthesis for $\mathcal{H}$, is not applicable in the present situation, and a different approach is needed. Following [BDH] the function F satisfying the assumptions of Theorem 3.18 is decomposed as

$$F=F_1+F_2,$$

where F_1 is holomorphic and F_2 antiholomorphic in tube direction $\mathcal{V}\mathbf{S}$. Next, for each of these functions, the results of Section 2 are applied.

The second section is the most difficult and original part of this paper. We consider the case of a polynomially growing analytic function that is holomorphic in tube direction $\mathcal{V}\mathbf{S}$, and satisfies one more equation generated by a special Laplacian on a subgroup of $\mathbf{N}(\Phi)$, which is the Heisenberg group. This permits us to use the results of the first section to obtain additional equations.

Later on the action of a maximal nilpotent subgroup N_0 in $\mathbf{S}$ on $\mathcal{D}$ enters into the picture and gives rise to some more equations on functions.

This is an interesting phenomenon, already observed in previous papers. Indeed, the behavior of the group N_0 was always crucial to characterizing pluriharmonicity by means of three invariant operators only [BDH], [DHMP]. The present paper shows a new aspect of this.

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1. Siegel half-space

1.1. *The Heisenberg group and the Siegel half-space.* We consider the group

$$\mathcal{H} = \mathbb{C}^n \times \mathbb{R} = \{(\xi, x) : \xi \in \mathbb{C}^n,\ x \in \mathbb{R}\}$$

with multiplication given by

$$(\xi, x) \circ (\zeta, y) = \big(\xi + \zeta, x + y + \tfrac{1}{2}\operatorname{Im} \xi\overline{\zeta}\big), \tag{1.1}$$

where

$$\xi\overline{\zeta} = \sum_{j=1}^{n} \xi_j \overline{\zeta}_j.$$

$\mathcal{H}$ is called the *Heisenberg group*. For $j = 1, \ldots, n$ let $\xi_j = x_j + iy_j$. The left-invariant vector fields

$$\mathcal{X}_j f(\xi, t) = \big(\big(\partial_{x_j} + \tfrac{1}{2} y_j \partial_t\big) f\big)(\xi, t), \tag{1.2}$$

$$\mathcal{Y}_j f(\xi, t) = \big(\big(\partial_{y_j} - \tfrac{1}{2} x_j \partial_t\big) f\big)(\xi, t), \tag{1.3}$$

$$\mathcal{T} f(\xi, t) = (\partial_t f)(\xi, t) \tag{1.4}$$

form a basis of the Lie algebra of $\mathcal{H}$.

Consider the semidirect product $\mathcal{S} = \mathcal{H} \times \mathbb{R}^+$ with multiplication

$$(\xi, u, a) \cdot (\xi_1, u_1, a_1) = ((\xi, u) \circ (a^{1/2}\xi_1, a u_1), a a_1). \tag{1.5}$$

In this section we study the so called *Siegel half-space*

$$\mathcal{U} = \big\{(\xi, z) : \xi \in \mathbb{C}^n,\ z \in \mathbb{C},\ \operatorname{Im} z > \tfrac{1}{4}|\xi|^2\big\},$$

which is biholomorphically equivalent to the unit complex ball $\mathcal{B}$ in $\mathbb{C}^{n+1}$.

The group $\mathcal{S}$ acts on $\mathcal{U}$ in the following way:

$$\begin{cases} (0, x) \circ (w, z) = (w, z + x), \\ (\xi, 0) \circ (w, z) = (w + \xi, z + 2i\phi(w, \xi) + i\phi(\xi, xi)), \\ a \circ (w, z) = (\sqrt{a}\, w, az), \end{cases} \tag{1.6}$$

where

$$\phi(\xi, w) = \frac{1}{4} \sum_{j=1}^{n} \xi_j \overline{w}_j$$

and the action is simply transitive. Let $\theta : \mathcal{S} \to \mathcal{U}$ be given by

$$\theta(s) = s \circ (0, i).$$

Putting $s = (\xi, t, a)$, we obtain

$$\theta((\xi, t, a)) = \big(\xi, t + i\big(a + \tfrac{1}{4}|\xi|^2\big)\big). \tag{1.7}$$

Let $\mathcal{Z}_j = \mathcal{X}_j - i\mathcal{Y}_j$, $\overline{\mathcal{Z}}_j = \mathcal{X}_j + i\mathcal{Y}_j$, where $\mathcal{X}_j$ and $\mathcal{Y}_j$ are given in (1.2) and (1.3). Then an easy computation shows that

$$\mathcal{Z}_j = 2\partial_{\xi_j} + \tfrac{1}{2} i \overline{\xi}_j \partial_t, \tag{1.8}$$

$$\overline{\mathcal{Z}}_j = 2\partial_{\overline{\xi}_j} - \tfrac{1}{2} i \xi_j \partial_t. \tag{1.9}$$

We consider the operator

$$\mathcal{L}_n = \sum_{j=1}^{n} \mathcal{Z}_j \overline{\mathcal{Z}}_j = \sum_{j=1}^{n} (\mathcal{X}_j^2 + \mathcal{Y}_j^2) - ni\mathcal{T}. \tag{1.10}$$

The function

$$\Phi(\xi, t) = \frac{2^{n-2}(n-1)!}{\pi^{n+1}} \log\left(\frac{|\xi|^2/4 - it}{|\xi|^2/4 + it}\right)(|\xi|^2/4 - it)^{-n} \tag{1.11}$$

is called the *relative fundamental solution* for $\mathcal{L}_n$, i.e.

$$f = \mathcal{L}_n f * \Phi + \mathcal{C} f \tag{1.12}$$

for f being a Schwartz function or distribution with compact support in $\mathcal{H}$. Here $\mathcal{C}$ is the *Cauchy–Szegö projection* ([S, Section XIII, §4]). Let $I = (I_1, \ldots, I_{2n+1})$ be a multiindex. Define

$$D^I = \mathcal{X}_1^{I_1} \ldots \mathcal{X}_n^{I_n} \mathcal{Y}_1^{I_{n+1}} \ldots \mathcal{Y}_n^{I_{2n}} \mathcal{T}^{I_{2n+1}}.$$

In the next section we will need the following characterization of polynomially growing $\mathcal{L}_n$-harmonic functions.

THEOREM 1.13. *Let $\widetilde{F}$ be a C^∞ function defined on $\mathcal{H}$, satisfying*

$$\mathcal{L}_n \widetilde{F}(\xi, t) = 0$$

and such that

$$|\widetilde{F}(\xi, t)| \le c(1 + |\xi| + |t|^{1/2})^N$$

for a constant $N \ge 0$. Moreover, assume that for all multiindices I we have

$$|D^I \widetilde{F}(\xi, t)| \le c_I (1 + |\xi| + |t|^{1/2})^N.$$

Then for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| > N + 2$ we have

$$\overline{\mathcal{Z}}^\alpha \widetilde{F}(\xi, t) = 0,$$

where $\overline{\mathcal{Z}}^\alpha$ denotes any operator $\overline{\mathcal{Z}}_{j_1} \ldots \overline{\mathcal{Z}}_{j_{|\alpha|}}$, $\mathcal{Z}_{j_k} \in \{\mathcal{Z}_1, \ldots, \mathcal{Z}_n\}$, such that $\mathcal{Z}_j$ appears α_j times.

Proof. Let $\phi \in C_c^\infty(\mathcal{H})$ and

$$\phi(\xi,t) = \begin{cases} 1, & \tau(\xi,t) \le 1, \\ 0, & \tau(\xi,t) > 2, \end{cases}$$

where

$$\tau(\xi,t) = |\xi| + |t|^{1/2}. \tag{1.14}$$

For $m \ge 1$ define $\phi_m(\xi,t) = \phi(\xi/m, t/m^2)$. By (1.12),

$$\phi_m \widetilde{F} = \mathcal{C}(\phi_m \widetilde{F}) + \mathcal{L}_n(\phi_m \widetilde{F}) * \varPhi,$$

and

$$\overline{\mathcal{Z}}_j(\mathcal{C}(\phi_m \widetilde{F})) = 0.$$

We also have

$$\overline{\mathcal{Z}}_j(|\xi|^2/4 - it) = 0, \qquad \overline{\mathcal{Z}}_j(|\xi|^2/4 + it) = \xi_j,$$

$$\overline{\mathcal{Z}}_j \log(|\xi|^2/4 + it) = \frac{\xi_j}{|\xi|^2/4 + it}.$$

Then

$$\overline{\mathcal{Z}}^\beta \varPhi(\xi,t) = c(\beta)\,\frac{\xi_1^{\beta_1} \ldots \xi_n^{\beta_n}}{(|\xi|^2/4 + it)^{|\beta|}(|\xi|^2/4 - it)^n},$$

So

$$|\overline{\mathcal{Z}}^\beta \varPhi(\xi,t)| \le c(\beta)\,\frac{\tau(\xi,t)^{|\beta|}}{\tau(\xi,t)^{2(|\beta|+n)}} = c(\beta)\tau(\xi,t)^{-|\beta|-2n}.$$

Fix (ξ,t). It is easy to notice that for m large enough we have

$$\overline{\mathcal{Z}}^\beta(\phi_m \widetilde{F}(\xi,t)) = \overline{\mathcal{Z}}^\beta \widetilde{F}(\xi,t).$$

Moreover,

$$\overline{\mathcal{Z}}^\beta(\phi_m \widetilde{F}(\xi,t)) = \int \mathcal{L}_n(\phi_m \widetilde{F})(w,s)\overline{\mathcal{Z}}^\beta \varPhi((w,s)^{-1}(\xi,t))\,dw\,ds.$$

Observe that

$$|\mathcal{L}_n(\phi_m \widetilde{F})(w,s)| \le cm^N, \qquad |\overline{\mathcal{Z}}^\beta \varPhi((w,s)^{-1}(\xi,t))| \le c(\beta,\xi,t)m^{-|\beta|-2n}$$

for $\tau(w,s) \le 2m$ and m large enough. Therefore,

$$|\overline{\mathcal{Z}}^\beta \widetilde{F}(\xi,t)| \le c \int\limits_{\{(w,s)\,:\,\tau(w,s)<2m\}} m^{N-|\beta|-2n}\,dw\,ds = c'm^{N+2-|\beta|}.$$

This proves the theorem. ∎

1.2. *Holomorphic functions on the Siegel half-space.* In this section we will characterize polynomially growing holomorphic functions defined on $\mathcal{U}$. In view of (1.7)–(1.9) we have

$$d\theta(\mathcal{Z}_j) = 2\partial_{\xi_j} + i\overline{\xi}_j\partial_z, \tag{1.15}$$

$$d\theta(\overline{\mathcal{Z}}_j) = 2\partial_{\bar{\xi}_j} - i\xi_j\partial_{\bar{z}}, \tag{1.16}$$

where $(d\theta(\mathcal{Z}_j)F)\circ\theta = \mathcal{Z}_j(F\circ\theta)$. For a function F defined on $\mathcal{U}$ we consider

$$\widetilde{F}((\xi,t)a) = F(\theta(\xi,t,a)). \tag{1.17}$$

Suppose that there is a positive integer N and a locally integrable function $\zeta(a)$ such that

$$|\widetilde{F}((\xi,t)a)| \le \zeta(a)(1+|\xi|+|t|^{1/2})^N. \tag{1.18}$$

Clearly by (1.7), if P is a polynomial defined on $\mathcal{U}$, then there are constants c, M such that $|\widetilde{P}((\xi,t)a)| \le c(1+a+|\xi|+|t|^{1/2})^M$, so the condition (1.18) is satisfied.

THEOREM 1.19. *Suppose that for a function F satisfying* (1.18) *we have*

$$(\partial_t + i\partial_a)\widetilde{F} = 0, \tag{1.20}$$

$$\mathcal{L}_n\widetilde{F} = (\mathcal{L} - in\mathcal{T})\widetilde{F} = 0, \tag{1.21}$$

where $\mathcal{L} = \sum_{j=1}^n(\mathcal{X}_j^2 + \mathcal{Y}_j^2)$. Then there is a polynomial W such that $F - W$ is a holomorphic function.

Proof. Notice that from (1.20) and (1.21),

$$(\mathcal{L} - n\partial_a)\widetilde{F} = (\mathcal{L} - in\mathcal{T})\widetilde{F} = 0 \quad \text{and} \quad (\partial_t^2 + \partial_a^2)\widetilde{F} = 0.$$

For a multiindex $I = (I_1,\ldots,I_{2n+1})$ let

$$D^I = (a\mathcal{X}_1)^{I_1}\ldots(a\mathcal{X}_n)^{I_n}(a\mathcal{Y}_1)^{I_{n+1}}\ldots(a\mathcal{Y}_n)^{I_{2n}}(a^2\mathcal{T})^{I_{2n+1}}$$

be a left-invariant differentiable operator defined on $\mathcal{S}$. Define

$$\mathbf{L} = a(\mathcal{L} - n\partial_a) + a^2(\partial_t^2 + \partial_a^2). \tag{1.22}$$

Then $\mathbf{L}$ is an elliptic operator with real polynomial coefficients and it annihilates the real and imaginary parts of $\widetilde{F}$, so $\Re\widetilde{F}$ and $\Im\widetilde{F}$ are real analytic ([N, §3.8]).

