ON THE k-CONVEXITY OF THE BESICOVITCH–ORLICZ SPACE OF ALMOST PERIODIC FUNCTIONS WITH THE ORLICZ NORM

BY

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Abstract. Boulahia and the present authors introduced the Orlicz norm in the class $B^\phi$-a.p. of Besicovitch–Orlicz almost periodic functions and gave several formulas for it; they also characterized the reflexivity of this space [Comment. Math. Univ. Carolin. 43 (2002)]. In the present paper, we consider the problem of $k$-convexity of $B^\phi$-a.p. with respect to the Orlicz norm; we give necessary and sufficient conditions in terms of strict convexity and reflexivity.

1. Introduction and preliminaries

1.1. Orlicz functions. In the following, the notation $\phi$ is used for an Orlicz function, i.e. a function $\phi : \mathbb{R} \to \mathbb{R}$ which is even, convex, satisfies $\phi(u) = 0$ iff $u = 0$, and $\lim_{u \to \infty} \phi(u)/u = \infty$, $\lim_{u \to 0} \phi(u)/u = 0$.

This function is said to be of $\Delta_2$-type when there exist constants $K > 2$ and $u_0 \geq 0$ such that

$$\phi(2u) \leq K \phi(u), \quad \forall u \geq u_0.$$  

The function $\psi(y) = \sup\{x | y| - \phi(x) : x \geq 0\}$ is called conjugate to $\phi$. It is an Orlicz function when $\phi$ is. The pair $(\phi, \psi)$ satisfies the Young inequality

$$xy \leq \phi(x) + \psi(y), \quad x \in \mathbb{R}, \ y \in \mathbb{R}.$$  

When both $\phi$ and $\psi$ are of $\Delta_2$-type we write $\phi \in \Delta_2 \cap \nabla_2$. Note that if $\psi$ is of $\Delta_2$-type then we have the following property (cf. [1]):

$$\forall \ell \in ]0, 1[, \forall u_0 \geq 0, \exists \beta = \beta(\ell) \in ]0, 1[, \ \phi(\ell u) \leq \ell (1 - \beta) \phi(u), \quad \forall u \geq u_0.$$  

Let now $\phi$ be strictly convex. Then (cf. [1]) for every $k > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\phi\left(\frac{u + v}{2}\right) \leq (1 - \delta)\left(\phi(u) + \phi(v)\right)$$  

for all $u, v \in \mathbb{R}$ satisfying $|u|, |v| \leq k$ and $|u - v| \geq \varepsilon$.

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A normed space $X$ is called \textit{strictly convex} when
\[ \forall x, y \in X, \quad \|x\| = \|y\| = 1, \quad \|x - y\| > 0 \quad \Rightarrow \quad \|x + y\| < 2. \]

$X$ is called \textit{$k$-convex} for $k \in \mathbb{N}$, $k \geq 2$ when, for each $\{x_n\} \subset B(X)$ (the closed unit ball of $X$), the following implication holds:
\[ (\|x_{n1} + \cdots + x_{nk}\| \rightarrow k \text{ as } n_1, \ldots, n_k \rightarrow \infty) \quad \Rightarrow \quad \{x_n\} \text{ is a Cauchy sequence in norm}. \]

When $(X, \| \cdot \|)$ is a Banach space, the right hand side of this implication means that $\{x_n\}$ is norm convergent to some $x \in X$.

The $k$-convexity has been introduced for $k = 2$ in [2]. In [4], it is shown that $k$-convexity for $k = 2$ implies approximate compactness, which in turn guarantees the existence of the projection of any element onto any convex and closed subset of the space.

Moreover it is known that if $X$ is $k$-convex then it is also $(k+1)$-convex, strictly convex and reflexive (cf. [1]). We can also easily see that uniform convexity implies $k$-convexity.

Let $X$ be a real linear space. A functional $\varphi : X \rightarrow [0, \infty]$ is a \textit{(pseudo) modular} if it satisfies
\begin{enumerate}[\textup{(i)}]
\item $\varphi(x) = 0$ iff $x = 0$ for a modular, and
\item $\varphi(0) = 0$ for a pseudomodular,
\item $\varphi(x) = \varphi(-x)$, $\forall x \in X$,
\item $\varphi(\alpha x + \beta y) \leq \varphi(x) + \varphi(y)$, $\forall \alpha, \beta \geq 0$, $\alpha + \beta = 1$, $x, y \in X$.
\end{enumerate}

When, in place of (iii), we have
\[ (\textup{iii})' \quad \varphi(\alpha x + \beta y) \leq \alpha \varphi(x) + \beta \varphi(y), \forall \alpha, \beta \geq 0, \alpha + \beta = 1, x, y \in X, \]
the (pseudo) modular $\varphi$ is called \textit{convex}.

