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ON THE k-CONVEXITY OF THE BESICOVITCH–ORLICZ SPACE OF ALMOST PERIODIC FUNCTIONS WITH THE ORLICZ NORM

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Abstract. Boulahia and the present authors introduced the Orlicz norm in the class B^{ϕ}-a.p. of Besicovitch–Orlicz almost periodic functions and gave several formulas for it; they also characterized the reflexivity of this space [Comment. Math. Univ. Carolin. 43 (2002)]. In the present paper, we consider the problem of k-convexity of B^{ϕ}-a.p. with respect to the Orlicz norm; we give necessary and sufficient conditions in terms of strict convexity and reflexivity.

1. Introduction and preliminaries

1.1. *Orlicz functions.* In the following, the notation ϕ is used for an *Orlicz function*, i.e. a function $\phi : \mathbb{R} \to \mathbb{R}$ which is even, convex, satisfies $\phi(u) = 0$ iff $u = 0$, and $\lim_{u\to\infty} \phi(u)/u = \infty$, $\lim_{u\to 0} \phi(u)/u = 0$.

This function is said to be of Δ_2-*type* when there exist constants $K > 2$ and $u_0 \geq 0$ such that

$$\phi(2u) \leq K\phi(u), \quad \forall u \geq u_0.$$

The function $\psi(y) = \sup\{x|y| - \phi(x) : x \geq 0\}$ is called *conjugate* to ϕ. It is an Orlicz function when ϕ is. The pair (ϕ, ψ) satisfies the *Young inequality*

$$xy \leq \phi(x) + \psi(y), \quad x \in \mathbb{R},\, y \in \mathbb{R}.$$

When both ϕ and ψ are of Δ_2-type we write $\phi \in \Delta_2 \cap \nabla_2$. Note that if ψ is of Δ_2-type then we have the following property (cf. [1]):

$$\forall \ell \in]0,1[,\ \forall u_0 \geq 0,\ \exists \beta = \beta(\ell) \in]0,1[, \quad \phi(\ell u) \leq \ell(1-\beta)\phi(u), \quad \forall u \geq u_0.$$

Let now ϕ be strictly convex. Then (cf. [1]) for every $k > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\phi\left(\frac{u+v}{2}\right) \leq (1-\delta)\left(\frac{\phi(u)+\phi(v)}{2}\right)$$

for all $u, v \in \mathbb{R}$ satisfying $|u|, |v| \leq k$ and $|u - v| \geq \varepsilon$.

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A normed space X is called *strictly convex* when

$$\forall x, y \in X, \quad \|x\| = \|y\| = 1,\ \|x - y\| > 0 \ \Rightarrow\ \|x + y\| < 2.$$

X is called *k-convex* for $k \in \mathbb{N}$, $k \geq 2$ when, for each $\{x_n\} \subset B(X)$ (the closed unit ball of X), the following implication holds:

$$(\|x_{n_1} + \dots + x_{n_k}\| \to k \text{ as } n_1, \dots, n_k \to \infty) \\ \Rightarrow \{x_n\} \text{ is a Cauchy sequence in norm.}$$

When $(X, \|\cdot\|)$ is a Banach space, the right hand side of this implication means that $\{x_n\}$ is norm convergent to some $x \in X$.

The k-convexity has been introduced for $k = 2$ in [2]. In [4], it is shown that k-convexity for $k = 2$ implies approximate compactness, which in turn guarantees the existence of the projection of any element onto any convex and closed subset of the space.

Moreover it is known that if X is k-convex then it is also $(k+1)$-convex, strictly convex and reflexive (cf. [1]). We can also easily see that uniform convexity implies k-convexity.

Let X be a real linear space. A functional $\varrho : X \to [0, \infty]$ is a (*pseudo*) *modular* if it satisfies

(i) $\varrho(x) = 0$ iff $x = 0$ for a modular, and
(i)′ $\varrho(0) = 0$ for a pseudomodular,
(ii) $\varrho(x) = \varrho(-x)$, $\forall x \in X$,
(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, $\forall \alpha, \beta \geq 0$, $\alpha + \beta = 1$, $x, y \in X$.

When, in place of (iii), we have

(iii)′ $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, $\forall \alpha, \beta \geq 0$, $\alpha + \beta = 1$, $x, y \in X$,

the (pseudo) modular ϱ is called *convex*.

The linear space $X_\varrho = \{x \in X : \lim_{\alpha\to 0} \varrho(\alpha x) = 0\}$ associated to the modular ϱ is called a *modular space*.

When ϱ is a convex (pseudo) modular, a (pseudo) norm is defined on X by the formula (cf. [10])

$$\|x\|_\varrho = \inf\{k > 0 : \varrho(x/k) \leq 1\}.$$

A sequence $\{x_n\} \subset X$ is called *modular convergent* to some $x \in X$ when $\lim_{n\to\infty} \varrho(x_n - x) = 0$. The definition of a modular Cauchy sequence is similar.