Using the Harnack inequality for $\mathbf{L}$, denoting by B some neighborhood in $\mathcal{S}$ of the unit element e we obtain

$$\begin{aligned} |D^I\widetilde{F}((\xi,t)a)| &\le c_I \int_B |\widetilde{F}((\xi,t,a)(w,s,b))|\,dm((w,s)b) \\ &\le \zeta_I(a)(1+|\xi|+|t|^{1/2})^N. \end{aligned} \tag{1.23}$$

Define

$$\mathcal{K} = \theta(\mathcal{L} - n\partial_a) = 4\sum_{j=1}^n \partial_{\xi_j}\partial_{\bar{\xi}_j} + 2i\sum_{j=1}^n \overline{\xi}_j\partial_z\partial_{\bar{\xi}_j} - 2i\sum_{j=1}^n \xi_j\partial_{\bar{z}}\partial_{\xi_j} + |\xi|^2\partial_z\partial_{\bar{z}}.$$

Then by (1.20) and (1.21) we have

$$\mathcal{K}F(\xi,z) = \Big(4\sum_{j=1}^n \partial_{\xi_j}\partial_{\bar{\xi}_j} + 2i\sum_{j=1}^n \overline{\xi}_j\partial_z\partial_{\bar{\xi}_j}\Big)F(\xi,z) = 0. \tag{1.24}$$

In view of Theorem 1.13, there is a number p such that for $1 \le j \le n$ we have $\overline{\mathcal{Z}}^{\beta}\widetilde{F}(\xi,t)=0$ if $|\beta|\ge p$. (We are going to skip $d\theta$ and write $\overline{\mathcal{Z}}^{\beta}$ on both $\mathcal{U}$ and $\mathcal{S}$). Together with (1.16) and (1.20) this implies

$$\partial_{\bar\xi}^{\beta}F(\xi,z)=0. \tag{1.25}$$

Put $w=(\xi,z)$. In a neighborhood U of $w_0=(0,i)\in\mathcal{U}$ we write

$$F(w)=\sum_{\alpha,\beta}c_{\alpha,\beta}(w-w_0)^{\alpha}(\overline{w}-\overline{w}_0)^{\beta},$$

where $w=(\xi,z)$ and $\xi=(\xi_1,\ldots,\xi_n)$, $z,\xi_j\in\mathbb{C}$ for $1\le j\le n$. Let $\beta=(\beta_1,\ldots,\beta_{n+1})$. In this notation $w^{\beta}=\xi_1^{\beta_1}\ldots\xi_n^{\beta_n}\cdot z^{\beta_{n+1}}$. Notice that if $c_{\alpha,\beta}\ne 0$ then

$$\sum_{j=1}^{n}\beta_j\le p-1,\qquad \beta_{n+1}=0.$$

Let $\beta_0=(\beta_1,\ldots,\beta_{n+1})$ be such that $\sum_j\beta_j=p-1$. So by (1.24),

$$\partial_z\partial_{\bar\xi}^{\beta_0}F=0. \tag{1.26}$$

Let

$$h(w)=\partial_{\bar\xi}^{\beta_0}F(w).$$

It is easy to see that $h(w)=h(\xi)$ is a holomorphic function which does not depend on z. Moreover,

$$|h(\xi)|=|\widetilde{h}(\xi,0,1)|=|\overline{\mathcal{Z}}_1^{\beta_1}\ldots\overline{\mathcal{Z}}_n^{\beta_n}\widetilde{F}(\xi,0,1)|\le c(1+|\xi|)^{N}.$$

and so h is a polynomial, i.e.

$$h(w)=W_{\beta_0}(w)=\beta_0!\sum_{\alpha}c_{\alpha,\beta_0}(w-w_0)^{\alpha}. \tag{1.27}$$

Moreover, W_{β_0} does not depend on z.

For smaller multiindices we need the following lemma:

Lemma 1.28. *Suppose that the function F satisfies the assumptions of Theorem* 1.19. *If*

$$\partial_{\bar\xi}^{m}F(\xi,z)=g(\xi,z)+W(\xi,z), \tag{1.29}$$

where g is a holomorphic function, W is a polynomial and $m=(m_1,\ldots,m_n)\ne(0,\ldots,0)$, then g is a polynomial.

Assuming that Lemma 1.28 holds, let us now finish the proof. Consider the function

$$\begin{aligned}g(w)&=\partial_{\bar\xi_1}^{\beta_1}\ldots\partial_{\bar\xi_j}^{\beta_j-1}\ldots\partial_{\bar\xi_n}^{\beta_n}\Big(F(w)-\sum_{\alpha}c_{\alpha,\beta_0}(w-w_0)^{\alpha}(\overline{w}-\overline{w}_0)^{\beta_0}\Big)\\&=\partial_{\bar\xi_1}^{\beta_1}\ldots\partial_{\bar\xi_j}^{\beta_j-1}\ldots\partial_{\bar\xi_n}^{\beta_n}F(w)+cW_{\beta_0}(w)\overline{\xi}_j,\end{aligned}$$

where W_{β_0} is as in (1.27). For $w \in U$ we have

$$g(w) = c\sum_{\alpha} c_{\alpha,\beta_1,\ldots,\beta_j-1,\ldots,\beta_n,0}(w-w_0)^{\alpha}.$$

Hence g is holomorphic. From Lemma 1.28 we see that g is a polynomial. Fix a multiindex $m = (m_1,\ldots,m_n,0)$ and assume that for all multiindices $r = (r_1,\ldots,r_n,0)$, $r \neq m$, $m_j \le r_j \le p_j - 1$,

$$\sum_{\alpha} c_{\alpha,r}(w-w_0)^{\alpha}(\overline{w}-\overline{w}_0)^{r}$$

is a polynomial. Then

$$g(w) = \partial_{\bar\xi}^{m}\Big(F(w) - \sum_{\alpha,r} c_{\alpha,r}(w-w_0)^{\alpha}(\overline{w}-\overline{w}_0)^{r}\Big) = \partial_{\bar\xi}^{m}F(w) + W_m(w),$$

where W_m is a polynomial. For $w \in U$ we have

$$g(w) = c\sum_{\alpha} c_{\alpha,m}(w-w_0)^{\alpha}.$$

So g is holomorphic, and by Lemma 1.28 it is a polynomial. Finally, we conclude that

$$F(w) = \sum_{\alpha} c_{\alpha,0}(w-w_0)^{\alpha} + W_0(w),$$

which completes the proof of Theorem 1.19. ∎

Proof of Lemma 1.28. Notice that $[\partial_{\bar\xi},\mathcal{K}] = 2i\partial_z\partial_{\bar\xi}$. Therefore

$$0 = \partial_{\bar\xi}^{m}\mathcal{K}F(\xi,z) = \mathcal{K}(\partial_{\bar\xi}^{m}F(\xi,z)) + 2i|m|\partial_z\partial_{\bar\xi}^{m}F(\xi,z),$$

where $|m| = \sum_{j=1}^{n} m_j$. From (1.29) we obtain

$$\mathcal{K}(g(\xi,z)) + \mathcal{K}(W(\xi,z)) + 2i|m|\partial_z g(\xi,z) + 2i|m|\partial_z W(\xi,z) = 0.$$

Since g is holomorphic, $\partial_z g(\xi,z)$ is a polynomial. Moreover, we can find a holomorphic polynomial P such that $\partial_z P = \partial_z g$. Then we have

$$\begin{aligned}
&\partial_z(g-P)(\xi,z) = 0,\\
&\partial_{\bar\xi_j}(g-P)(\xi,z) = 0, \qquad 1 \le j \le n,\\
&\partial_{\bar z}(g-P)(\xi,z) = 0.
\end{aligned}$$

Therefore, the function $h_1(\xi) = (g-P)(\xi,z)$ is holomorphic and independent of z. Then by (1.29) we get

$$\begin{aligned}
|h_1(\xi)| = |\widetilde{h}_1(\xi,0,1)| &= |\overline{\mathcal{Z}}^{m}\widetilde{F}(\xi,0,1) - \widetilde{W}(\xi,0,1) - \widetilde{P}(\xi,0,1)|\\
&\le \zeta_m(1)(1+|\xi|)^{M}
\end{aligned}$$

and so g is a polynomial. ∎

2. Type two Siegel domain over the cone of Hermitian matrices

2.1. *Definition and the basic properties.* Suppose we are given a cone Ω in a Euclidean space $\mathcal{V}$, a complex vector space $\mathcal{Z}$ and a Hermitian symmetric bilinear mapping

$$\Phi : \mathcal{Z} \times \mathcal{Z} \to \mathcal{V}^{\mathbb{C}} = \mathcal{V} + i\mathcal{V}$$

such that

$$\begin{gathered} \Phi(\xi,\xi) \in \overline{\Omega} \quad \text{ for } \xi \in \mathcal{Z}, \\ \text{if} \quad \Phi(\xi,\xi) = 0 \quad \text{then} \quad \xi = 0, \\ \Phi(\xi,w) = \overline{\Phi(w,\xi)}. \end{gathered}$$

The *Siegel domain* associated with these data is the set

$$\mathcal{D} = \{(\xi,z) \in \mathcal{Z} \oplus \mathcal{V}^{\mathbb{C}} : \Im z - \Phi(\xi,\xi) \in \Omega\},$$

where $\Im(x+iy) = y$ for $x+iy \in \mathcal{V}^{\mathbb{C}}$. In this paper $\mathcal{V}$ is the space of hermitian $n \times n$ matrices, considered as a linear space over $\mathbb{R}$. Then $\mathcal{V}^{\mathbb{C}}$ is the space of complex-valued $n \times n$ matrices. The space $\mathcal{Z}$ consists of the complex-valued $n \times m$ matrices and Ω is the cone of positive definite matrices in $\mathcal{V}^{\mathbb{C}}$. The bilinear mapping Φ is given by

$$\Phi(\xi,w) = \xi \overline{w}^t.$$

The elements $w \in \mathcal{Z}, x \in \mathcal{V}$ act on $\mathcal{D}$ in the following way:

$$(\xi,z) \mapsto w \circ (\xi,z) = (\xi + w, z + 2i\Phi(\xi,w) + i\Phi(w,w)), \tag{2.1}$$

$$(\xi,z) \mapsto x \circ (\xi,z) = (\xi, z + x). \tag{2.2}$$

All the mappings of the form (2.1) and (2.2) form a group which will be denoted by $\mathbf{N}(\Phi)$. The multiplication in $\mathbf{N}(\Phi)$ is given by

$$(\xi,x)(\zeta,y) = (\xi + \zeta, x + y + 2\Im\Phi(\xi,\zeta)). \tag{2.3}$$

Clearly, $\mathcal{Z} \oplus \mathcal{V}$ is the Lie algebra of $\mathbf{N}(\Phi)$. Let $\mathbf{S}$ be the group of lower triangular complex $n \times n$ matrices with positive entries on the diagonal and let

$$\sigma(s)\xi = s\xi, \qquad s \in \mathbf{S},\ \xi \in \mathcal{Z}.$$

Notice that

$$\Phi(\sigma(s)\xi, \sigma(s)w) = s\Phi(\xi,w)\overline{s}^t. \tag{2.4}$$

Therefore, s acts on $\mathcal{D}$ as follows:

$$(\xi,z) \mapsto s \circ (\xi,z) = (\sigma(s)\xi, sz\overline{s}^t). \tag{2.5}$$

$\mathbf{N}(\Phi)$ and $\mathbf{S}$ generate a solvable Lie group which is their semidirect product with multiplication

$$(\xi,x,s) \circ (\xi_1,x_1,s_1) = ((\xi,x)(\sigma(s)\xi_1, sx_1\overline{s}^t), ss_1). \tag{2.6}$$

Then the Lie algebra of $\mathbf{N}(\Phi)\mathbf{S}$ is $\mathcal{Z}\oplus\mathcal{V}\oplus\mathcal{S}$, where $\mathcal{S}$ is the Lie algebra of $\mathbf{S}$. Clearly the group $\mathbf{N}(\Phi)\mathbf{S}$ acts simply transitively on the domain $\mathcal{D}$, and the function

$$\theta(\xi, vs) = (\xi, vs)\circ(0,i) \tag{2.7}$$

is a diffeomorphism of $\mathbf{N}(\Phi)\mathbf{S}$ onto $\mathcal{D}$. So we are going to identify the group $\mathbf{N}(\Phi)\mathbf{S}$ with the domain $\mathcal{D}$. If $n=1$, then $\mathcal{D}$ is just the Siegel half-space, so from now on we assume that $n\geq 2$. Let $|\xi|$ and $|v|$ be Euclidean norms in $\mathcal{Z}$ and $\mathcal{V}$ respectively, and $\|s\|$ the norm of the linear transformation $(\xi,v)\mapsto(\sigma(s)\xi, sv)$.