The linear space $X_\varphi = \{x \in X : \lim_{\alpha \rightarrow 0} \varphi(\alpha x) = 0\}$ associated to the modular $\varphi$ is called a \textit{modular space}.

When $\varphi$ is a convex (pseudo) modular, a (pseudo) norm is defined on $X$ by the formula (cf. [10])
\[ \|x\|_\varphi = \inf\{k > 0 : \varphi(x/k) \leq 1\}. \]

A sequence $\{x_n\} \subset X$ is called \textit{modular convergent} to some $x \in X$ when $\lim_{n \rightarrow \infty} \varphi(x_n - x) = 0$. The definition of a modular Cauchy sequence is similar.

\subsection*{1.2. The Besicovitch–Orlicz space of almost periodic functions.}

Let $M(\mathbb{R})$ be the set of real Lebesgue measurable functions on $\mathbb{R}$. The functional
\[ \varphi_{B\phi} : M(\mathbb{R}) \rightarrow [0, \infty], \quad \varphi_{B\phi}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{T} \phi(|f(t)|) \, dt, \]

is a convex pseudomodular (cf. [6]–[8]). The associated modular space
\[ B^\phi(\mathbb{R}) = \{ f \in M(\mathbb{R}) : \lim_{\alpha \to 0} \varrho_{B^\phi}(\alpha f) = 0 \} = \{ f \in M(\mathbb{R}) : \varrho_{B^\phi}(\lambda f) < \infty \text{ for some } \lambda > 0 \} \]
is called the \textit{Besicovitch–Orlicz space}. This space is endowed with the Luxemburg pseudonorm (cf. [6]–[8])
\[ \| f \|_{B^\phi} = \inf \{ k > 0 : \varrho_{B^\phi}(f/k) \leq 1 \}, \quad f \in B^\phi(\mathbb{R}). \]

Let now \( \mathcal{A} \) be the set of generalized trigonometric polynomials, i.e.
\[ \mathcal{A} = \left\{ P(t) = \sum_{j=1}^{n} \alpha_j \exp(i\lambda_j t) : \lambda_j \in \mathbb{R}, \alpha_j \in \mathbb{C}, n \in \mathbb{N} \right\}. \]
The \textit{Besicovitch–Orlicz space of almost periodic functions}, denoted \( B^\phi \text{-a.p.} \), is the closure of \( \mathcal{A} \) in \( B^\phi(\mathbb{R}) \) with respect to the pseudonorm \( \| \cdot \|_{B^\phi} \):
\[ B^\phi \text{-a.p.} = \{ f \in B^\phi(\mathbb{R}) : \exists \{ p_n \}_{n=1}^{\infty} \subset \mathcal{A}, \lim_{n \to \infty} \| f - p_n \|_{B^\phi} = 0 \}. \]
In the case \( \phi(x) = |x| \), we use the notation \( B^1 \text{-a.p.} \). Some structural and topological properties of this space are considered in [6]–[8].

Besides the Luxemburg norm, we may endow this space with the \textit{Orlicz pseudonorm} (cf. [9])
\[ \| f \|_{B^\psi} = \sup \{ M(|fg|) : g \in B^\psi \text{-a.p.}, \varrho_{B^\phi}(g) \leq 1 \} \]
where \( \psi \) denotes the conjugate function to \( \phi \) and
\[ M(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) d\mu \quad \text{for } f \in B^1 \text{-a.p.} \]
The Orlicz norm \( \| \cdot \|_{B^\psi} \) satisfies (cf. [9])
\[ \| f \|_{B^\phi} = \inf \left\{ \frac{1}{k} (1 + \varrho_{B^\phi}(kf)) : k > 0 \right\}. \]
More precisely,
\[ \| f \|_{B^\phi} = \frac{1}{k} (1 + \varrho_{B^\phi}(kf)) \quad \text{for some } k \in ]0, \infty[, \]
which means that the set
\[ K(f) = \left\{ k > 0 : \| f \|_{B^\phi} = \frac{1}{k} (1 + \varrho_{B^\phi}(kf)) \right\} \]
is not empty. Moreover, these two norms are equivalent (cf. [9]):
\[ \| f \|_{B^\phi} \leq \| f \|_{B^\psi} \leq 2 \| f \|_{B^\phi}. \]