1.2. *The Besicovitch–Orlicz space of almost periodic functions.* Let $M(\mathbb{R})$ be the set of real Lebesgue measurable functions on $\mathbb{R}$. The functional

$$\varrho_{B^\phi} : M(\mathbb{R}) \to [0, \infty], \qquad \varrho_{B^\phi}(f) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} \phi(|f(t)|)\, dt,$$

is a convex pseudomodular (cf. [6]–[8]). The associated modular space

$$B^{\phi}(\mathbb{R}) = \{f \in M(\mathbb{R}) : \lim_{\alpha\to 0} \varrho_{B^{\phi}}(\alpha f) = 0\}$$
$$= \{f \in M(\mathbb{R}) : \varrho_{B^{\phi}}(\lambda f) < \infty \text{ for some } \lambda > 0\}$$

is called the *Besicovitch–Orlicz space.* This space is endowed with the *Luxemburg pseudonorm* (cf. [6]–[8])

$$\|f\|_{B^{\phi}} = \inf\{k > 0 : \varrho_{B^{\phi}}(f/k) \le 1\}, \qquad f \in B^{\phi}(\mathbb{R}).$$

Let now $\mathcal{A}$ be the set of generalized trigonometric polynomials, i.e.

$$\mathcal{A} = \Big\{P(t) = \sum_{j=1}^{n} \alpha_j \exp(i\lambda_j t) : \lambda_j \in \mathbb{R},\ \alpha_j \in \mathbb{C},\ n \in \mathbb{N}\Big\}.$$

The *Besicovitch–Orlicz space of almost periodic functions*, denoted B^{ϕ}-a.p., is the closure of $\mathcal{A}$ in $B^{\phi}(\mathbb{R})$ with respect to the pseudonorm $\|\cdot\|_{B^{\phi}}$:

$$B^{\phi}\text{-a.p.} = \{f \in B^{\phi}(\mathbb{R}) : \exists \{p_n\}_{n=1}^{\infty} \subset \mathcal{A},\ \lim_{n\to\infty} \|f - p_n\|_{B^{\phi}} = 0\}.$$

In the case $\phi(x) = |x|$, we use the notation B^1-a.p. Some structural and topological properties of this space are considered in [6]–[8].

Besides the Luxemburg norm, we may endow this space with the *Orlicz pseudonorm* (cf. [9])

$$|||f|||_{B^{\phi}} = \sup\{M(|fg|) : g \in B^{\psi}\text{-a.p.},\ \varrho_{B^{\psi}}(g) \le 1\}$$

where ψ denotes the conjugate function to ϕ and

$$M(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,d\mu \qquad \text{for } f \in B^1\text{-a.p.}$$

The Orlicz norm $|||\cdot|||_{B^{\phi}}$ satisfies (cf. [9])

$$|||f|||_{B^{\phi}} = \inf\Big\{\frac{1}{k}\,(1 + \varrho_{B^{\phi}}(kf)) : k > 0\Big\}.$$

More precisely,

$$|||f|||_{B^{\phi}} = \frac{1}{k}\,(1 + \varrho_{B^{\phi}}(kf)) \qquad \text{for some } k \in \,]0, \infty[, \tag{1.1}$$

which means that the set

$$K(f) = \Big\{k > 0 : |||f|||_{B^{\phi}} = \frac{1}{k}\,(1 + \varrho_{B^{\phi}}(kf))\Big\}$$

is not empty. Moreover, these two norms are equivalent (cf. [9]):

$$\|f\|_{B^{\phi}} \le |||f|||_{B^{\phi}} \le 2\|f\|_{B^{\phi}}.$$

Note also the important fact that when $f \in B^{\phi}$-a.p., the limit in the expression of $\varrho_{B^{\phi}}(f)$ exists (cf. [6]).

The following technical result is used in the proof of the necessity conditions of our main theorem.

Let $\{A_i\}_{i\geq 1} \subset \mathbb{R}$ be measurable subsets such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\bigcup_{i\geq 1} A_i \subset [0,\alpha], \alpha < 1$. Let $f = \sum_{i\geq 1} a_i \chi_{A_i}$ with $\sum_{i\geq 1} \phi(a_i)\mu(A_i) < \infty$ and let $\widetilde{f}$ be the periodic extension of f to the whole $\mathbb{R}$ (with period 1). Then there exists a sequence $\{P_m\}_{m\geq 1} \subset \mathcal{A}$ such that (cf. [6])

$$(1.2) \qquad \varrho_{B^\phi}\left(\frac{\widetilde{f} - P_m}{4}\right) \to 0 \quad \text{as } m \to \infty.$$

2. Results. We first give some convergence results which we will use extensively in different proofs.

Let $\Sigma = \Sigma(\mathbb{R})$ be the Σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. We define the set function

$$\overline{\mu}(A) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} \chi_A(t)\, dt = \overline{\lim_{T\to\infty}} \frac{1}{2T}\, \mu([-T,T] \cap A), \qquad A \in \Sigma,$$

where μ is the Lebesgue measure. Clearly, $\overline{\mu}$ is not σ-additive and $\overline{\mu}(A) = 0$ when $A \in \Sigma$ with $\mu(A) < \infty$. As usual, a sequence $\{f_k\}_{k\geq 1}$ of Σ-measurable functions will be called *$\overline{\mu}$-convergent* to a measurable function f when, for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \overline{\mu}\{t \in \mathbb{R} : |f_k(t) - f(t)| \geq \varepsilon\} = 0.$$

Similarly, we define a $\overline{\mu}$-Cauchy sequence.