The Lie algebra $\mathcal{Z}\oplus\mathcal{V}\oplus\mathcal{S}$ is identified with matrices:

$$\mathcal{Z}=\{(\xi_{ij}) : \xi_{ij}\in\mathbb{C},\ i=1,\dots,n,\ j=1,\dots,m\}$$
$$\mathcal{V}=\{(v_{ij}) : v_{ij}\in\mathbb{C},\ v_{ji}=\overline{v}_{ij},\ i,j=1,\dots,n\}$$
$$\mathcal{S}=\{(w_{i,j}) : w_{ij}\in\mathbb{C},\ w_{ij}=0 \text{ for } i<j,\ w_{jj}\in\mathbb{R},\ i,j=1,\dots,n.\}$$

We will need commutation relations in the algebra $\mathcal{Z}\oplus\mathcal{V}\oplus\mathcal{S}$. We consider the following basis of it. For $\mathcal{S}$ we choose the matrices

$$H_k=(h_{pq})_{p,q=1,\dots,n}, \quad k=1,\dots,n, \quad \text{where} \quad h_{pq}=\tfrac{1}{2}\delta_{pq,kk},$$

and the matrices

$$Y^\alpha_{kj}=(y_{pq})_{p,q=1,\dots,n}, \quad 1\leq k<j\leq n,\ \alpha\in\{1,i\}, \quad \text{where} \quad y_{pq}=\frac{\alpha}{\sqrt{2}}\,\delta_{pq,jk}.$$

A basis of $\mathcal{V}$ consists of the matrices

$$X_{kk}=(x_{pq})_{p,q=1,\dots,n}, \quad k=1,\dots,n, \quad \text{where} \quad x_{pq}=\delta_{pq,kk},$$

and the matrices

$$X^\alpha_{kl}=(x_{pq})_{p,q=1,\dots,n}, \quad 1\leq k<l\leq n,\ \alpha\in\{1,i\},$$
$$\text{where} \quad x_{pq}=\frac{\alpha}{\sqrt{2}}\,\delta_{pq,lk}+\frac{\overline{\alpha}}{\sqrt{2}}\,\delta_{pq,kl}.$$

On the other hand, it is convenient to take in $\mathcal{Z}$ the matrices

$$\mathcal{X}_{kl}=(x_{pq})_{\substack{p=1,\dots,n\\ q=1,\dots,m}}, \quad k=1,\dots,n,\ l=1,\dots,m, \quad \text{where} \quad x_{pq}=\frac{1}{2}\,\delta_{pq,kl},$$
$$\mathcal{Y}_{kl}=(y_{pq})_{\substack{p=1,\dots,n\\ q=1,\dots,m}}, \quad k=1,\dots,n,\ l=1,\dots,m, \quad \text{where} \quad y_{pq}=\frac{i}{2}\,\delta_{pq,kl}.$$

Now we calculate brackets in the algebra $\mathcal{S}$. It is easy to see that the only nonzero commutators in $\mathcal{S}$ are

$$[H_k, Y^\alpha_{kj}]=-\tfrac{1}{2}Y^\alpha_{kj}, \quad [H_j, Y^\alpha_{kj}]=\tfrac{1}{2}Y^\alpha_{kj} \tag{2.8}$$

for $\alpha=1,i$, $1\leq k<j\leq n$.

Notice that $\mathcal{V}$ is abelian so a commutator in $\mathcal{V}\oplus\mathcal{S}$ is given by

$$[(x,s),(x_1,s_1)]=(sx_1+x_1\overline{s}^t-s_1x-x\overline{s}_1^t, [s,s_1]). \tag{2.9}$$

The only nonzero commutators are given by

$$(2.10)\qquad \begin{aligned} &[H_k, X^\alpha_{jk}] = \tfrac{1}{2} X^\alpha_{jk}, && \text{for } 1 \le j < k \le n,\\ &[H_k, X^\alpha_{kj}] = \tfrac{1}{2} X^\alpha_{kj}, && \text{for } 1 \le k < j \le n,\\ &[H_k, X_{kk}] = X_{kk}, && \text{for } 1 \le k \le n,\\ &[Y^\alpha_{kj}, X_{kk}] = X^\alpha_{kj}, && \text{for } 1 \le k < j \le n,\\ &[Y^\alpha_{kj}, X^\alpha_{kj}] = X_{jj}, && \text{for } 1 \le k < j \le n,\\ &[Y^\alpha_{kj}, X^\alpha_{pk}] = \frac{\alpha^2}{\sqrt{2}}\, X^1_{pj}, && \text{for } 1 \le p < k < j \le n,\ \alpha = 1, i,\\ &[Y^\alpha_{kj}, X^\beta_{pk}] = \frac{1}{\sqrt{2}}\, X^i_{pj}, && \text{for } 1 \le p < k < j \le n,\ \alpha \ne \beta. \end{aligned}$$

Notice that the bracket in $\mathbf{N}(\Phi)$ is given by

$$(2.11)\qquad [(\zeta, x), (\zeta_1, x_1)] = (0, 4\Im\Phi(\zeta, \zeta_1)).$$

Using (2.11) we may easily find the nonzero commutators in $\mathbf{N}(\Phi)$:

$$(2.12)\qquad \begin{aligned} &[\mathcal{X}_{kl}, \mathcal{X}_{pl}] = \frac{1}{\sqrt{2}}\, X^i_{kp} && \text{for } 1 \le k < p \le n,\\ &[\mathcal{X}_{kl}, \mathcal{Y}_{kl}] = -X_{kk} && \text{for } k = 1, \dots, n,\\ &[\mathcal{X}_{kl}, \mathcal{Y}_{pl}] = -\frac{1}{\sqrt{2}}\, X^1_{\min(k,p),\max(k,p)} && \text{for } k, p = 1, \dots, n,\ k \ne p,\\ &[\mathcal{Y}_{kl}, \mathcal{Y}_{pl}] = \frac{1}{\sqrt{2}}\, X^i_{kp} && \text{for } 1 \le k < p \le n, \end{aligned}$$

where $l = 1, \dots, m$.

We do the same with commutators in $\mathcal{Z} \oplus \mathcal{S}$. It is easy to see that in this algebra

$$[(0, s), (\xi, 0)] = s\xi,$$

where $s \in \mathcal{S}$ and $\xi \in \mathcal{Z}$. Therefore the only nonzero commutators are

$$(2.13)\qquad \begin{aligned} &[Y^1_{kj}, \mathcal{X}_{kl}] = \frac{1}{\sqrt{2}}\, \mathcal{X}_{jl}, && [Y^1_{kj}, \mathcal{Y}_{kl}] = \frac{1}{\sqrt{2}}\, \mathcal{Y}_{jl},\\ &[Y^i_{kj}, \mathcal{X}_{kl}] = \frac{1}{\sqrt{2}}\, \mathcal{Y}_{jl} && [Y^i_{kj}, \mathcal{Y}_{kl}] = -\frac{1}{\sqrt{2}}\, \mathcal{X}_{jl}, \end{aligned}$$

where $1 \le k < j \le n$ and $l = 1, \dots, m$.

We are left with the commutators

$$(2.14)\qquad [H_k, \mathcal{X}_{kl}] = \tfrac{1}{2}\mathcal{X}_{kl}, \qquad [H_k, \mathcal{Y}_{kl}] = \tfrac{1}{2}\mathcal{Y}_{kl},$$

where $k = 1, \dots, n$ and $l = 1, \dots, m$.

Remark 2.15. The basis of the algebra $\mathcal{Z} \oplus \mathcal{V} \oplus \mathcal{S}$ just chosen is consistent with [BDH, Section 2.2], the elements of $\mathcal{V} \oplus \mathcal{S}$ being denoted identically. If we fix k then the vector fields

$$\mathcal{X}_{k1},\ldots,\mathcal{X}_{km},\mathcal{Y}_{k1},\ldots,\mathcal{Y}_{km}$$

form a basis of the root space, denoted by $\mathcal{Z}_k$ in [BDH], and they correspond to the vectors $\mathcal{X}_k^\alpha,\mathcal{Y}_k^\alpha$ in [BDH]. The structure of a Jordan algebra in $\mathcal{V}$ is given by multiplication

$$x\circ x_1=\tfrac{1}{2}(xx_1+x_1x)$$

and the scalar product is

$$\langle x,x_1\rangle=\operatorname{tr}(xx_1).$$

2.2. *Holomorphic functions on $\mathcal{D}$.* On $\mathbf{N}(\Phi)\mathbf{S}$ we will consider a number of differential operators. Let

$$\begin{aligned}
\overline{\mathcal{W}}_k&=X_{kk}+iH_k, && k=1,\ldots,n,\\
\overline{\mathcal{V}}_{jk}^\alpha&=X_{jk}^\alpha+iY_{jk}^\alpha, && \alpha=1\text{ and }1\le j<k\le n,\\
\overline{\mathcal{V}}_{jk}^\beta&=X_{jk}^\beta+iY_{jk}^\beta, && \beta=i\text{ and }1\le j<k\le n,
\end{aligned}$$

and

$$\mathcal{L}_1=\sum_{k=1}^m\mathcal{Z}_k\overline{\mathcal{Z}}_k,$$

where

$$\mathcal{Z}_k=\mathcal{X}_{1k}-i\mathcal{Y}_{1k},\qquad k=1,\ldots,m.$$

For a function F on $\mathcal{D}$ we define

$$\widetilde{F}(\xi,z)=F\left((\xi,xs)\circ(0,i)\right)=F\circ\theta(\xi,xs). \tag{2.16}$$

Suppose that $\widetilde{F}$ has the following properties:

$$\widetilde{F}\text{ is annihilated by the operators }\mathcal{L}_1,\overline{\mathcal{W}}_j\text{ and }\overline{\mathcal{V}}_{jk}^\alpha, \tag{2.17}$$

and for a submultiplicative function η,

$$|\widetilde{F}(\xi,xs)|\le c\eta(s)(1+|\xi|+|x|^{1/2})^M. \tag{2.18}$$

A *submultiplicative function* is a function bounded on compact sets and satisfying $\eta(s_1s_2)\le\eta(s_1)\eta(s_2)$. We have the following characterization of functions $\widetilde{F}$.

THEOREM 2.19. *Let $\widetilde{F}$ be an analytic function which satisfies* (2.17) *and* (2.18). *Then there is a polynomial W such that $F-W$ is holomorphic. If $\mathcal{F}$ is a family of functions satisfying* (2.18) *for a given M, then the polynomials W may be chosen to have degrees uniformly bounded.*

The rest of the section is devoted to the proof of Theorem 2.19. Notice that

$$\mathcal{L}_1=\sum_{k=1}^m(\mathcal{X}_{1k}^2+\mathcal{Y}_{1k}^2)-miX_{11}.$$

For a function satisfying (2.17) we also have

$$\mathcal{L}_1\widetilde{F} = \Big(\sum_{k=1}^{m}(\mathcal{X}_{1k}^2 + \mathcal{Y}_{1k}^2) - mH_1\Big)\widetilde{F}.$$

Define

$$\Delta_j = \mathcal{W}_j\overline{\mathcal{W}}_j = X_{jj}^2 + H_j^2 - H_j,$$
$$\Delta_{jk}^{\alpha} = \mathcal{V}_{jk}^{\alpha}\overline{\mathcal{V}}_j^{\alpha} = (X_{kj}^{\alpha})^2 + (Y_{jk}^{\alpha})^2 - H_k,$$

for $\alpha = 1, i$ and $1 \leq j < k \leq n$. Let

$$\mathcal{M} = \sum_{k=1}^{m}(\mathcal{X}_{1k}^2 + \mathcal{Y}_{1k}^2) - mH_1 + \sum_{j=1}^{n}\Delta_j + \sum_{j<k,\alpha}\Delta_{jk}^{\alpha}. \tag{2.20}$$

Then $\mathcal{M}\widetilde{F} = 0$. From (2.13) the vector fields $\mathcal{X}_{1k}, \mathcal{Y}_{1k}$ for $k = 1, \ldots, n$ and X_{jj}, H_j for $j = 1, \ldots, n$, and $X_{jk}^{\alpha}, Y_{jk}^{\alpha}$ for $1 \leq j < k \leq n$ and $\alpha = 1, i$ generate the Lie algebra of the group $\mathbf{N}(\Phi)\mathbf{S}$. This means that the operator $\mathcal{M}$ satisfies the Hörmander condition. We will use the Harnack inequality for the operator $\mathcal{M}$ to estimate the derivatives

$$|D\widetilde{F}(\xi, vs)| \leq c\eta(s)(1 + \|s\|)^M(1 + |\xi| + |v|^{1/2})^M,$$

where $D = P_1 \ldots P_r$, $r \in \mathbf{N}$, $P_j \in \{\mathcal{X}_{pq}, \mathcal{Y}_{pq}, X_{kl}^{\alpha}, Y_{kl}^{\alpha}, X_{kl}^{\beta}, Y_{kl}^{\beta}, X_{kk}, H_k : 1 \leq q \leq m,\ 1 \leq p \leq n,\ 1 \leq k < l \leq n\}$, and $\|s\|$ is the norm of s as a linear map on $\mathcal{V}^{\mathbb{C}}$.