Note also the important fact that when \( f \in B^\phi \text{-a.p.} \), the limit in the expression of \( \varrho_{B^\phi}(f) \) exists (cf. [6]).
The following technical result is used in the proof of the necessity conditions of our main theorem.

Let \( \{A_i\}_{i \geq 1} \subset \mathbb{R} \) be measurable subsets such that \( A_i \cap A_j = \emptyset \) if \( i \neq j \) and \( \bigcup_{i \geq 1} A_i \subset [0, \alpha], \alpha < 1 \). Let \( f = \sum_{i \geq 1} a_i \chi_{A_i} \), with \( \sum_{i \geq 1} \phi(a_i)\mu(A_i) < \infty \) and let \( \tilde{f} \) be the periodic extension of \( f \) to the whole \( \mathbb{R} \) (with period 1). Then there exists a sequence \( \{P_m\}_{m \geq 1} \subset A \) such that (cf. [6])

\[
\varrho_{B^\phi} \left( \frac{\tilde{f} - P_m}{4} \right) \to 0 \quad \text{as} \quad m \to \infty.
\]

\( \tag{1.2} \)

2. Results. We first give some convergence results which we will use extensively in different proofs.

Let \( \Sigma = \Sigma(\mathbb{R}) \) be the \( \Sigma \)-algebra of Lebesgue measurable subsets of \( \mathbb{R} \). We define the set function

\[
\overline{\mu}(A) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \chi_A(t) \, dt = \lim_{T \to \infty} \frac{1}{2T} \mu([-T, T] \cap A), \quad A \in \Sigma,
\]

where \( \mu \) is the Lebesgue measure. Clearly, \( \overline{\mu} \) is not \( \sigma \)-additive and \( \overline{\mu}(A) = 0 \) when \( A \in \Sigma \) with \( \mu(A) < \infty \). As usual, a sequence \( \{f_k\}_{k \geq 1} \) of \( \Sigma \)-measurable functions will be called \( \overline{\mu} \)-convergent to a measurable function \( f \) when, for all \( \varepsilon > 0 \),

\[
\lim_{k \to \infty} \overline{\mu}\{t \in \mathbb{R} : |f_k(t) - f(t)| \geq \varepsilon\} = 0.
\]

Similarly, we define a \( \overline{\mu} \)-Cauchy sequence.

**Lemma 1** ([6]–[8]). Let \( \{f_n\}_{n \geq 1} \subset B^\phi(\mathbb{R}) \). Then:

1. If \( \{f_n\}_{n \geq 1} \) is modular convergent to some \( f \in B^\phi(\mathbb{R}) \) then it is also \( \overline{\mu} \)-convergent to \( f \).
2. If \( \{f_n\}_{n \geq 1} \) is \( \overline{\mu} \)-convergent to some \( f \in B^\phi(\mathbb{R}) \) and there exists \( g \in B^\phi \)-a.p. such that \( \max(|f_k(x)|, |f(x)|) \leq g(x) \) for all \( x \in \mathbb{R} \), then \( \lim_{n \to \infty} \varrho_{B^\phi}(f_n) = \varrho_{B^\phi}(f) \).

**Lemma 2.** Let \( \{f_n\}, \{g_n\} \subset B^\phi \)-a.p. with \( \|f_n\|_{B^\phi} = 1, \|g_n\|_{B^\phi} = 1 \) and \( \lim_{n,m \to \infty} \|f_n + g_n\|_{B^\phi} = 2 \). Let \( \{k_n\}_{n \geq 1} \) and \( \{h_n\}_{n \geq 1} \) be sequences of scalars such that the norms of \( f_n \) and \( g_n \) are attained in formula (1.1) at the points \( k_n \) and \( h_n \) respectively. If \( \phi \) is strictly convex and \( b = \sup_n \{k_n, h_n\} \) is finite, then \( k_nf_n - h_ng_m \to 0 \) in \( \overline{\mu} \).