LEMMA 1 ([6]–[8]). *Let $\{f_n\}_{n\geq 1} \subset B^\phi(\mathbb{R})$. Then:*

1. *If $\{f_n\}_{n\geq 1}$ is modular convergent to some $f \in B^\phi(\mathbb{R})$ then it is also $\overline{\mu}$-convergent to f.*
2. *If $\{f_n\}_{n\geq 1}$ is $\overline{\mu}$-convergent to some $f \in B^\phi(\mathbb{R})$ and there exists $g \in B^\phi$-a.p. such that $\max(|f_k(x)|, |f(x)|) \leq g(x)$ for all $x \in \mathbb{R}$, then $\lim_{n\to\infty} \varrho_{B^\phi}(f_n) = \varrho_{B^\phi}(f)$.*

LEMMA 2. *Let $\{f_n\}, \{g_n\} \subset B^\phi$-a.p. with $\|\|f_n\|\|_{B^\phi} = 1$, $\|\|g_n\|\|_{B^\phi} = 1$ and $\lim_{n,m\to\infty} \|\|f_n + g_m\|\|_{B^\phi} = 2$. Let $\{k_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ be sequences of scalars such that the norms of f_n and g_n are attained in formula* (1.1) *at the points k_n and h_n respectively. If ϕ is strictly convex and $b = \sup_n\{k_n, h_n\}$ is finite, then $k_n f_n - h_m g_m \to 0$ in $\overline{\mu}$.*

Proof. Indeed, in the opposite case, we may assume that $\overline{\mu}(E_{n,m}) > \theta$ where $E_{n,m} = \{t \in \mathbb{R} : |k_n f_n(t) - h_m g_m(t)| \geq r\}$ and r, θ are some fixed positive numbers.

From easy computations we can show the following:

$$\forall \varepsilon > 0,\ \exists \sigma > 0,\ \forall A \in \Sigma, \quad \overline{\mu}(A) \geq \varepsilon \ \Rightarrow\ \|\chi_A\|_{B^\phi} > \sigma.$$

Let now $k > 1$ be such that $\overline{\mu}(A) \geq \theta/4 \Rightarrow \|\chi_A\|_{B^\phi} \geq 1/k$ and define

$$A_n = \{t \in \mathbb{R} : |f_n(t)| \geq k\}, \qquad B_n = \{t \in \mathbb{R} : |g_n(t)| \geq k\}.$$

We have

$$1 = |||f_n|||_{B^\phi} \geq \|f_n\|_{B^\phi} \geq \|f_n \chi_{A_n}\|_{B^\phi} \geq k\|\chi_{A_n}\|_{B^\phi},$$

i.e. $\|\chi_{A_n}\|_{B^\phi} \leq 1/k$ and so $\overline{\mu}(A_n) \leq \theta/4$. By similar computations we also get $\overline{\mu}(B_n) \leq \theta/4$.

From the strict convexity of ϕ, there exists $\delta > 0$ such that

$$\phi(ru + (1-r)v) \leq (1-\delta)[r\phi(u) + (1-r)\phi(v)]$$

for each $r \in [1/(1+b), b/(b+1)]$ and $|u|, |v| \leq bk$, $|u - v| \geq r$ (see [1]).

Since $k_n/(k_n + h_m)$ and $h_m/(k_n + h_m)$ are in $[1/(1+b), b/(b+1)]$, for $t \in E_{n,m} \setminus (A_n \cup B_m)$ we have

$$\phi\left(\frac{k_n h_m}{k_n + h_m}\,(f_n(t) + g_m(t))\right) \leq (1-\delta)\left[\frac{h_m}{k_n + h_m}\,\phi(k_n f_n(t)) + \frac{k_n}{k_n + h_m}\,\phi(h_m g_m(t))\right]. \tag{2.1}$$

Then using (1.1) it follows that

$$\begin{aligned}
2 - |||f_n + g_m|||_{B^\phi}
&\geq \frac{1}{k_n}\,(1 + \varrho_{B^\phi}(k_n f_n)) + \frac{1}{h_m}\,(1 + \varrho_{B^\phi}(h_m g_m)) \\
&\quad - \frac{k_n + h_m}{k_n h_m}\left(1 + \varrho_{B^\phi}\left(\frac{k_n h_m}{k_n + h_m}\,(f_n(t) + g_m(t))\right)\right) \\
&\geq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{k_n + h_m}{k_n h_m}\left[\frac{h_m}{k_n + h_m}\,\phi(k_n f_n(t)) + \frac{k_n}{k_n + h_m}\,\phi(h_m g_m(t))\right. \\
&\quad \left. - \phi\left(\frac{k_n h_m}{k_n + h_m}\,(f_n(t) + g_m(t))\right)\right] dt \\
&\geq \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{(E_{n,m}\setminus(A_n\cup B_m))\cap[-T,T]} \left[\frac{\delta}{k_n}\,\phi(k_n f_n(t)) + \frac{\delta}{h_m}\,\phi(h_m g_m(t))\right] dt \\
&\geq \frac{2\delta}{b}\,\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{(E_{n,m}\setminus(A_n\cup B_m))\cap[-T,T]} \left[\phi\left(\frac{|k_n f_n(t) - h_n g_n(t)|}{2}\right)\right] dt
\end{aligned}$$

$$\geq \frac{2\delta}{b}\,\phi\left(\frac{r}{2}\right)\overline{\mu}(E_{n,m}\setminus(A_n\cup B_m)) \geq \frac{\delta}{b}\,\phi\left(\frac{r}{2}\right)(\overline{\mu}(E_{n,m})-\overline{\mu}(A_n)-\overline{\mu}(B_m))$$

$$\geq \frac{2\delta}{b}\,\phi\left(\frac{r}{2}\right)\frac{\theta}{2} \geq \frac{\delta}{b}\,\phi\left(\frac{r}{2}\right)\theta.$$

This contradicts the assumption that $\|\!|f_n+g_n|\!\|_{B^\phi}\to 2$. ∎

Lemma 3. *Let $f\in B^\phi$-a.p. and $E\in\Sigma$. Then the function*

$$F:]0,\infty[\to\mathbb{R},\qquad F(\lambda)=\varrho_{B^\phi}(f\chi_E/\lambda),$$

is continuous on $]0,\infty[$.