Indeed, by the Harnack inequality for $\mathcal{M}$ ([VSC, Section III]) we have

$$|D\widetilde{F}(0, e)| \leq c\int_B |\widetilde{F}(\chi, wr)|\, d\chi\, dw\, dr,$$

where B is a fixed bounded neighborhood of $(0, e)$. Since $\mathcal{M}$ is left-invariant, we may write

$$\begin{aligned} |D\widetilde{F}(\xi, vs)| &\leq c\int_B |\widetilde{F}((\xi, vs) \circ (\chi, wr))|\, d\chi\, dw\, dr \\ &\leq c\eta(s)\int_B \eta(r)(1 + |\xi + s\chi| + |x + sws^t + 2\Im\Phi(\xi, s\chi)|^{1/2})^M\, d\chi\, dw\, dr \\ &\leq c\eta(s)(1 + \|s\|)^M(1 + |\xi| + |x|^{1/2})^M\int_B \eta(r)(1 + |\chi| + |w|^{1/2})^M\, d\chi\, dw\, dr. \end{aligned} \tag{2.21}$$

Consider the Heisenberg group $\mathcal{H}$ generated by $\mathcal{X}_{1k}, \mathcal{Y}_{1k}$, $k = 1, \ldots, m$, and X_{11}. The function $\widetilde{F}$ restricted to $\mathcal{H}$ satisfies the assumptions of Theorem 1.13. Hence for $m_1 = |\gamma| = M + 3$,

$$\overline{\mathcal{Z}}^{\gamma}\widetilde{F} = 0 \tag{2.22}$$

on the group $\mathcal{H}$, where $\overline{\mathcal{Z}}^{\gamma}$ is as in Theorem 1.13. We will show that (2.22) is satisfied on $\mathbf{N}(\Phi)\mathbf{S}$. To prove that, we consider

$$G(\xi, xs) = \widetilde{F}((\chi, vs_1)(\xi, xs)).$$

Condition (2.17) is satisfied for G. Moreover, by (2.6),

$$\begin{aligned}|G(\xi,xs)| &\le c\eta(s_1)(1+\|s_1\|)^M(1+|\chi|+|v|^{1/2})^M\eta(s)(1+|\xi|+|x|^{1/2})^M\\ &= C(\chi,vs_1)\eta(s)(1+|\xi|+|x|^{1/2})^M,\end{aligned}$$

where $C(\chi,vs_1)$ is a constant depending on χ, v and s_1. By Theorem 1.13,

$$\overline{\mathcal{Z}}^\gamma G(\xi,x)=0$$

for $|\gamma|=M+3$ and $(\xi,x)\in\mathcal{H}$. But

$$(\overline{\mathcal{Z}}^\gamma G)(\xi,x)=\overline{\mathcal{Z}}^\gamma(\widetilde{F}((\eta,vs_1)(\xi,x)))=(\overline{\mathcal{Z}}^\gamma\widetilde{F})((\eta,vs_1)(\xi,x)),$$

and (2.22) follows on $\mathbf{N}(\Phi)\mathbf{S}$.

We need to write $\overline{\mathcal{W}}_j, \overline{\mathcal{V}}_{jk}^\alpha$ for $\alpha=1,i$ and $\overline{\mathcal{Z}}_k$ in coordinates on the domain $\mathcal{D}$. Introduce on $\mathcal{V}^{\mathbb{C}}$ the following coordinates:

$$Z=Z^\alpha+iZ^\beta,\qquad (z_{kl})=(z_{kl}^\alpha)+i(z_{kl}^\beta)$$

where $z_{kl}^\alpha=(z_{kl}+z_{lk})/2$ and $z_{kl}^\beta=(-z_{kl}+z_{lk})/2i$ and $1\le k<l\le n$. Moreover, notice that the terms z_{kk}^α and z_{kk}^β are the real and imaginary parts of z_{kk}, respectively. We write every element ξ of $\mathcal{Z}$ as $\xi=(\xi_{ij})$. The element $(\xi,z)\in\mathcal{D}$ will always be written in coordinates as $\xi_{kl}, z_{kl}^\alpha, z_{kl}^\beta$. Moreover, we assume that

$$\xi_{kl}=x_{kl}+iy_{kl},\qquad z_{kl}^\alpha=x_{kl}^\alpha+iy_{kl}^\alpha,\quad z_{kl}^\beta=x_{kl}^\beta+iy_{kl}^\beta.$$

Each $s\in\mathbf{S}$ will be written as

$$s=wa, \tag{2.23}$$

where $w=[w_{kj}]$ is a lower triangular matrix with $w_{kk}=1$ and a is a diagonal matrix with strictly positive entries $a_1,\dots,a_n$. In the theorem below writing w, w_{kj}, a_k we mean $w\circ\theta^{-1}, w_{kj}\circ\theta^{-1}, a_k\circ\theta^{-1}$.

THEOREM 2.24. *Let θ be as in* (2.7). *Then*

$$d\theta(X_{kk}+iH_k)=a_k\Big(2\partial_{\bar z_{kk}}+\sum_{h=k}^n\sum_{j=k}^n(c_{hj}^\alpha(w)\partial_{\bar z_{hj}^\alpha}+c_{hj}^\beta(w)\partial_{\bar z_{hj}^\beta})\Big), \tag{2.25}$$

where $c_{kk}^\alpha(w)=c_{kk}^\beta(w)=0$ and $k=1,\dots,n$. For $1\le k<l\le n$,

$$d\theta(X_{kl}^\alpha+iY_{kl}^\alpha)=\sqrt{a_ka_l}\Big(\sqrt2\,\partial_{\bar z_{kl}^\alpha}+\sum_{h=k}^n\sum_{j=l}^n(b_{hj}^\alpha(w)\partial_{\bar z_{hj}^\alpha}+b_{hj}^\beta(w)\partial_{\bar z_{hj}^\beta})\Big), \tag{2.26}$$

where $b_{kl}^\alpha(w)=b_{kl}^\beta(w)=0$. For $1\le k<l\le n$,

$$d\theta(X_{kl}^\beta+iY_{kl}^\beta)=\sqrt{a_ka_l}\Big(\sqrt2\,\partial_{\bar z_{kl}^\beta}+\sum_{h=k}^n\sum_{j=l}^n(d_{hj}^\alpha(w)\partial_{\bar z_{hj}^\alpha}+d_{hj}^\beta(w)\partial_{\bar z_{hj}^\beta})\Big), \tag{2.27}$$

where $d^{\alpha}_{kl}(w) = d^{\beta}_{kl}(w) = 0$,

$$d\theta(\mathcal{X}_{1k} + i\mathcal{Y}_{1k}) = \sqrt{a_1}\Big(\sum_{l=1}^{n} \overline{w}_{l1}\Big(\partial_{\bar{\xi}_{lk}} - 2i\sum_{j=1}^{n} \xi_{jk}\partial_{\bar{z}_{lj}}\Big)\Big), \tag{2.28}$$

where $w_{11} = 1$,

$$\overline{w}_{k1} \circ \theta^{-1} = \frac{i(\bar{z}_{k1} - z_{1k}) + 2(\xi, \overline{\xi}^t)_{1k}}{2(a_1 \circ \theta^{-1})}, \tag{2.29}$$

$(\xi, \overline{\xi}^t)_{1k} = \sum_{j=1}^{m} \xi_{1j} \cdot \overline{\xi}_{kj}$ *and* $a_1 \circ \theta^{-1} = \Im z_{11} - |\xi_1|^2$, *where* $|\xi_1|^2 = (\xi, \overline{\xi}^t)_{11}$. *All the terms* $b^{\alpha}_{kl}, b^{\beta}_{lk}, c^{\alpha}_{kl}, c^{\beta}_{kl}, d^{\alpha}_{kl}, d^{\beta}_{kl}$ *are nonzero polynomials of degree* 2 *depending on* w *and* $\overline{w}$.

The proof of this theorem is standard and it is left to the reader.

The aim of this section is to show that for every j, k, $\partial_{\bar{\xi}_{jk}} F$ is a polynomial. We start with the following observations. Notice that for every $1 \le j < k \le n$ and $l = 1, \ldots, m$ we have $\overline{\mathcal{Z}}_l(a_j) = 0$, $\mathcal{Z}_l(a_j) = 0$ and $\overline{\mathcal{Z}}_l(w_{jk}) = 0$, $\mathcal{Z}_l(w_{jk}) = 0$. This means that the factors a_j are not important in our case. So we have the analogous equalities on the domain. For example $d\theta(\overline{\mathcal{Z}}_l)(a_j \circ \theta^{-1}) = 0$. *To simplify notation we will identify the vector fields and functions on the group and on the domain denoting them identically, i.e. if we write* $\overline{\mathcal{Z}}_k F$, *we mean* $d\theta(\overline{\mathcal{Z}}_k)F$. This will not lead to confusion. For $k = 1, \ldots, m$ set $w = (1, w_{21}, \ldots, w_{n1})$ and $\eta_k = (\xi_{1k}, \ldots, \xi_{nk})$. Then

$$\overline{w}^{\alpha} = \overline{w}_{21}^{\alpha_2} \ldots \overline{w}_{n1}^{\alpha_n} \tag{2.30}$$

and

$$\partial^{\alpha}_{\bar{\eta}_k} = \partial^{\alpha_1}_{\bar{\xi}_{1k}} \ldots \partial^{\alpha_n}_{\bar{\xi}_{nk}}.$$

THEOREM 2.31. *Suppose that the function* $\widetilde{F}$ *satisfies conditions* (2.17) *and* (2.18), *and* $m_1 = m(M+3)$. *Then*

(a) $\partial_{\bar{z}_{kl}} F = 0$ *for* $k, l = 1, \ldots, n$,
(b) $\overline{\mathcal{Z}}^q_k F = a_1^{q/2}(\sum_{|\alpha|=q} c_{\alpha}\overline{w}^{\alpha}\partial^{\alpha}_{\bar{\eta}_k})F$, *where* $c_{\alpha} > 0$.
(c) $\partial^{\gamma}_{\bar{\xi}} F(\xi, z) = 0$ *for every multiindex* γ *with* $|\gamma| = m_1$.

For every compact subset $\mathcal{K} \subset \mathcal{V} \oplus \mathcal{S}$ *and every multiindex* γ *there are constants* $c(\gamma), M(\gamma)$ *such that if* $(\xi, z) \in \theta(\mathcal{Z} \times \mathcal{K})$, *then*

(d) $|\partial^{\gamma}_{\bar{\xi}} F(\xi, z)| \le c(\gamma)(1 + |\xi|)^{M(\gamma)}$.

Proof. (a) will be proved by induction on $k+l$. Notice that for $k = l = n$, from (2.25) and (2.17) we obtain

$$0 = d\theta(\overline{\mathcal{W}}_n)F(\xi, z) = 2a_n\partial_{\bar{z}_{nn}}F(\xi, z).$$

Therefore $\partial_{\bar{z}_{nn}}F(\xi, z) = 0$.

Assume that the formula holds for $2n \geq k+l > t$. Then for $k+l=t$ from (2.17) we have

$$\begin{aligned}
0 &= d\theta(X_{kl}^{\alpha} + iY_{kl}^{\alpha})F(\xi,z)\\
&= \sqrt{a_k a_l}\Big(\sqrt{2}\,\partial_{\bar{z}_{kl}^{\alpha}} + \sum_{p=k+1}^{n}(b_{pl}^{\alpha}(w)\partial_{\bar{z}_{pl}^{\alpha}} + b_{pl}^{\beta}(w)\partial_{\bar{z}_{pl}^{\beta}})\\
&\quad + \sum_{q=l+1}^{n}(b_{kq}^{\alpha}(w)\partial_{\bar{z}_{kq}^{\alpha}} + b_{kq}^{\beta}(w)\partial_{\bar{z}_{kq}^{\beta}})\\
&\quad + \sum_{p=k+1}^{n}\sum_{q=l+1}^{n}(b_{pq}^{\alpha}(w)\partial_{\bar{z}_{pq}^{\alpha}} + b_{pq}^{\beta}(w)\partial_{\bar{z}_{pq}^{\beta}})\Big)F(\xi,z).
\end{aligned}$$

Notice that we can apply the inductive assumption to the terms containing sums. Therefore we obtain

$$\partial_{\bar{z}_{kl}^{\alpha}}F(\xi,z) = 0.$$

By the same method we prove the rest of (a).

Now we prove (b) by induction on q. For $q=1$, by (a) we get

$$\begin{aligned}
\overline{\mathcal{Z}}_k F &= \sqrt{a_1}\Big(\sum_{l=1}^{n} w_{l1}\Big(\partial_{\bar{\xi}_{lk}} - 2i\sum_{j=1}^{n}\xi_{jk}\partial_{\bar{z}_{lj}}\Big)\Big)F\\
&= \sqrt{a_1}\Big(\sum_{l=1}^{n} w_{l1}\partial_{\bar{\xi}_{lk}}\Big)F.
\end{aligned} \tag{2.32}$$

Assume now that the formula holds for q. Then

$$\overline{\mathcal{Z}}_k^{q+1}F = a_1^{q/2}\,\overline{\mathcal{Z}}_k\Big(\sum_{|\alpha|=q} c_\alpha \overline{w}^{\alpha}\partial_{\bar{\eta}_k}^{\alpha}\Big)F.$$

Notice that $\overline{\mathcal{Z}}_k\overline{w}^{\alpha} = 0$ and $\partial_{\bar{z}_{lj}}F = 0$ by (a). Therefore

$$\overline{\mathcal{Z}}_k^{q+1}F = a_1^{(q+1)/2}\Big(\sum_{|\alpha|=q+1} c_\alpha \overline{w}^{\alpha}\partial_{\bar{\eta}_k}^{\alpha}\Big)F \tag{2.33}$$

and c_α are strictly positive, which finishes the proof of (b).