**Proof.** Indeed, in the opposite case, we may assume that \( \overline{\mu}(E_{n,m}) > \theta \) where \( E_{n,m} = \{t \in \mathbb{R} : |k_nf_n(t) - h_ng_m(t)| \geq r\} \) and \( r, \theta \) are some fixed positive numbers.
From easy computations we can show the following:
\[ \forall \varepsilon > 0, \ \exists \sigma > 0, \ \forall A \in \Sigma, \ \overline{m}(A) \geq \varepsilon \Rightarrow \|X_A\|_{B^\phi} > \sigma. \]
Let now \( k > 1 \) be such that \( \overline{m}(A) \geq \theta/4 \Rightarrow \|X_A\|_{B^\phi} \geq 1/k \) and define
\[ A_n = \{ t \in \mathbb{R} : |f_n(t)| \geq k \}, \quad B_n = \{ t \in \mathbb{R} : |g_n(t)| \geq k \}. \]
We have
\[ 1 = \|f_n\|_{B^\phi} \geq \|f_n\|_{B^\phi} \geq \|f_nX_{A_n}\|_{B^\phi} \geq k\|X_{A_n}\|_{B^\phi}, \]
i.e. \( \|X_{A_n}\|_{B^\phi} \leq 1/k \) and so \( \overline{m}(A_n) \leq \theta/4 \). By similar computations we also get \( \overline{m}(B_n) \leq \theta/4 \).

From the strict convexity of \( \phi \), there exists \( \delta > 0 \) such that
\[ \phi(ru + (1 - r)v) \leq (1 - \delta)[r\phi(u) + (1 - r)\phi(v)] \]
for each \( r \in [1/(1 + b), b/(b + 1)] \) and \( |u|, |v| \leq bk, |u - v| \geq r \) (see [1]).
Since \( k_n/(k_n + h_m) \) and \( h_m/(k_n + h_m) \) are in \( [1/(1 + b), b/(b + 1)] \), for \( t \in E_{n,m} \setminus (A_n \cup B_m) \) we have
\[ (2.1) \quad \phi \left( \frac{k_nh_m}{k_n + h_m} (f_n(t) + g_m(t)) \right) \]
\[ \leq (1 - \delta) \left[ \frac{h_m}{k_n + h_m} \phi(k_nf_n(t)) + \frac{k_n}{k_n + h_m} \phi(h_ng_m(t)) \right]. \]
Then using (1.1) it follows that
\[ 2 - \|f_n + g_m\|_{B^\phi} \]
\[ \geq \frac{1}{k_n} (1 + \varepsilon_{B^\phi}(k_nf_n)) + \frac{1}{h_m} (1 + \varepsilon_{B^\phi}(h_ng_m)) \]
\[ - \frac{k_n + h_m}{k_nh_m} \left( 1 + \varepsilon_{B^\phi} \left( \frac{k_nh_m}{k_n + h_m} (f_n(t) + g_m(t)) \right) \right) \]
\[ \geq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[ \frac{h_m}{k_n + h_m} \phi(k_nf_n(t)) + \frac{k_n}{k_n + h_m} \phi(h_ng_m(t)) \right] \]
\[ - \phi \left( \frac{k_nh_m}{k_n + h_m} (f_n(t) + g_m(t)) \right) \] \[ dt \]
\[ \geq \lim_{T \to \infty} \frac{1}{2T} \int_{(E_{n,m} \setminus (A_n \cup B_m)) \cap [-T,T]} \left[ \frac{\delta}{k_n} \phi(k_nf_n(t)) + \frac{\delta}{h_m} \phi(h_ng_m(t)) \right] \]
\[ \geq \frac{2\delta}{b} \lim_{T \to \infty} \frac{1}{2T} \int_{(E_{n,m} \setminus (A_n \cup B_m)) \cap [-T,T]} \phi \left( \frac{|k_nf_n(t) - h_ng_n(t)|}{2} \right) \] \[ dt \]
\[
\geq \frac{2\delta}{b} \phi \left( \frac{r}{2} \right) \overline{\mu}(E_{n,m} \setminus (A_n \cup B_m)) \geq \frac{\delta}{b} \phi \left( \frac{r}{2} \right) (\overline{\mu}(E_{n,m}) - \overline{\mu}(A_n) - \overline{\mu}(B_m)) \\
\geq \frac{2\delta}{b} \phi \left( \frac{r}{2} \right) \theta \geq \frac{\delta}{b} \phi \left( \frac{r}{2} \right) \theta.
\]