Proof. Let $\lambda_0\in\ ]0,\infty[$ and $\{\lambda_n\}$ be a sequence of scalars such that $\lim_{n\to\infty}\lambda_n=\lambda_0$. We have

$$\varrho_{B^\phi}\left[\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_0}\right)f\chi_E\right] \leq \left|\frac{1}{\lambda_n}-\frac{1}{\lambda_0}\right|\varrho_{B^\phi}(f\chi_E)\to 0 \quad \text{as } n\to\infty,$$

so $\{(1/\lambda_n)f\chi_E\}$ is modular convergent to $(1/\lambda_0)f\chi_E$. Moreover, we have

$$\max\left(\frac{1}{|\lambda_n|}|f|\chi_E,\frac{1}{|\lambda_0|}|f|\chi_E\right)\leq M|f|\in B^\phi a.p.$$

for some constant M. Now, using Lemma 1, we get

$$\lim_{n\to\infty}\varrho_{B^\phi}\left(\frac{f\chi_E}{\lambda_n}\right)=\varrho_{B^\phi}\left(\frac{f\chi_E}{\lambda_0}\right),$$

which means that F is continuous at λ_0. ∎

Remark 1. We already know that (cf. [6])

$$\varrho_{B^\phi}(f)\leq 1 \;\Leftrightarrow\; \|f\|_{B^\phi}\leq 1 \quad \text{for any } f\in B^\phi\text{-a.p.}$$

From Lemma 3 it follows that also

$$\varrho_{B^\phi}(f\chi_E)\leq 1 \;\Leftrightarrow\; \|f\chi_E\|_{B^\phi}\leq 1 \quad \text{for any } f\in B^\phi\text{-a.p. and } E\in\Sigma.$$

Remark 2. In the same way, we know from [6] that

$$\forall\varepsilon>0,\exists\delta>0,\forall f\in B^\phi\text{-a.p.},\quad \varrho_{B^\phi}(f)\leq\delta \;\Rightarrow\; \|f\|_{B^\phi}\leq\varepsilon.$$

From Lemma 3 it follows that the same holds for $f\chi_E$ instead of f.

Lemma 4. *Assume $\phi\in\Delta_2$. Then for all $L>0$ and $\varepsilon>0$ there exists a $\delta>0$ such that if $f,g\in B^\phi$-a.p. and $E\in\Sigma$, then*

$$\varrho_{B^\phi}(f\chi_E)\leq L,\ \varrho_{B^\phi}(g\chi_E)\leq\delta \;\Rightarrow\; |\varrho_{B^\phi}((f+g)\chi_E)-\varrho_{B^\phi}(f\chi_E)|<\varepsilon.$$

Proof. Using Lemma 3, the arguments are the same as those for the Orlicz space case (see [1, Lemma 1.40]), so we omit the proof. ∎

Lemma 5.

(1) *If ϕ is of Δ_2-type, then*

$$\inf\{k\in K(f): \|\!|f|\!\|_{B^\phi}=1,\ f\in B^\phi\text{-}a.p.\}=d>1.$$

(2) *If the conjugate* ψ *to* ϕ *is of* Δ_2*-type, then, for each* $a, b > 0$, *the set* $Q = \{K(f) : a \le \|\!|f|\!\|_{B^\phi} \le b,\ f \in B^\phi\text{-}a.p.\}$ *is bounded.*

Proof. The arguments are exactly the same as those used in the Orlicz space case (see [1]), so we omit the proof. ∎

LEMMA 6. *Suppose* $\phi \in \Delta_2 \cap \nabla_2$ *and let* $\{f_n\}, \{g_n\} \subset B^\phi$*-a.p. be such that* $\|\!|f_n|\!\|_{B^\phi}, \|\!|g_n|\!\|_{B^\phi} \le 1$, $n = 1, 2, \ldots$, *and* $\lim_{n,m\to\infty} \|\!|f_n + g_m|\!\|_{B^\phi} = 2$. *Then for every* $\varepsilon \in (0, 1)$ *there are* $\delta > 0$ *and* $n_0 \in \mathbb{N}$ *such that for all* $n, m \ge n_0$ *and all* $E \in \Sigma$ *we have* $\varrho_{B^\phi}(g_m\chi_E) \le \delta \Rightarrow \varrho_{B^\phi}(f_n\chi_E) \le \varepsilon$.