For (c), by (b) we get

$$0 = \overline{\mathcal{Z}}_k^{m_1}F = \sum_{|\alpha|=m_1} c_\alpha \overline{w}^{\alpha}\partial_{\bar{\eta}_k}^{\alpha}F.$$

Notice that if $\alpha \neq \beta$ then $\overline{w}^{\alpha} \neq \overline{w}^{\beta}$. So, if we differentiate the equation m_1 times with respect to $\bar{z}_{n1}$, we obtain $\partial_{\bar{\xi}_{nk}}^{m_1}F = 0$. Induction gives $\partial_{\bar{\eta}_k}^{\beta}F = 0$, if we apply $\partial_{\bar{z}}^{\beta}$, where $z = (1, z_{21}, \ldots, z_{n1})$, $|\beta| = M+3$ and $j = 1,\ldots,m$. Notice that for $|\alpha| = m_1$ we will just have $\partial_{\bar{\xi}}^{\alpha}F = 0$.

Now, we show (d). Let $s = wa$ be as in (2.23). Notice that

$$\partial_{\bar{\xi}}^{\alpha} = \sum_{\alpha,\beta} W_{\alpha,\beta}(\xi, vwa) D^{\beta},$$

where $W_{\alpha,\beta}$ is a polynomial in $\xi, v, w, a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}$. If vs is in the compact set $\mathcal{K}$, then from (2.21),

$$(2.34)\qquad |\partial_{\bar{\xi}}^{\alpha} \widetilde{F}(\xi, vs)| \le c_{\mathcal{K}}(1+|\xi|)^{M(\alpha)} \sum_{\gamma} |D^{\gamma} \widetilde{F}(\xi, vs)| \le c(1+|\xi|)^{M(\alpha)+M}.$$

Because of (2.7) the image of $\partial_{\bar{\xi}_{kl}}$ on the domain is $\partial_{\bar{\xi}_{kl}} + \sum_{p,q} W_{pq}(\xi)\partial_{\bar{z}_{pq}}$ and so we obtain

$$\partial_{\bar{\xi}_{kl}} \widetilde{F}(\xi, vs) = \Big(\Big(\partial_{\bar{\xi}_{kl}} + \sum_{pq} W_{pq}(\xi)\partial_{\bar{z}_{pq}}\Big)F\Big)(\theta(\xi, vs)) = \partial_{\bar{\xi}_{kl}} F(\xi, z).$$

Therefore, on $\theta(\mathcal{Z} \times \mathcal{K})$ we have

$$|\partial_{\bar{\xi}}^{\alpha} F(\xi, z)| \le c(1 + |\xi|)^{M(\alpha)+M},$$

which finishes the proof of Theorem 2.31. ∎

Every polynomially growing function which is holomorphic in $\mathbb{C}^n$ is polynomial. We will need the following generalization of this fact to our situation.

LEMMA 2.35. *Suppose that a function f is analytic on $\mathcal{D}$ and satisfies the following conditions*:

(1) *For all k, l, we have $\partial_{\bar{z}_{kl}} f(\xi, z) = 0$.*
(2) *For every k, l there is β_{kl} such that $\partial_{z_{kl}}^{\beta_{kl}} f(\xi, z) = 0$.*
(3) *There is p such that for all $|\gamma| = p$, we have $\partial_{\bar{\xi}}^{\gamma} f(\xi, z) = 0$.*
(4) *There is a nonpositive integer N such that for every compact set K there is a constant c such that $|f(\xi, z)| \le c(1 + |\xi|)^N$ if $(\xi, z) \in \theta(\mathcal{Z} \times K)$.*
(5) *There is a nonpositive integer N such that for every $|\gamma| \le p$ and every compact set K there is a constant c such that $|\partial_{\bar{\xi}}^{\gamma} f(\xi, z)| \le c(1+|\xi|)^N$ if $(\xi, z) \in \theta(\mathcal{Z} \times K)$.*

Then f is a polynomial.

Let $\mathcal{F}$ be a family of functions which satisfy the above conditions for given p, N, β_{kl}. Then the degrees of the polynomials in $\mathcal{F}$ have a common bound.

The proof of Lemma 2.35 is an elementary calculation.

In the next step we will try to get some more equations satisfied by the function F. We know that $\overline{\mathcal{Z}}^{\alpha} F = 0$ for $|\alpha| = m_1$. We will need some formulas for $\overline{\mathcal{Z}}^{\alpha}$ for $|\alpha| < m_1$. To do this we will use commutators.

Notice that for $k=1,\dots,m$ the vector fields $\overline{\mathcal{Z}}_k$ have the properties

$$[\mathcal{Z}_k,\overline{\mathcal{Z}}_j]=0,\qquad k\neq j, \tag{2.36}$$

$$[\mathcal{Z}_k,\overline{\mathcal{Z}}_k]F=[\mathcal{Z}_1,\overline{\mathcal{Z}}_1]F=-ia_1\Big(\sum_{j=1}^n w_{j1}\Big(\sum_{l=1}^n \overline{w}_{l1}\partial_{z_{jl}}\Big)\Big)F, \tag{2.37}$$

where $w_{11}=1$, and F is as in (a) of Theorem 2.31. Indeed,

$$\begin{aligned}[\mathcal{Z}_k,\overline{\mathcal{Z}}_j]&=[\mathcal{X}_{1k}-i\mathcal{Y}_{1k},\mathcal{X}_{1j}+i\mathcal{Y}_{1j}]\\ &=[\mathcal{X}_{1k},\mathcal{X}_{1j}]+[\mathcal{Y}_{1k},\mathcal{Y}_{1j}]-i([\mathcal{Y}_{1k},\mathcal{X}_{1j}]-[\mathcal{X}_{1k},\mathcal{Y}_{1j}])=0\end{aligned}$$

by (2.12). On the other hand, if $k=j$, then $[\mathcal{X}_{1k},\mathcal{Y}_{1k}]=-X_{11}$. Therefore

$$[\mathcal{Z}_k,\overline{\mathcal{Z}}_k]=-2iX_{11}=-i(\mathcal{W}_1+\overline{\mathcal{W}}_1). \tag{2.38}$$

Using the fact that $\overline{\mathcal{W}}_1F=0$, we obtain

$$[\mathcal{Z}_k,\overline{\mathcal{Z}}_k]F=-i\mathcal{W}_1F.$$

A direct computation using the action of $(\xi,vs)\in\mathbf{N}(\Phi)\mathbf{S}$ on $\mathcal{D}$ given by

$$(\xi,vs)\circ(\chi,u)=(\xi+s\chi,v+su\overline{s}^t+2i\Phi(s\chi,\xi)+i\Phi(\xi,\xi)). \tag{2.39}$$

shows that

$$\mathcal{W}_1=a_1\sum_{j=1}^n w_{j1}\Big(\sum_{l=1}^n\overline{w}_{l1}\partial_{z_{jl}}\Big),$$

and so we get (2.37).

Lemma 2.40.

$$\overline{\mathcal{Z}}_k^l\mathcal{Z}_k\overline{\mathcal{Z}}_kF=(l[\mathcal{Z}_k,\overline{\mathcal{Z}}_k]\overline{\mathcal{Z}}_k^l+\mathcal{Z}_k\overline{\mathcal{Z}}_k^{l+1})F.$$

Proof. The proof is by induction. For $l=1$ we have

$$\overline{\mathcal{Z}}_k\mathcal{Z}_k\overline{\mathcal{Z}}_kF=(\mathcal{Z}_k\overline{\mathcal{Z}}_k^2+[\mathcal{Z}_k,\overline{\mathcal{Z}}_k]\overline{\mathcal{Z}}_k)F.$$

Assume that (2.40) holds for l. Then

$$\overline{\mathcal{Z}}_k^{l+1}\mathcal{Z}_k\overline{\mathcal{Z}}_kF=\overline{\mathcal{Z}}_k(l[\mathcal{Z}_k,\overline{\mathcal{Z}}_k]\overline{\mathcal{Z}}_k^l+\mathcal{Z}_k\overline{\mathcal{Z}}_k^{l+1})F.$$

Notice that

$$[[\mathcal{Z}_k,\overline{\mathcal{Z}}_k],\overline{\mathcal{Z}}_k]=[[\mathcal{Z}_k,\overline{\mathcal{Z}}_k],\mathcal{Z}_k]=0.$$

Therefore,

$$\begin{aligned}\overline{\mathcal{Z}}_k^{l+1}\mathcal{Z}_k\overline{\mathcal{Z}}_kF&=(l[\mathcal{Z}_k,\overline{\mathcal{Z}}_k]\overline{\mathcal{Z}}_k^{l+1}+[\mathcal{Z}_k,\overline{\mathcal{Z}}_k]\overline{\mathcal{Z}}_k^{l+1}+\mathcal{Z}_k\overline{\mathcal{Z}}_k^{l+2})F\\ &=((l+1)[\mathcal{Z}_k,\overline{\mathcal{Z}}_k]\overline{\mathcal{Z}}_k^{l+1}+\mathcal{Z}_k\overline{\mathcal{Z}}_k^{l+2})F,\end{aligned}$$

which finishes the proof. ■

From Lemma 2.40, (2.36) and (2.37) we obtain

$$\overline{\mathcal{Z}}^\alpha\mathcal{L}_1F=\overline{\mathcal{Z}}^\alpha\sum_{k=1}^m\mathcal{Z}_k\overline{\mathcal{Z}}_kF=|\alpha|[\mathcal{Z}_1,\overline{\mathcal{Z}}_1]\overline{\mathcal{Z}}^\alpha F+\sum_{k=1}^m\mathcal{Z}_k\overline{\mathcal{Z}}^{\beta_k}F, \tag{2.41}$$

where $\beta_k = \alpha + e_k$ and $e_k = (0, \dots, 1, \dots, 0)$, with 1 in the kth place. Moreover, notice that if $|\alpha| = m_1 - 1$, then $\overline{\mathcal{Z}}_k^{\beta_k} = 0$ for every $k = 1, \dots, m$. Therefore we get an additional equation

$$[\mathcal{Z}_1, \overline{\mathcal{Z}}_1]\overline{\mathcal{Z}}^\alpha F = 0. \tag{2.42}$$

Since $[[\mathcal{Z}_1, \overline{\mathcal{Z}}_1], \overline{\mathcal{Z}}_k]F = 0$, (2.38) gives

$$\overline{\mathcal{Z}}^\alpha\Big(\sum_{j=1}^n w_{j1}\Big(\sum_{k=1}^n \overline{w}_{k1}\partial_{z_{jk}}\Big)\Big)F = 0. \tag{2.43}$$

Hence

$$\sum_{j=1}^n w_{j1}\overline{\mathcal{Z}}^\alpha\Big(\sum_{k=1}^n \overline{w}_{k1}\partial_{z_{jk}}\Big)F = 0. \tag{2.44}$$

Notice that for $j = 2, \dots, n$ and $k = 1, \dots, n$ from (2.29) and (2.28) we have

$$\begin{aligned}
\partial_{\bar{z}_{1j}} w_{k1} &= 0 \quad \text{ for } j \neq k,\\
\partial_{\bar{z}_{1j}} w_{j1} &= \frac{-i}{2a_1} \neq 0,\\
[\partial_{\bar{z}_{1j}}, \overline{\mathcal{Z}}_l] &= 0 \quad \text{ for } l = 1, \dots, m,\\
\partial_{\bar{z}_{1j}} \overline{w}_{k1} &= 0.
\end{aligned}$$

Applying $\partial_{\bar{z}_{1j}}$ to both sides of (2.44), for a function F that satisfies (a) from Theorem 2.31 we have

$$\begin{aligned}
0 &= \sum_{l=1}^n (\partial_{\bar{z}_{1j}} w_{l1})\overline{\mathcal{Z}}^\alpha\Big(\sum_{k=1}^n \overline{w}_{k1}\partial_{z_{lk}}\Big)F + \sum_{l=1}^n w_{l1}\overline{\mathcal{Z}}^\alpha\Big(\sum_{k=1}^n \overline{w}_{k1}\partial_{z_{lk}}\Big)\partial_{\bar{z}_{1j}}F\\
&= \frac{i}{2a_1}\,\overline{\mathcal{Z}}^\alpha\Big(\sum_{k=1}^n \overline{w}_{k1}\partial_{z_{jk}}\Big)F,
\end{aligned}$$

For $|\alpha| = m_1 - 1$ we get the equation

$$\overline{\mathcal{Z}}^\alpha\Big(\sum_{k=1}^n \overline{w}_{k1}\partial_{z_{jk}}\Big)F = 0 \quad \text{ for } 1 \leq j \leq n. \tag{2.45}$$

We will consider the case when $j = 1$ (for other indices the proof is the same). The equation (2.45) for $|\alpha| = m_1 - 1$ gives

$$\overline{\mathcal{Z}}^\alpha\Big(\sum_{k=1}^n \overline{w}_{k1}\partial_{z_{1k}}\Big)F = \sum_{k=1}^n \overline{w}_{k1}\overline{\mathcal{Z}}^\alpha\partial_{z_{1k}}F = 0. \tag{2.46}$$

We will need an expression for $\overline{\mathcal{Z}}^\alpha$, similar to (2.33). We will show by induction that if $\partial_{\bar{z}_{kl}}F = 0$ and $k, l = 1, \dots, n$, then

$$\overline{\mathcal{Z}}^\alpha F = a^{|\alpha|/2}\sum_{|\beta|=|\alpha|}\overline{w}^\beta\Big(\sum_{|\gamma|=|\alpha|} c_{\alpha,\beta,\gamma}\partial_{\bar{\xi}}^\gamma F\Big), \tag{2.47}$$

where $\overline{w}^\beta$ is as in (2.30), and γ is a multiindex related to partial derivatives given by $\xi = (\xi_{pq})_{\substack{1\le p\le n\\ 1\le q\le m}}$. Notice that some $c_{\alpha,\beta,\gamma}$ may vanish and for $|\alpha| = 1$ we simply have (2.32).