This contradicts the assumption that \[\| f_n + g_n \|_{B^\phi} \to 2.\]

**Lemma 3.** Let \( f \in B^\phi\)-a.p. and \( E \in \Sigma \). Then the function \[ F : [0, \infty[ \to \mathbb{R}, \quad F(\lambda) = \varrho_{B^\phi}(f \chi_E/\lambda), \]

is continuous on \([0, \infty[.\]

**Proof.** Let \( \lambda_0 \in [0, \infty[ \) and \( \{\lambda_n\} \) be a sequence of scalars such that \( \lim_{n \to \infty} \lambda_n = \lambda_0 \). We have
\[
\varrho_{B^\phi}\left( \frac{1}{\lambda_n} - \frac{1}{\lambda_0} \right) f \chi_E \leq \left| \frac{1}{\lambda_n} - \frac{1}{\lambda_0} \right| \varrho_{B^\phi}(f \chi_E) \to 0 \quad \text{as } n \to \infty,
\]

so \( \{(1/\lambda_n)f \chi_E\} \) is modular convergent to \( (1/\lambda_0)f \chi_E \). Moreover, we have
\[
\max\left( \frac{1}{|\lambda_n|} |f| \chi_E, \frac{1}{|\lambda_0|} |f| \chi_E \right) \leq M|f| \in B^\phi\text{-a.p.}
\]

for some constant \( M \). Now, using Lemma 1, we get
\[
\lim_{n \to \infty} \varrho_{B^\phi}\left( \frac{f \chi_E}{\lambda_n} \right) = \varrho_{B^\phi}\left( \frac{f \chi_E}{\lambda_0} \right),
\]

which means that \( F \) is continuous at \( \lambda_0 \). \( \blacksquare \)

**Remark 1.** We already know that (cf. [6])
\[
\varrho_{B^\phi}(f) \leq 1 \iff \|f\|_{B^\phi} \leq 1 \quad \text{for any } f \in B^\phi\text{-a.p.}
\]

From Lemma 3 it follows that also
\[
\varrho_{B^\phi}(f \chi_E) \leq 1 \iff \|f \chi_E\|_{B^\phi} \leq 1 \quad \text{for any } f \in B^\phi\text{-a.p. and } E \in \Sigma.
\]

**Remark 2.** In the same way, we know from [6] that
\[
\forall \varepsilon > 0, \exists \delta > 0, \forall f \in B^\phi\text{-a.p., } \varrho_{B^\phi}(f) \leq \delta \Rightarrow \|f\|_{B^\phi} \leq \varepsilon.
\]

From Lemma 3 it follows that the same holds for \( f \chi_E \) instead of \( f \).

**Lemma 4.** Assume \( \phi \in \Delta_2 \). Then for all \( L > 0 \) and \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( f, g \in B^\phi\)-a.p. and \( E \in \Sigma \), then
\[
\varrho_{B^\phi}(f \chi_E) \leq L, \quad \varrho_{B^\phi}(g \chi_E) \leq \delta \Rightarrow |\varrho_{B^\phi}((f + g) \chi_E) - \varrho_{B^\phi}(f \chi_E)| < \varepsilon.
\]

**Proof.** Using Lemma 3, the arguments are the same as those for the Orlicz space case (see [1, Lemma 1.40]), so we omit the proof. \( \blacksquare \)

**Lemma 5.**

(1) If \( \phi \) is of \( \Delta_2\)-type, then
\[
\inf\{k \in K(f) : \|f\|_{B^\phi} = 1, f \in B^\phi\text{-a.p.}\} = d > 1.
\]
(2) If the conjugate $\psi$ to $\phi$ is of $\Delta_2$-type, then, for each $a, b > 0$, the set 
$Q = \{K(f) : a \leq \|f\|_{B^\phi} \leq b, f \in B^\phi$-a.p.$ \}$ is bounded.

Proof. The arguments are exactly the same as those used in the Orlicz space case (see [1]), so we omit the proof. 