Proof. Let $u' > 0$ be such that $\phi(u') < \varepsilon/2$, and put $E_n = \{t \in \mathbb{R} : |f_n(t)| < u'\}$. Then

$$\varrho_{B^\phi}(f_n\chi_{E\cap E_n}) \le \phi(u')\overline{\mu}(E \cap E_n) \le \varepsilon$$

for any $E \in \Sigma$. Hence we may assume that $|f_n(t)| \ge u'$ for all $t \in \mathbb{R}$.

Let $k_n \in K(f_n)$ and $h_n \in K(g_n)$. Then

$$\frac{h_n}{k_n + h_n} \in \left[\frac{1}{1+b}, \frac{b}{1+b}\right] \subset \,]0, 1[,$$

where $b = \sup_n\{k_n, h_n\} < \infty$. We may suppose that $\inf_n\{k_n, h_n\} \ge a > 0$.

Since $\phi \in \nabla_2$ there exists $\beta > 0$ such that (cf. [1])

$$\phi\left(\frac{bu}{1+b}\right) \le \frac{b(1-\beta)}{1+b}\,\phi(u), \quad \forall |u| \ge u', \tag{2.2}$$

and using the fact that the function $\ell \mapsto \phi(\ell u)/\ell u$ is increasing, we obtain

$$\phi(\ell u) \le \ell(1-\beta)\phi(u), \quad \forall \ell \in \left[\frac{1}{1+b}, \frac{b}{1+b}\right], \forall |u| \ge u'.$$

Given any $\alpha > 0$, from Lemma 4, there exists $\delta' > 0$ such that

$$\varrho_{B^\phi}(f) \le 1,\ \varrho_{B^\phi}(g) \le \delta' \ \Rightarrow\ |\varrho_{B^\phi}(f + g) - \varrho_{B^\phi}(f)| < \alpha. \tag{2.3}$$

Since ϕ is of Δ_2-type, we may choose $\delta > 0$ such that $\varrho_{B^\phi}(g) \le \delta \Rightarrow \varrho_{B^\phi}\big(\frac{b^2}{2a}g\big) \le \delta'$ and hence

$$\varrho_{B^\phi}(g_m\chi_E) \le \delta \ \Rightarrow\ \varrho_{B^\phi}\left(\frac{k_n h_n}{k_n + h_n}\, g_m\chi_E\right) \le \varrho_{B^\phi}\left(\frac{b^2}{2a}\, g_m\chi_E\right) \le \delta'.$$

Now, from (2.3), we get

$$\begin{aligned}\varrho_{B^\phi}\left(\frac{k_n h_m}{k_n + h_m}\,(f_n + g_m)\chi_E\right) &\le \varrho_{B^\phi}\left(\frac{k_n h_m}{k_n + h_m}\, f_n\chi_E\right) + \alpha \\ &\le \frac{h_m}{k_n + h_m}\,(1-\beta)\varrho_{B^\phi}(k_n f_n\chi_E) + \alpha.\end{aligned}$$

Take an integer n' such that

$$n, m \ge n' \ \Rightarrow\ 2 - \|\!|f_n + g_m|\!\|_{B^\phi} < \alpha.$$

Using the convexity of ϕ, for $n, m \geq n'$ we have

$$\begin{aligned}
&\alpha \geq 2 - \|f_n + g_m\|_{B^\phi} \\
&\geq \frac{1}{k_n}\,\varrho_{B^\phi}(k_n f_n) + \frac{1}{h_m}\,\varrho_{B^\phi}(h_m g_m) - \frac{k_n + h_m}{k_n h_m}\,\varrho_{B^\phi}\left(\frac{k_n h_m}{k_n + h_m}\,(f_n + g_m)\right) \\
&\geq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\frac{1}{k_n}\,\phi(k_n f_n) + \frac{1}{h_m}\,\phi(h_m g_m) \right.\\
&\qquad\qquad\qquad\qquad \left. - \frac{k_n + h_m}{k_n h_m}\,\phi\left(\frac{k_n h_m}{k_n + h_m}\,(f_n + g_m)\right)\right] d\mu \\
&\geq \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{E\cap[-T,T]} \left[\frac{1}{k_n}\,\phi(k_n f_n) + \frac{1}{h_m}\,\phi(h_m g_m) \right.\\
&\qquad\qquad\qquad\qquad \left. - \frac{k_n + h_m}{k_n h_m}\,\phi\left(\frac{k_n h_m}{k_n + h_m}\,(f_n + g_m)\right)\right] d\mu \\
&\geq \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{E\cap[-T,T]} \left[\frac{1}{k_n}\,\phi(k_n f_n) + \frac{1}{h_m}\,\phi(h_m g_m)\right] d\mu \\
&\quad - \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{E\cap[-T,T]} \frac{k_n + h_m}{k_n h_m}\,\phi\left(\frac{k_n h_m}{k_n + h_m}\,(f_n + g_m)\right) d\mu \\
&\geq \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{E\cap[-T,T]} \left[\frac{1}{k_n}\,\phi(k_n f_n) + \frac{1}{h_m}\,\phi(h_m g_m)\right] d\mu \\
&\quad - \frac{1}{k_n}\,(1-\beta)\,\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{E\cap[-T,T]} \phi(k_n f_n)\, d\mu - \frac{k_n + h_m}{k_n h_m}\,\alpha \\
&\geq \frac{\beta}{k_n}\,\varrho_{B^\phi}(k_n f_n \chi_E) - \frac{2b}{a^2}\,\alpha \geq \frac{\beta}{b}\,\varrho_{B^\phi}(a f_n \chi_E) - \frac{2b}{a^2}\,\alpha.
\end{aligned}$$

Now, since $\alpha > 0$ is arbitrary and ϕ is of Δ_2-type, we get the desired result. ∎

LEMMA 7. *Let $\{f_n\}_n \subset B^\phi$-a.p. be such that* $\sup_n \varrho_{B^\phi}(f_n) < \infty$. *Then for every $\theta > 0$ there exists $A > 0$ such that* $\sup_n \overline{\mu}(\{t \in \mathbb{R} : |f_n(t)| \geq A\}) < \theta$.