Indeed, from (2.28),

$$\overline{\mathcal{Z}}_k(\overline{\mathcal{Z}}^\alpha F) = a^{|\alpha|/2} \sum_{|\beta|=|\alpha|} \overline{w}^\beta \Big(\sum_{|\gamma|=|\alpha|} c_{\alpha,\beta,\gamma} \overline{\mathcal{Z}}_k \partial_{\bar\xi}^\gamma F\Big)$$
$$= a^{|\alpha|/2} \sum_{|\beta|=|\alpha|} \overline{w}^\beta \Big(\sum_{|\gamma|=|\alpha|} c_{\alpha,\beta,\gamma} \Big(\sum_{l=1}^n \overline{w}_{l1} \partial_{\bar\xi_{lk}} \Big) \partial_{\bar\xi}^\gamma F\Big).$$

Rearranging terms we will get (2.47). Let

$$f_{\alpha,\beta} = \sum_{|\gamma|=|\alpha|} c_{\alpha,\beta,\gamma} \partial_{\bar\xi}^\gamma F. \tag{2.48}$$

If $|\alpha| = 1$, $\overline{\mathcal{Z}}^\alpha = \overline{\mathcal{Z}}_k$ and $\overline{w}^\beta = \overline{w}_{j1}$ then by (2.32),

$$f_{\alpha,\beta} = \partial_{\bar\xi_{jk}} F. \tag{2.49}$$

Notice that if $\partial_{\bar z_{kl}} F = 0$, then also $\partial_{\bar z_{kl}} \partial_{z_{1k}} F = 0$. Hence

$$\overline{\mathcal{Z}}^\alpha \partial_{z_{1k}} F = a^{|\alpha|/2} \sum_{|\beta|=|\alpha|} \overline{w}^\beta \Big(\sum_{|\gamma|=|\alpha|} c_{\alpha,\beta,\gamma} \partial_{\bar\xi}^\gamma \partial_{z_{1k}} F\Big) = a^{|\alpha|/2} \sum_{|\beta|=|\alpha|} \overline{w}^\beta \partial_{z_{1k}} f_{\alpha,\beta}.$$

The equation (2.46) gives

$$\sum_{k=1}^n \sum_{|\beta|=|\alpha|} \overline{w}_{k1} \overline{w}^\beta \partial_{z_{1k}} f_{\alpha,\beta} = 0. \tag{2.50}$$

Analogously for $j = 2, \dots, n$,

$$\sum_{k=1}^n \sum_{|\beta|=|\alpha|} \overline{w}_{k1} \overline{w}^\beta \partial_{z_{jk}} f_{\alpha,\beta} = 0.$$

We will show that for every α the functions $f_{\alpha,\beta}$ are polynomials in ξ and z. In view of Lemma 2.35 it is enough to prove that for every z_{jk} there is γ_{jk} such that $\partial_{z_{jk}}^{\gamma_{jk}} f_{\alpha,\beta} = 0$. *We start with* $|\alpha| = m_1 - 1$ *and then proceed by downward induction on the length of* α*, finally getting the conclusion for* $|\alpha| = 1$*. Then Theorem* 2.19 *follows immediately by* (2.49).

Now we prove that the $f_{\alpha,\beta}$ are polynomials.

LEMMA 2.51. *For* $|\alpha| = m_1 - 1$, $f_{\alpha,\beta}$ *is a polynomial. If* $\mathcal{F}$ *is a family of functions* $\widetilde{F}$ *satisfying condition* (2.18) *for fixed* M, *then the degrees of the corresponding polynomials* $f_{\alpha,\beta}$ *have a common bound.*

To prove this we need the following lemma:

LEMMA 2.52. *Let $|\alpha| = m_1 - 1$. For every $|\beta| = |\alpha|$ and every j,*

$$\partial_{z_{11}}^{|\alpha|-j+1} f_{\alpha,\beta} = 0. \tag{2.53}$$

Proof. We proceed by induction on $\beta_1 = j$. As before for $k = 2, \dots, n$ we apply $\partial_{\bar{z}_{k1}} \partial_{\bar{z}_{21}}^{\beta_2} \dots \partial_{\bar{z}_{n1}}^{\beta_n}$ to (2.50) to get n equations

$$\partial_{z_{1k}} f_{\alpha,(\beta_1,\dots,\beta_n)} + \sum_{j\neq k} \partial_{z_{1j}} f_{\alpha,(\beta_1,\dots,\beta_j-1,\dots,\beta_k+1,\dots,\beta_n)} = 0, \quad 1 \le k \le n, \tag{2.54}$$

with the convention that if some $\beta_l = 0$, then

$$\partial_{z_{1l}} f_{\alpha,(\beta_1,\dots,\beta_l-1,\dots,\beta_k+1,\dots,\beta_n)} = 0.$$

For $j = |\alpha|$ by (2.54) we have

$$\partial_{z_{11}} f_{\alpha,(|\alpha|,0,\dots,0)} = 0.$$

Assume that (2.53) is true for every β such that $\beta_1 = j$. We will show that it is true for $j-1$. By (2.54) for $k = 1$ and $\beta_1 = j-1$ we obtain

$$\partial_{z_{11}} f_{\alpha,(j-1,\beta_2,\dots,\beta_n)} + \sum_{l>1} \partial_{z_{1l}} f_{\alpha,(j,\beta_2,\dots,\beta_l-1,\dots,\beta_n)} = 0.$$

Applying $\partial_{z_{11}}^{|\alpha|-j+1}$ to both sides of the above equality and using the inductive assumption we get

$$\partial_{z_{11}}^{|\alpha|-(j-1)+1} f_{\alpha,(j-1,\beta_2,\dots,\beta_n)} = 0.$$

This finishes the proof of Lemma 2.52. ∎

Proof of Lemma 2.51. Notice that the same method may be used for the other equations, and we get the same result as for the variable z_{11}, i.e. for $|\alpha| = m_1 - 1$, every β and every k there is $l \le |\alpha| + 1$ such that

$$\partial_{z_{1k}}^{l} f_{\alpha,\beta} = 0. \tag{2.55}$$

Moreover, notice that by (2.45), we may use the same argument for other variables z_{jk}, $j = 2, \dots, n$. Therefore by (2.35) for $|\alpha| = m_1 - 1$ and every β, $f_{\alpha,\beta}$ is a polynomial.

Notice that if $\mathcal{F}$ is a family of functions $\widetilde{F}$ satisfying condition (2.18) for fixed M, then the corresponding polynomials $f_{\alpha,\beta}$ satisfy the assumptions of Lemma 2.35 with the same p, M and β_{kl}. Therefore the degrees of the $f_{\alpha,\beta}$ have a common bound. ∎

We want to get a similar conclusion for $0 < |\alpha| < m_1 - 1$.

THEOREM 2.56. *For $0 < |\alpha| \le m_1 - 1$ and $|\beta| = |\alpha|$, $f_{\alpha,\beta}$ is a polynomial. If $\mathcal{F}$ is a family of functions F satisfying condition* (2.18) *for fixed M, then the degrees of the corresponding polynomials $f_{\alpha,\beta}$ have a common bound.*

Proof. The proof is by induction on $|\alpha|$. By Lemma 2.51, $a_1^{|\alpha|/2} \overline{\mathcal{Z}}^{\alpha} F$ is a polynomial in ξ, z. Assume that the assertion is true for $|\alpha| = K+1 \le m_1 - 1$.

We want to show that (2.56) is true for $|\alpha| = K$. By (2.41) we have

$$0 = c_\alpha[\mathcal{Z}_1, \overline{\mathcal{Z}}_1]\overline{\mathcal{Z}}^\alpha F + \sum_{k=1}^m \mathcal{Z}_k \overline{\mathcal{Z}}^{\delta_k} F,$$

where $\delta_k = |\alpha| + 1$. By the inductive assumption, $a_1^{(|\alpha|+2)/2}\mathcal{Z}_k\overline{\mathcal{Z}}^{\delta_k}F$ is a polynomial $W(\xi, z)$. Indeed, for $|\delta| = |\alpha| + 2$, $a_1^{|\alpha|+2}\overline{w}^\delta$ is a polynomial in ξ, z, i.e.

$$a_1^{(|\alpha|+2)/2}[\mathcal{Z}_1, \overline{\mathcal{Z}}_1]\overline{\mathcal{Z}}^\alpha F = W(\xi, z). \tag{2.57}$$

By (2.38) and the fact that $\overline{\mathcal{W}}_1 F = 0$ we obtain

$$a_1^{(|\alpha|+4)/2}\overline{\mathcal{Z}}^\alpha\Big(\sum_{j=1}^n w_{j1}\Big(\sum_{k=1}^n \overline{w}_{k1}\partial_{z_{jk}}\Big)\Big)F(\xi, z) = W(\xi, z).$$

As before this equation is equivalent to

$$a_1^{(|\alpha|+2)/2}\overline{\mathcal{Z}}^\alpha\Big(\sum_{k=1}^n \overline{w}_{k1}\partial_{z_{jk}}\Big)F(\xi, z) = W_j(\xi, z), \quad 1 \le j \le n, \tag{2.58}$$

where W_j is a polynomial in ξ, z.

Assume now $j = 1$ (for other j's the proof is analogous). First by (2.58) and (2.47) we get, for $|\alpha| = K$,

$$a_1^{|\alpha|+1}\Big(\sum_{k=1}^n \sum_{|\beta|=K} \overline{w}_{k1}\overline{w}^\beta \partial_{z_{1k}} f_{\alpha,\beta}\Big) = W_1,$$

where $\overline{\mathcal{Z}}^\alpha F = a_1^{|\alpha|/2}\sum_{|\beta|=K}\overline{w}^\beta f_{\alpha,\beta}$.

Applying $\partial_{\bar{z}_{k1}}\partial_{\bar{z}_{21}}^{\beta_2}\ldots\partial_{\bar{z}_{n1}}^{\beta_n}$ for $1 < k \le n$, we get for $1 \le k \le n$ the equations

$$a_1^Q\Big(\partial_{z_{1k}} f_{\alpha,(\beta_1,\beta_2,\ldots,\beta_n)} + \sum_{j\ne k}\partial_{z_{1j}} f_{\alpha,(\beta_1,\ldots,\beta_j-1,\ldots,\beta_k+1,\ldots,\beta_n)}\Big) = \widetilde{W}_{1,\beta}, \tag{2.59}$$

where Q is a positive integer $\le |\alpha| + 1$ and if $\beta_j = 0$ for some j, then $\partial_{z_{1j}} f_{\alpha,(\beta_1,\ldots,\beta_j-1,\ldots,\beta_k+1,\ldots,\beta_n)}$ is meant to be zero. Applying $\partial_{\bar{z}_{11}^Q}$, we get

$$\Big(\partial_{z_{1k}} f_{\alpha,(\beta_1,\beta_2,\ldots,\beta_n)} + \sum_{j\ne k}\partial_{z_{1j}} f_{\alpha,(\beta_1,\ldots,\beta_j-1,\ldots,\beta_k+1,\ldots,\beta_n)}\Big) = W_{1,\beta} \tag{2.60}$$

and if $\beta_j = 0$ for some j, then $\partial_{z_{1j}} f_{\alpha,(\beta_1,\ldots,\beta_j-1,\ldots,\beta_k+1,\ldots,\beta_n)}$ is meant to vanish.