Lemma 6. Suppose $\phi \in \Delta_2 \cap \nabla_2$ and let $\{f_n\}, \{g_n\} \subset B^\phi$-a.p. be such 
that $\|f_n\|_{B^\phi}, \|g_n\|_{B^\phi} \leq 1$, $n = 1, 2, \ldots$, and $\lim_{n,m \to \infty} \|f_n + g_m\|_{B^\phi} = 2$. 
Then for every $\varepsilon \in (0,1)$ there are $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all 
$n, m \geq n_0$ and all $E \in \Sigma$ we have $\varrho_{B^\phi}(g_m\chi_E) \leq \delta \Rightarrow \varrho_{B^\phi}(f_n\chi_E) \leq \varepsilon$.

Proof. Let $u' > 0$ be such that $\phi(u') < \varepsilon/2$, and put $E_n = \{t \in \mathbb{R} : |f_n(t)| < u'\}$. Then

$$
\varrho_{B^\phi}(f_n\chi_{E_n \cap E_n}) \leq \phi(u')\overline{\mu}(E \cap E_n) \leq \varepsilon
$$

for any $E \in \Sigma$. Hence we may assume that $|f_n(t)| \geq u'$ for all $t \in \mathbb{R}$.

Let $k_n \in K(f_n)$ and $h_n \in K(g_n)$. Then

$$
\frac{h_n}{k_n + h_n} \in \left[ \frac{1}{1 + b'}, \frac{b}{1 + b} \right] \subset \{0, 1[,
$$

where $b = \sup_n \{k_n, h_n\} < \infty$. We may suppose that $\inf_n \{k_n, h_n\} \geq a > 0$.

Since $\phi \in \nabla_2$ there exists $\beta > 0$ such that (cf. [1])

$$
(2.2) \quad \phi \left( \frac{bu}{1 + b} \right) \leq \frac{b(1 - \beta)}{1 + b} \phi(u), \quad \forall |u| \geq u',
$$

and using the fact that the function $\ell \mapsto \phi(\ell u)/\ell u$ is increasing, we obtain

$$
\phi(\ell u) \leq \ell (1 - \beta) \phi(u), \quad \forall \ell \in \left[ \frac{1}{1 + b'}, \frac{b}{1 + b} \right], \forall |u| \geq u'.
$$

Given any $\alpha > 0$, from Lemma 4, there exists $\delta' > 0$ such that

$$
(2.3) \quad \varrho_{B^\phi}(f) \leq 1, \varrho_{B^\phi}(g) \leq \delta' \Rightarrow |\varrho_{B^\phi}(f + g) - \varrho_{B^\phi}(f)| < \alpha.
$$

Since $\phi$ is of $\Delta_2$-type, we may choose $\delta > 0$ such that $\varrho_{B^\phi}(g) \leq \delta \Rightarrow \varrho_{B^\phi}(\frac{b^2}{2a}g) \leq \delta'$ and hence

$$
\varrho_{B^\phi}(g_m\chi_E) \leq \delta \Rightarrow \varrho_{B^\phi}\left( \frac{k_n h_n}{k_n + h_n} g_m\chi_E \right) \leq \varrho_{B^\phi}\left( \frac{b^2}{2a} g_m\chi_E \right) \leq \delta'.
$$

Now, from (2.3), we get

$$
\varrho_{B^\phi}\left( \frac{k_n h_m}{k_n + h_m} (f_n + g_m)\chi_E \right) \leq \varrho_{B^\phi}\left( \frac{k_n h_m}{k_n + h_m} f_n\chi_E \right) + \alpha
\leq \frac{h_m}{k_n + h_m} (1 - \beta) \varrho_{B^\phi}(k_n f_n\chi_E) + \alpha.
$$

Take an integer $n'$ such that

$$
n, m \geq n' \Rightarrow 2 - \|f_n + g_m\|_{B^\phi} < \alpha.
$$
Using the convexity of \( \phi \), for \( n, m \geq n' \) we have