Proof. In fact, in the opposite case we have

$$\lim_{N\to\infty} \sup_n \overline{\mu}(\{t \in \mathbb{R} : |f_n(t)| \geq N\}) \neq 0 \tag{2.4}$$

(note that the sequence is decreasing, so its limit exists). Putting $E_{n,N} = \{t \in \mathbb{R} : |f_n(t)| \geq N\}$, we then get

$$\varrho_{B^\phi}(f_n) \geq \varrho_{B^\phi}(f_n \chi_{E_{n,N}}) \geq N\overline{\mu}(E_{n,N}),$$

and taking the supremum over n gives

$$\sup_n \varrho_{B^\phi}(f_n) \geq \sup_n \varrho_{B^\phi}(f_n \chi_{E_{n,N}}) \geq \sup_n N\overline{\mu}(E_{n,N}) = N \sup_n \overline{\mu}(E_{n,N}). \tag{2.5}$$

Finally, letting $N \to \infty$ in (2.5) and using again (2.4), we obtain $\sup_n \varrho_{B^\phi}(f_n) = \infty$. This contradicts the assumption. ■

Lemma 8. *Let $\{f_n\}_n$ be a sequence in B^ϕ-a.p. satisfying the $\overline{\mu}$-Cauchy condition and modular equicontinuous, i.e. for every $\varepsilon > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\overline{\mu}(E) < \delta \;\Rightarrow\; \varrho_{B^\phi}(f_n \chi_E) \le \varepsilon,\ \forall n \ge n_0.$$

If $\sup_n \varrho_{B^\phi}(f_n) < \infty$, then $\{\varrho_{B^\phi}(f_n)\}_{n\ge 1}$ is a Cauchy sequence in $\mathbb{R}$.

Proof. First, we show the assertion for $\phi(u) = |u|$. Set $E_{n,m} = \{t \in \mathbb{R} : |f_n(t) - f_m(t)| > \varepsilon/2\}$. The sequence $\{f_n\}$ being equicontinuous, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have

$$\overline{\mu}(E) < \delta \;\Rightarrow\; \varrho_{B^1}(f_n \chi_E) \le \varepsilon/4.$$

Since $\{f_n\}$ is a $\overline{\mu}$-Cauchy sequence, there exists $n_1 \in \mathbb{N}$ such that $\overline{\mu}(E_{n,m}) < \delta$ for $n, m \ge n_1$. Taking $n, m \ge \max(n_0, n_1)$ we get

$$\begin{aligned}
|\varrho_{B^1}(f_n) - \varrho_{B^1}(f_m)| &= \left| \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f_n(t)|\, d\mu - \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f_m(t)|\, d\mu \right| \\
&\le \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f_n(t) - f_m(t)|\, d\mu \\
&\le \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{[-T,T]\cap E_{n,m}} |f_n(t) - f_m(t)|\, d\mu \\
&\quad + \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{[-T,T]\cap E^{\mathrm{c}}_{n,m}} |f_n(t) - f_m(t)|\, d\mu \\
&\le \varrho_{B^1}(f_n \chi_{E_{n,m}}) + \varrho_{B^1}(f_m \chi_{E_{n,m}}) + \frac{\varepsilon}{2}\, \overline{\mu}(E^{\mathrm{c}}_{n,m}) \\
&\le \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Now, for an arbitrary Orlicz function ϕ, it is sufficient to show that $(\phi(f_n))_n$ is a $\overline{\mu}$-Cauchy sequence; the result follows then from the case $\phi(x) = |x|$.

By Lemma 7, we know that if $\sup_n \varrho_{B^\phi}(f_n) < \infty$ then for every $\theta > 0$, there exists $M > 0$ such that $\overline{\mu}(\{t \in \mathbb{R} : |f_n(t)| \ge M\}) < \theta$ for all n.

Put $G_n = \{t \in \mathbb{R} : |f_n(t)| \le M\}$ and let $\varepsilon > 0$. Since ϕ is uniformly continuous on $[-M, M]$, there exists $\eta > 0$ such that

$$|\phi(t_1) - \phi(t_2)| \ge \varepsilon \;\Rightarrow\; |t_1 - t_2| > \eta.$$

Now since for all $t \in G_n \cap G_m$, we have $f_n(t), f_m(t) \in [-M, M]$, it follows that

$$|\phi(f_n(t)) - \phi(f_m(t))| \ge \varepsilon \;\Rightarrow\; |f_n(t) - f_m(t)| > \eta,$$

whence, for any $\varepsilon, \theta > 0$,

$$\begin{aligned}\overline{\mu}\{t \in \mathbb{R} : |\phi(f_n(t)) - \phi(f_m(t))| \geq \varepsilon\} \\ &\leq \overline{\mu}\{t \in G_n \cap G_m : |\phi(f_n(t)) - \phi(f_m(t))| \geq \varepsilon\} \\ &\quad + \overline{\mu}\{t \in (G_n \cap G_m)^{\mathrm{c}} : |\phi(f_n(t)) - \phi(f_m(t))| \geq \varepsilon\} \\ &\leq \overline{\mu}\{t \in G_n \cap G_m : |f_n(t) - f_m(t)| \geq \eta\} + 2\theta.\end{aligned}$$