LEMMA 2.61. *For $|\beta| = |\alpha| = K$ there is γ such that*

$$\partial_{z_{11}}^\gamma f_{\alpha,\beta} = 0.$$

Proof. We use induction on β_1. Notice that for $\beta_1 = K$ and $k = 1$, by (2.60), we get

$$\partial_{z_{11}} f_{\alpha,(K,0,\ldots,0)} = W_{1,(K,0,\ldots,0)}.$$

Therefore there is γ_1 such that $\partial_{z_{11}}^{\gamma_1} f_{\alpha,(K,0,\ldots,0)} = 0$. Now, assume that the conclusion is true for every β such that $\beta_1 = j+1$, i.e. there is γ_{j+1} such that $\partial_{z_{11}}^{\gamma_{j+1}} f_{\alpha,\beta} = 0$. In view of (2.60) for $k=1$ we have

$$\partial_{z_{11}} f_{\alpha,(j,\beta_2,\ldots,\beta_n)} + \sum_{l>j} \partial_{z_{1l}} f_{\alpha,(j+1,\beta_2,\ldots,\beta_j-1,\ldots,\beta_n)} = W_{1,\beta},$$

where $W_{1,\beta}$ are polynomials. Apply $\partial_{z_{11}}^{\gamma_{j+1}}$ to this equation. Then by the inductive assumption we obtain

$$\partial_{z_{11}}^{\gamma_{j+1}+1} f_{\alpha,(j,\beta_2,\ldots,\beta_n)} = V_{1,\beta}$$

for some polynomials $V_{1,\beta}$. Then there is γ_j such that $\partial_{z_{11}}^{\gamma_j} f_{\alpha,\beta} = 0$, which finishes the proof of Lemma 2.61. ∎

Notice that an analogous argument gives a similar result for other coefficients, i.e. for $|\beta| = K$ and all j,k, there is δ_{jk} such that

$$\partial_{z_{jk}}^{\delta_{jk}} f_{\alpha,\beta} = 0.$$

Moreover, notice that if $\mathcal{F}$ is a family of functions $\widetilde{F}$ satisfying (2.18) for the same M, then the degrees of the polynomials W in (2.57) will have a common bound, so the δ_{jk} will be uniformly bounded. Therefore, the $f_{\alpha,\beta}$ satisfy the assumptions of Lemma 2.35 uniformly and so their degrees have a common bound. ∎

3. Pluriharmonic functions on $\mathcal{D}$. Our purpose is to characterize polynomially growing pluriharmonic functions by three invariant differential operators (Theorem 3.19). On $\mathbf{N}(\Phi)\mathbf{S}$ we consider the operators

$$\begin{aligned} \Delta_j &= X_{jj}^2 + H_j^2 - H_j, && j \le n,\\ \mathcal{L}_{kl} &= \mathcal{X}_{kl}^2 + \mathcal{Y}_{kl}^2 - H_k, && k \le n,\ l \le m,\\ \Delta_{kl}^{\alpha} &= (X_{kl}^{\alpha})^2 + (Y_{kl}^{\alpha})^2 - H_l, && k < l \le n,\ \alpha = 1, i, \end{aligned}$$

$$\mathcal{L} = \sum_{j=1}^{n} \gamma_j \mathcal{L}_j, \qquad \gamma_j > 0, \tag{3.1}$$

where $\mathcal{L}_j = \sum_{k=1}^m \mathcal{L}_{jk}$,

$$\mathbf{H}\widetilde{F} = \sum_j \alpha_j \Delta_j, \qquad \text{where } \alpha_j > 0.$$

and

$$L = \sum_{j=1}^{n} d_j \Delta_j + \sum_{k<l}^{n} c_{kl}^{\alpha} \Delta_{kl}^{\alpha}, \tag{3.2}$$

where $d_j, c_{kl}^{\alpha} > 0$. As L acts on the right, it is well defined also on $\mathcal{V}\mathbf{S}$. One can choose the coefficients $d_j, c_{k,l}^{\alpha}$ in such a way that the maximal boundary is $\mathcal{V}$ ([BDH]), which means that every bounded L-harmonic function $\widetilde{G}$ on $\mathcal{V}\mathbf{S}$ is the Poisson integral

$$\widetilde{G}(xs) = \int_{\mathcal{V}} g(x + sy\overline{s}^t)P_L(y) =: g * P_L^s, \tag{3.3}$$

where $g \in L^\infty(\mathcal{V})$, P_L is the Poisson kernel for L, and P_L^s is given by a proper change of variables ([BDH], [DH1]).

P_L is a smooth, positive function on $\mathcal{V}$ with integral 1, and $\widetilde{G} \leftrightarrow g$ is a one-one mapping of $L^\infty(\mathcal{V})$ onto the space of bounded functions L-harmonic on $\mathcal{V}\mathbf{S}$. Moreover,

$$\lim_{a\to 0} \widetilde{G}(xwa) = g(x)$$

in the weak sense and

$$\|\widetilde{G}\|_{L^\infty} = \|g\|_{L^\infty}.$$

Writing $a \to 0$ we mean that $a_j \to 0$ for every $j = 1, \ldots, n$.

Assume that for a real smooth function $\widetilde{F}$ defined on $\mathbf{N}(\Phi)\mathbf{S}$ we have

$$|\widetilde{F}((\xi, x)s)| \le c(1 + |\xi|)^M, \tag{3.4}$$

$$L\widetilde{F} = 0. \tag{3.5}$$

Then for every ξ there exists $f_\xi \in L^\infty(\mathcal{V})$ such that

$$\widetilde{F}((\xi, x)s) = \int_{\mathcal{V}} f_\xi(x + sy\overline{s}^t)P_L(y)\, dy. \tag{3.6}$$

f_ξ is called *the boundary value* of $\widetilde{F}((\xi,\cdot)\cdot)$.

Moreover, assume for a while that there exists ε such that for every ξ,

$$\operatorname{supp} \widehat{f_\xi} \subset \{\lambda : \varepsilon \le |\lambda| \le \varepsilon^{-1}\}. \tag{3.7}$$

Later on we will get rid of this assumption.

Set $F = \widetilde{F} \circ \theta^{-1}$. The first approximation of Theorem 3.19 is

Lemma 3.8. *If $\widetilde{F} \in C^\infty(\mathbf{N}(\Phi)\mathbf{S})$ satisfies assumptions* (3.4), (3.5), (3.7), *and is annihilated by $\mathbf{H}$, $\mathcal{L}$, $\mathcal{L}_1$, then there is a polynomial W such that $F - W$ is pluriharmonic.*

Proof. Notice that for every ξ, $\widetilde{F}((\xi,\cdot)\cdot)$ is a bounded function on $\mathcal{V}\mathbf{S}$, annihilated by L and $\mathbf{H}$, so by Theorem 5.1 and Corollary 5.9 of [BDH],

$$\operatorname{supp} \widehat{f_\xi}(\cdot) \subset \overline{\Omega} \cup -\overline{\Omega},$$

where Ω is the cone of positive hermitian matrices.

Let φ_1 be a Schwartz function defined on $\mathcal{V}$ such that $\widehat{\varphi}_1 \in C_c^\infty(\mathcal{V})$ is real-valued. Moreover, assume that $\widehat{\varphi}_1 \equiv 1$ on a neighborhood of $\operatorname{supp} f_\xi \cap \overline{\Omega}$

and $\operatorname{supp}\varphi_1 \cap -\overline{\Omega} = \emptyset$. Then

$$\widetilde{F}_1((\xi,x)s) = \int_{\mathcal{V}} \varphi_1(v)\widetilde{F}((\xi,x-v)s)\,dv = \int_{\mathcal{V}} \varphi_1(v)\widetilde{F}((0,-v)\cdot(\xi,x)s)\,dv$$

satisfies the assumptions of the present lemma. Set $f_{1,\xi} = \varphi_1 *_{\mathcal{V}} f_\xi$. Then $f_{1,\xi}$ is the boundary value of $\widetilde{F}_1$ and $\operatorname{supp}\widehat{f}_{1,\xi} \subset \overline{\Omega}$.

Using the same arguments as in Section 5 of [BDH], we prove that $\widetilde{F}_1 \circ \theta^{-1}$ is holomorphic, and so

$$(X_{jj} + iH_j)\widetilde{F}_1 = 0, \quad (X_{kl}^\alpha + iY_{kl}^\alpha)\widetilde{F}_1 = 0 \tag{3.9}$$

for $j = 1,\ldots,n$ and $1 \le k < l \le n$, $\alpha = 1, i$. Moreover, $\mathcal{L}_j\widetilde{F}_1 = 0$ for $j = 1,\ldots,n$.

Let $\widehat{\varphi}_2(\lambda) = \widehat{\varphi}_1(-\lambda)$. Analogously, we define $\widetilde{F}_2 = \varphi_2 *_{\mathcal{V}} \widetilde{F}$. Then, of course, $\widetilde{F} = \widetilde{F}_1 + \widetilde{F}_2$, and we prove that $\widetilde{F}_2 \circ \theta^{-1}$ is antiholomorphic, therefore

$$(X_{jj} - iH_j)\widetilde{F_1} = 0, \quad (X_{kl}^\alpha - iY_{kl}^\alpha)\widetilde{F}_1 = 0$$

for $j = 1,\ldots,n$, $1 \le k < l \le n$, $\alpha = 1, i$. Moreover $\mathcal{L}_1\widetilde{F}_2 = 0$. Set $F_1 = \widetilde{F}_1 \circ \theta^{-1}$ and $F_2 = \widetilde{F}_2 \circ \theta^{-1}$. Then in view of (1.19) there are polynomials W_1, W_2 such that $F_1 - W_1 = H_1$ is holomorphic and $F_2 - W_2 = H_2$ is antiholomorphic. Then

$$F = F_1 + F_2 = H_1 + H_2 + W_1 + W_2 = H_1 + H_2 + W,$$

where W is a polynomial, which finishes the proof. ∎

The above lemma can be made considerably stronger.

Lemma 3.10. *Suppose that $\widetilde{F} \in C^\infty(\mathbf{N}(\Phi)\mathbf{S})$ satisfies* (3.5), (3.6), (3.7) *and is annihilated by $\mathbf{H}$ and $\mathcal{L}$. Then there is a polynomial W such that $F - W$ is pluriharmonic.*

Proof. As in the proof of Lemma 3.8 we disintegrate $\widetilde{F}$ as $\widetilde{F} = \widetilde{F}_1 + \widetilde{F}_2$, where

$$(X_j + iH_j)\widetilde{F}_1 = 0 \quad \text{and} \quad (X_{kl}^\alpha + iY_{kl}^\alpha)\widetilde{F}_1 = 0 \tag{3.11}$$

for $j = 1,\ldots,n$, $1 \le k < l \le n$ and $\alpha = 1, i$, and

$$\mathcal{L}\widetilde{F}_1 = 0. \tag{3.12}$$

We will show that (3.12) gives $\mathcal{L}_j\widetilde{F} = 0$ for every $j = 1,\ldots,n$. We use the left-invariant vector fields on $\mathbf{N}(\Phi)$, which are identified with $\partial_{\mathbf{x}_{jj}}$, $\partial_{\mathbf{x}_{jk}^\alpha}$, $\partial_{x_{kl}}$, $\partial_{y_{kl}}$ at e. For $1 \le j < k \le n$ and $l = 1,\ldots,m$ denote them by $\widetilde{X}_{jj}, \widetilde{X}_{jk}^\alpha, \widetilde{\mathcal{X}}_{kl}, \widetilde{\mathcal{Y}}_{kl}$. Notice that

$$X\widetilde{F}_1((\xi,x)s) = (\operatorname{Ad}_s \widetilde{X})\widetilde{F}_1^s(\xi,x), \tag{3.13}$$

where for a fixed s, $\widetilde{F}_1^s(\xi,x)=\widetilde{F}_1((\xi,x)s)$. If we write $s=wa$, then

$$\Delta_j\widetilde{F}_1((\xi,x)wa)=a_j^2((\mathrm{Ad}_w\,\widetilde{X}_j)^2+\partial_{a_j}^2)\widetilde{F}_1^s(\xi,x),$$

$$\mathcal{L}_{jk}\widetilde{F}_1((\xi,x)wa)=a_j((\mathrm{Ad}_w\,\widetilde{\mathcal{X}}_{jk})^2+(\mathrm{Ad}_w\,\widetilde{\mathcal{Y}}_{jk})^2-\partial_{a_j})\widetilde{F}_1^s(\xi,x).$$

From (3.11) and (3.13) we obtain

$$\partial_{a_j}\widetilde{F}_1((\xi,x)wa)=i\,\mathrm{Ad}_w(\widetilde{X}_{jj})\widetilde{F}_1^s(\xi,x).$$

Using the fact that $\mathcal{L}\widetilde{F}=0$ we have

$$\Big(\sum_{j=1}^n\gamma_j a_j\Big(\sum_{k=1}^m(\mathrm{Ad}_w(\widetilde{\mathcal{X}}_{jk}))^2+(\mathrm{Ad}_w(\widetilde{\mathcal{Y}}_{jk}))^2-\Im(\mathrm{Ad}_w(\widetilde{X}_{jj}))\Big)\Big)\widetilde{F}_1^s=0. \tag{3.14}$$

Now, we let a_j go to zero. To do this, we have to ensure some regularity of the boundary value f_1 of $\widetilde{F}_1^s$. Instead of $\widetilde{F}_1^s$ consider

$$\varrho*\widetilde{F}_1^s(\xi,x)=\int_{\mathbf{N}(\Phi)}\varrho((\xi,x)(\eta,u)^{-1})\widetilde{F}_1^s(\eta,u)\,d\eta\,du$$

for $\varrho\in C_c^\infty(\mathbf{N}(\Phi))$. Then

$$\Big(\sum_{j=1}^n\gamma_j a_j\Big(\sum_{k=1}^m(\mathrm{Ad}_w(\widetilde{\mathcal{X}}_{jk}))^2+(\mathrm{Ad}_w(\widetilde{\mathcal{Y}}_{jk}))^2-\Im(\mathrm{Ad}_w(\widetilde{X}_{jj}))\Big)\Big)(\varrho*\widetilde{F}_1^s)=0,$$

and the boundary value for $\varrho*\widetilde{F}_1^s$ is $\varrho*f_1$.

Fix w and j. Let $a_j=t$, and $a_k=t^2$ for $j\neq k$. If we divide (3.14) by t and let $t\to 0$, we obtain

$$\begin{aligned}&D_{j,w}(\varrho*f_1)\\&\quad=\Big(\sum_{k=1}^m((\mathrm{Ad}_w(\widetilde{\mathcal{X}}_{jk}))^2+(\mathrm{Ad}_w(\widetilde{\mathcal{Y}}_{jk}))^2)-\Im(\mathrm{Ad}_w(\widetilde{X}_{jj}))\Big)(\varrho*f_1)=0.\end{aligned} \tag{3.15}$$

We do this for every j. $D_{j,w}$ is a left-invariant operator on $\mathbf{N}(\Phi)$. We will show that (3.15) implies

$$D_{j,w}\widetilde{F}_1^s=0, \tag{3.16}$$

If $s:=wa$ in (3.15) then we get the assertion of Lemma 3.10.

Let

$$G((\xi,x)s)=\varrho*\widetilde{F}_1^s(\xi,x),\qquad g(\xi,x)=\varrho*f_1(\xi,x)=g_\xi(x).$$

Then $LG=0$ and

$$G^s(\xi,x)=G((\xi,x)s)=g_\xi*_{\mathcal{V}}P_L^s(x).$$

We only need to show the following property: *If D is a left-invariant operator defined on $\mathbf{N}(\Phi)$, then*

$$DG^s(\xi,x)=\int_{\mathcal{V}}(Dg)(\xi,x-v)P_L^s(v)\,dv=\int_{\mathcal{V}}(Dg)((0,-v)(\xi,x))P_L^s(v)\,dv.$$

This equality holds because we can change the order of integration and differentiation. This is allowed because

$$Dg(\xi,x) = D(\varrho * f_1)(\xi,x)$$

is dominated by $c(1+|\xi|)^N$ for some N. So if $D_{j,w}(\varrho * \widetilde{F}_1^s) = 0$ for every ϱ and s, then the equality (3.16) is proved. ∎

The class of operators which characterize the polynomially pluriharmonic functions can be extended even more. We replace the operator L by a more general one. Namely, let

$$L' = \sum_{j=1}^{n} d_j \varDelta_j + \sum_{k<l}^{n} c_{kl}^{\alpha} \varDelta_{kl}^{\alpha}, \tag{3.17}$$

where $d_j, c_{kl}^{\alpha} > 0$.

THEOREM 3.18. *Assume that $\widetilde{F}$ satisfies* (3.4) *and* $L'\widetilde{F} = \mathbf{H}\widetilde{F} = \mathcal{L}\widetilde{F} = 0$. *Then there is a well defined boundary value $f \in L^\infty(\mathbf{N}(\varPhi))$ of $\widetilde{F}$. Moreover, assume that f satisfies* (3.7). *Then there is a polynomial W such that $F-W$ is pluriharmonic.*

Proof. If $L'\widetilde{F} = 0$, $\mathbf{H}\widetilde{F} = 0$ and (3.4) holds, then the arguments of Section 3 of [BDH] show that $\widetilde{\varDelta}_j F = 0$ for every j, and adding to L' a suitable linear combination $\sum_{j=1}^n \eta_j \varDelta_j$, $\eta_j \geq 0$, we can get the operator $L = L' + \sum_{j=1}^n \eta_j \varDelta_j$ (see Proposition 3.5 in [BDH] and Lemma 2.1 in [DHMP]). Then, as before, there is $f \in L^\infty(\mathbf{N}(\varPhi))$ such that (3.6) holds for $f_\xi(x) = f(\xi,x)$. Now the theorem follows from Lemma 3.10. ∎

Our purpose is to remove the condition (3.7), i.e. we want to prove the following main theorem:

THEOREM 3.19. *Suppose that $\widetilde{F}$ satisfies* (3.4) *and* $L'\widetilde{F} = 0$, $\mathbf{H}\widetilde{F} = 0$ *and* $\mathcal{L}\widetilde{F} = 0$. *Then there is a polynomial W such that $F - W$ is pluriharmonic.*

Proof. Let φ be a Schwartz function on $\mathcal{V}$ with

$$\widehat{\varphi}(-\lambda) = \widehat{\varphi}(\lambda), \qquad \widehat{\varphi}(\lambda) = \begin{cases} 1, & |\lambda| \leq 1, \\ 0, & |\lambda| \geq 2. \end{cases}$$

Consider the family of functions $p^{-n}\varphi(x/p)$, $p \in \mathbf{N}$. For every bounded function g we may choose a subsequence p_k such that the limit

$$\lim_{p_k \to \infty} \int_{\mathcal{V}} p_k^{-n} \varphi(x/p_k) g(x)\, dx$$

exists. Let $\{\xi_j\}$ be a dense countable subset in $\mathcal{Z}$. By the diagonal method we choose p_k such that the limit

$$\lim_{p_k \to \infty} \int_{\mathcal{V}} p_k^{-n} \varphi(x/p_k) f_{\xi_j}(x)\, dx = H(\xi_j) \tag{3.20}$$

exists for every ξ_j. Define

$$\psi_k(x) = p_k^{-n}\varphi(x/p_k), \qquad \varphi_k(x) = p_k^n\varphi(p_k x),$$

and consider

$$\widetilde{F}_k^1((\xi,x)s) = \int_{\mathcal{V}} \varphi_k(v)\widetilde{F}((0,-v)(\xi,x)s)\,dv = \int_{\mathcal{V}} \varphi_k(v)\widetilde{F}((\xi,x-v)s)\,dv,$$
$$\widetilde{F}_k^2((\xi,x)s) = \int_{\mathcal{V}} \psi_k(v)\widetilde{F}((0,-v)(\xi,x)s)\,dv = \int_{\mathcal{V}} \psi_k(v)\widetilde{F}((\xi,x-v)s)\,dv.$$

The condition (3.4) guarantees that F_k^1 and F_k^2 are well defined and uniformly bounded on every compact set in $\mathbf{N}(\Phi)\mathbf{S}$. Moreover, for $j = 1,2$,

$$\mathbf{H}\widetilde{F}_k^j = 0, \qquad L'\widetilde{F}_k^j = 0, \qquad \mathcal{L}\widetilde{F}_k^j = 0$$

and $\lim_{k\to\infty}\widetilde{F}_k^1 = \widetilde{F}$ uniformly on compact sets in $\mathbf{N}(\Phi)\mathbf{S}$.

Notice that the boundary values for $\widetilde{F}_k^1$ and $\widetilde{F}_k^2$ are $\varphi_k * f_\xi$ and $\psi_k * f_\xi$, respectively, where $f_\xi(x) = f(\xi,x)$ is the boundary value for $\widetilde{F}$.

Since $\widehat{\varphi}(\lambda/p_k) - \widehat{\psi}(p_k\lambda) = 0$ if $|\lambda| \le 1/p_k$ or $|\lambda| \ge 2p_k$, we have

$$\widehat{\varphi_k * f_\xi} - \widehat{\psi_k * f_\xi} = 0$$

there. This means that $\widetilde{F}_k^1 - \widetilde{F}_k^2$ satisfies the assumptions of Theorem 3.18. So, we have a family of functions $F_k^1 - F_k^2$ and a family of polynomials W_k (see Theorem 2.19) with degrees bounded by the same constant M_1 such that $F_k^1 - F_k^2 - W_k$ are pluriharmonic functions. As usual $F_k^j = \widetilde{F}_k^j \circ \theta^{-1}$.

Let $w = (\xi,z)$ and for $k = nm + n^2$ let $w_1,\dots,w_k$ be coordinates in $\mathcal{Z}\oplus\mathcal{V}^C$. If $|\alpha| \ge M_1$, then for all k and j we have

$$\begin{aligned} 0 &= \partial_{\overline{w}_j}\partial_w^\alpha(F_k^1 - F_k^2 - W_k) = \partial_{\overline{w}_j}\partial_w^\alpha(F_k^1 - F_k^2),\\ 0 &= \partial_{w_j}\partial_{\overline{w}}^\alpha(F_k^1 - F_k^2 - W_k) = \partial_{w_j}\partial_{\overline{w}}^\alpha(F_k^1 - F_k^2). \end{aligned} \tag{3.21}$$

Now assume that

$$\lim_{k\to\infty}\widetilde{F}_k^2((\xi,xs)) = H(\xi)$$

exists and is a polynomial, hence $\lim_{k\to\infty}(\widetilde{F}_k^1 - \widetilde{F}_k^2) = \widetilde{F} - H$. Then we can take the limit of both sides of (3.21), and if $\alpha,\beta \ne 0$, and $|\alpha| + |\beta| > \max(M_1, \deg H)$, then

$$\partial_{w_j}^\alpha\partial_{\overline{w}_j}^\beta(F) = \partial_{w_j}^\alpha\partial_{\overline{w}_j}^\beta(F - H) = \lim_{k\to\infty}\partial_{w_j}^\alpha\partial_{\overline{w}_j}^\beta(F_k^1 - F_k^2) = 0.$$

This means that there exists a polynomial W such that $F - W$ is pluriharmonic. To finish the proof of Theorem 3.19 we have to show the following

LEMMA 3.22. *The limit*

$$\lim_{k\to\infty}\widetilde{F}_k^2((\xi,xs)) = H(\xi)$$

exists, is independent of x,s, and it is a polynomial in ξ, $\overline{\xi}$.

Proof. Changing variables $sy\overline{s}^t = u$, we write (3.6) as

$$\widetilde{F}((\xi,x)s) = \int_{\mathcal{V}} f_\xi(x-u)P_L^s(u)\,du.$$

Since φ_k is even, we have

$$\begin{aligned}\widetilde{F}_k^2((\xi_j,x)s) &= \int_{\mathcal{V}} \varphi_k(v)\int_{\mathcal{V}} f_{\xi_j}(x-v-y)P_L^s(y)\,dy\,dv\\ &= \int_{\mathcal{V}} \varphi_k(v)\int_{\mathcal{V}} f_{\xi_j}(y)P_L^s(x-v-y)\,dy\,dv\\ &= \int_{\mathcal{V}} \varphi_k(v)\int_{\mathcal{V}} f_{\xi_j}(y)P_L^s(x+v-y)\,dy\,dv.\end{aligned}$$

Let $\gamma(y) = P_L^s(x+y)$. Then

$$\widetilde{F}_k^2((\xi_j,x)s) = \int_{\mathcal{V}} \varphi_k(v) f_{\xi_j} * \gamma(v)\,dv.$$

From Lemma 4.3 of [BDH] and (3.20),

$$\lim_{k\to\infty} \widetilde{F}_k^2((\xi_j,x)s) = H(\xi_j)\int_{\mathcal{V}} \gamma(v)\,dv = H(\xi_j).$$

For every $(\xi,x)s$ we choose a sequence $\xi_j \to \xi$. Fix a bounded neighborhood U of $(\xi,x)s$ such that $(\xi_j,x)s \in U$. By (3.4) and the Harnack inequality the first derivatives of F_k^2 are bounded on U by the same constant, so there exists a constant C such that

$$|\widetilde{F}_k^2((\xi,x)s) - \widetilde{F}_k^2((\xi_j,x)s)| \le C|\xi-\xi_j|.$$

The density of ξ_j in $\mathcal{Z}$ implies that

$$\lim_{k\to\infty} \widetilde{F}_k^2((\xi,x)s) = H(\xi)$$

exists and is independent of x,s. Moreover, on compact sets, the functions $\widetilde{F}_k^2$ are bounded by the same constant, so the convergence is in the sense of distributions. Then we have

$$\mathcal{L}H = \lim_{k\to\infty} \mathcal{L}\widetilde{F}_k^2 = 0.$$

Because H is independent of x,s we have

$$\sum_{j=1}^n \gamma_j\Big(\sum_{k=1}^m \partial^2_{x_{jk}} + \partial^2_{y_{jk}}\Big)H(\xi) = 0,$$

which shows that H is a polynomial. ∎

Finally, the proof of Theorem 3.19 is complete. ∎

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ul. Zaciszna 7/16
55-200 Oława, Poland
E-mail: m.gilzynska@prawo.uni.wroc.pl

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