Letting $n, m \to \infty$, we get

$$\forall\varepsilon > 0,\ \forall\theta > 0, \quad \overline{\mu}\{t \in \mathbb{R} : |\phi(f_n(t)) - \phi(f_m(t))| \geq \varepsilon\} \leq 2\theta.$$

Finally, since θ is arbitrary, we get the desired result. ∎

LEMMA 9. *Let $\{f_n\} \subset B^{\phi}$-a.p. be a $\overline{\mu}$-Cauchy sequence equicontinuous in norm. Then $\{f_n\}$ is a modular Cauchy sequence. In particular, if $\phi \in \Delta_2$, the sequence $\{f_n\}$ is norm convergent to some $f \in B^{\phi}$-a.p.*

Proof. Set $E_{n,m} = \{t \in \mathbb{R} : |f_n(t) - f_m(t)| > \varepsilon/2\}$. The sequence $\{f_n\}$ being equicontinuous in norm, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\overline{\mu}(E) < \delta \ \Rightarrow\ \varrho_{B^{\phi}}(2f_n\chi_E) \leq \varepsilon/2.$$

Since $\{f_n\}$ satisfies the $\overline{\mu}$-Cauchy condition, there exists $n_1 \in \mathbb{N}^*$ such that $n, m \geq n_1 \Rightarrow \overline{\mu}(E_{n,m}) < \delta$. Taking $n, m \geq \max(n_0, n_1)$ we get

$$\begin{aligned}\varrho_{B^{\phi}}(f_n - f_m) &\leq \varrho_{B^{\phi}}((f_n - f_m)\chi_{E_{n,m}}) + \varrho_{B^{\phi}}((f_n - f_m)\chi_{(E_{n,m})^{\mathrm{c}}}) \\ &\leq \frac{1}{2}\left[\varrho_{B^{\phi}}(2f_n\chi_{E_{n,m}}) + \varrho_{B^{\phi}}(2f_m\chi_{E_{n,m}})\right] + \frac{\varepsilon}{2}\,\overline{\mu}((E_{n,m})^{\mathrm{c}}) \\ &\leq \frac{1}{2}\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = \varepsilon. \ ∎\end{aligned}$$

LEMMA 10. *Let $f \in E^{\phi}([0,1])$, where $E^{\phi}([0,1])$ is the Orlicz class*

$$E^{\phi}([0,1]) = \{f \text{ measurable} : \varrho_{\phi}(\lambda f) < \infty,\ \forall\lambda > 0\},$$

and let ϱ_{ϕ} be the usual Orlicz modular. Then:

(1) *If $\widetilde{f}$ is the 1-periodic extension of f to the whole $\mathbb{R}$, then $\widetilde{f} \in B^{\phi}$-a.p.*
(2) *The injection $i : E^{\phi}([0,1]) \to B^{\phi}$-a.p., $i(f) = \widetilde{f}$, is an isometry with respect to the modular and for the respective Orlicz norms.*

Proof. (1) Let $f = \sum_{i=1}^{n} a_i\chi_{A_i}$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{n} A_i \subset [0,\alpha]$, $0 < \alpha < 1$. Let $m \in \mathbb{N}$. Since $\sum_{i=1}^{n} \phi(ma_i)\mu(A_i) < \infty$, it follows from (1.2) that there exists $P_m \in \mathcal{P}$ (the set of generalized trigonometric polynomials) for which

$$\varrho_{B^{\phi}}\left(\frac{m}{4}\,(\widetilde{f} - P_m)\right) \leq \frac{1}{m},$$

where $\widetilde{f}$ is the 1-periodic extension of f.

Let $\lambda > 0$ and $m_0 \in \mathbb{N}$ be such that $\lambda \le m_0/4$. Then

$$\varrho_{B^\phi}(\lambda(\widetilde{f} - P_m)) \le \varrho_{B^\phi}\left(\frac{m}{4}(\widetilde{f} - P_m)\right) \le \frac{1}{m}, \quad \forall m \ge m_0.$$

This means that $\lim_{m\to\infty} \|\widetilde{f} - P_m\|_{B^\phi} = 0$, i.e. $\widetilde{f} \in B^\phi$-a.p.

Consider now the general case of $f \in E^\phi([0,1])$. It is known (see [1]) that the step functions are dense in $E^\phi([0,1])$, hence given $\varepsilon > 0$, there is a $g_\varepsilon = \sum_{i=1}^n a_i \chi_{A_i}$ for which $\|g_\varepsilon - f\|_\phi \le \varepsilon/4$. Since f is absolutely continuous, we may choose $\delta > 0$ such that $\mu(A) \le \delta \Rightarrow \|f\chi_A\|_\phi \le \varepsilon/4$. We take $\alpha > 0$ with $1 - \alpha \le \delta$ and put $A_i^\alpha = A_i \cap [0,\alpha]$, $i = 1, n$. Then the function $g_\varepsilon^\alpha = \sum_{i=1}^n a_i \chi_{A_i^\alpha}$ belongs to $E^\phi([0,1])$. If $\widetilde{f}$ and $\widetilde{g}_\varepsilon^\alpha$ are the respective 1-periodic extensions, then

$$\begin{aligned}\|\widetilde{f} - \widetilde{g}_\varepsilon^\alpha\|_{B^\phi} &= \|f - g_\varepsilon^\alpha\|_\phi \le \|(f - g_\varepsilon^\alpha)\chi_{[0,\alpha]}\|_\phi + \|(f - g_\varepsilon^\alpha)\chi_{[\alpha,1]}\|_\phi \\ &\le \|f - g_\varepsilon\|_\phi + \|f\chi_{[\alpha,1]}\|_\phi \le \varepsilon/4 + \varepsilon/4 = \varepsilon/2.\end{aligned}$$

Now, since $\widetilde{g}_\varepsilon^\alpha \in B^\phi$-a.p., there exists $P_\varepsilon \in \mathcal{P}$ for which $\|\widetilde{g}_\varepsilon^\alpha - P_\varepsilon\|_{B^\phi} \le \varepsilon/2$. Finally,

$$\|\widetilde{f} - P_\varepsilon\|_{B^\phi} \le \|\widetilde{f} - \widetilde{g}_\varepsilon^\alpha\|_{B^\phi} + \|\widetilde{g}_\varepsilon^\alpha - P_\varepsilon\|_{B^\phi} \le \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

i.e. $\widetilde{f} \in B^\phi$-a.p.

(2) It is clear that $i : E^\phi([0,1]) \to B^\phi$-a.p. is a modular isometry. The fact that it is also an isometry for the Orlicz norms follows immediately since

$$|\!|\!| f |\!|\!|_\phi = \inf_{k>0}\left\{\frac{1}{k}(1 + \varrho_\phi(kf))\right\} = \inf_{k>0}\left\{\frac{1}{k}(1 + \varrho_{B^\phi}(k\widetilde{f}\,))\right\} = |\!|\!| \widetilde{f}\, |\!|\!|_{B^\phi}. \ \blacksquare$$

We can now state our main result.

THEOREM 1. *The space $(B^\phi$-a.p.$, \|\cdot\|_{B^\phi})$ is k-convex iff $\phi \in \Delta_2 \cap \nabla_2$ and ϕ is strictly convex.*

Proof. Necessity. As known for general Banach spaces, k-convexity implies strict convexity and reflexivity. From [9], reflexivity of B^ϕ-a.p. implies that $\phi \in \Delta_2 \cap \nabla_2$. It remains to show that ϕ is strictly convex. Indeed, strict convexity of ϕ is necessary for strict convexity of the Orlicz class $E^\phi([0,1])$ (cf. [1]) and using Proposition 10, we deduce that it is also necessary for strict convexity of B^ϕ-a.p.

For the sufficiency, let $\{f_n\} \subset B^\phi$-a.p. with $|\!|\!| f_n |\!|\!|_{B^\phi} = 1$ and $|\!|\!| f_n + f_m |\!|\!|_{B^\phi} \to 2$ as $n, m \to \infty$. Given any $\varepsilon > 0$, take n_0 and δ as in Lemma 6. Since $f_{n_0} \in B^\phi$-a.p. there is a $\delta' > 0$ such that $\overline{\mu}(E) < \delta' \Rightarrow \varrho_{B^\phi}(f_{n_0}\chi_E) \le \delta$ and then by Lemma 6 we obtain $\varrho_{B^\phi}(f_m \chi_E) \le \varepsilon$ for all $m \ge n_0$.

On the other hand, since $|\!|\!| f_n + f_m |\!|\!|_{B^\phi} \to 2$ as $n, m \to \infty$, from Lemma 2 it follows that $\{k_n f_n\}$ is a $\overline{\mu}$-Cauchy sequence. Now, we will show that it is also modular equicontinuous.

Given any $\varepsilon > 0$, from Remark 2 there is $\delta > 0$ such that $\varrho_{B^\phi}(f_n \chi_E) \le \delta \Rightarrow \|k_n f_n \chi_E\|_{B^\phi} \le \varepsilon$ and then from the arguments presented above we also have the implication $\overline{\mu}(E) < \delta' \Rightarrow \|k_n f_n \chi_E\|_{B^\phi} \le \varepsilon, \forall n \ge n_0$ for some δ'. This means that the sequence $\{k_n f_n\}_n$ is norm equicontinuous.

Moreover, from Lemma 8, $\{\varrho_{B^\phi}(k_n f_n)\}_{n\ge 1}$ is a Cauchy sequence in $\mathbb{R}$, whence it converges to some $l \in \mathbb{R}$.

Now, using (1.1), we may write $|||f_n|||_{B^\phi} = (1/k_n)(1 + \varrho_{B^\phi}(k_n f_n))$ and letting $n \to \infty$ we get $\lim_{n\to\infty} k_n = 1 + l$.

Finally, from Lemma 9, the sequence $(k_n f_n)_n$ is modular Cauchy and again by the Δ_2-condition it is a norm Cauchy sequence, i.e. it converges in norm to some $g \in B^\phi$-a.p.

Consequently, $\{f_n\}$ is norm convergent to $g/(1+l)$. ■

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