\[
\alpha \geq 2 - \| f_n + g_m \|_{B^\phi} \\
\geq \frac{1}{k_n} \varrho_{B^\phi}(k_n f_n) + \frac{1}{h_m} \varrho_{B^\phi}(h_m g_m) - \frac{k_n + h_m}{k_n h_m} \varrho_{B^\phi}(\frac{k_n h_m}{k_n + h_m} (f_n + g_m)) \\
\geq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[ \frac{1}{k_n} \phi(k_n f_n) + \frac{1}{h_m} \phi(h_m g_m) - \frac{k_n + h_m}{k_n h_m} \phi(\frac{k_n h_m}{k_n + h_m} (f_n + g_m)) \right] d\mu \\
\geq \lim_{T \to \infty} \frac{1}{2T} \int_{E \cap [-T, T]} \left[ \frac{1}{k_n} \phi(k_n f_n) + \frac{1}{h_m} \phi(h_m g_m) - \frac{k_n + h_m}{k_n h_m} \phi(\frac{k_n h_m}{k_n + h_m} (f_n + g_m)) \right] d\mu \\
- \lim_{T \to \infty} \frac{1}{2T} \int_{E \cap [-T, T]} \frac{k_n + h_m}{k_n h_m} \phi(\frac{k_n h_m}{k_n + h_m} (f_n + g_m)) d\mu \\
\geq \lim_{T \to \infty} \frac{1}{2T} \int_{E \cap [-T, T]} \left[ \frac{1}{k_n} \phi(k_n f_n) + \frac{1}{h_m} \phi(h_m g_m) \right] d\mu \\
- \frac{1}{k_n} (1 - \beta) \lim_{T \to \infty} \frac{1}{2T} \int_{E \cap [-T, T]} \phi(k_n f_n) d\mu - \frac{k_n + h_m}{k_n h_m} \alpha \\
\geq \frac{\beta}{k_n} \varrho_{B^\phi}(k_n f_n \chi_E) - \frac{2b}{\alpha^2} \alpha \geq \frac{\beta}{b} \varrho_{B^\phi}(a f_n \chi_E) - \frac{2b}{\alpha^2} \alpha.
\]

Now, since \( \alpha > 0 \) is arbitrary and \( \phi \) is of \( \Delta_2 \)-type, we get the desired result. \( \blacksquare \)

**Lemma 7.** Let \( \{f_n\}_n \subset B^\phi \)-a.p. be such that \( \sup_n \varrho_{B^\phi}(f_n) < \infty \). Then for every \( \theta > 0 \) there exists \( A > 0 \) such that \( \sup_n \overline{\mu}(\{ t \in \mathbb{R} : |f_n(t)| \geq A \}) < \theta \).

**Proof.** In fact, in the opposite case we have

\[
(2.4) \quad \lim_{N \to \infty} \sup_n \overline{\mu}(\{ t \in \mathbb{R} : |f_n(t)| \geq N \}) \neq 0
\]

(note that the sequence is decreasing, so its limit exists). Putting \( E_{n,N} = \{ t \in \mathbb{R} : |f_n(t)| \geq N \} \), we then get

\[
\varrho_{B^\phi}(f_n) \geq \varrho_{B^\phi}(f_n \chi_{E_{n,N}}) \geq N \overline{\mu}(E_{n,N}),
\]

and taking the supremum over \( n \) gives

\[
(2.5) \quad \sup_n \varrho_{B^\phi}(f_n) \geq \sup_n \varrho_{B^\phi}(f_n \chi_{E_{n,N}}) \geq \sup_n N \overline{\mu}(E_{n,N}) = N \sup_n \overline{\mu}(E_{n,N}).
\]
Finally, letting $N \to \infty$ in (2.5) and using again (2.4), we obtain $\sup_n \varphi_{B^\phi}(f_n) = \infty$. This contradicts the assumption.}\]

**Lemma 8.** Let $\{f_n\}_n$ be a sequence in $B^\phi$-a.p. satisfying the $\overline{\mu}$-Cauchy condition and modular equicontinuous, i.e. for every $\varepsilon > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\overline{\mu}(E) < \delta \Rightarrow \varphi_{B^\phi}(f_n \chi_E) \leq \varepsilon, \forall n \geq n_0.$$  

If $\sup_n \varphi_{B^\phi}(f_n) < \infty$, then $\{\varphi_{B^\phi}(f_n)\}_{n \geq 1}$ is a Cauchy sequence in $\mathbb{R}$.

**Proof.** First, we show the assertion for $\phi(u) = |u|$. Set $E_{n,m} = \{t \in \mathbb{R} : |f_n(t) - f_m(t)| > \varepsilon/2\}$. The sequence $\{f_n\}$ being equicontinuous, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\overline{\mu}(E) < \delta \Rightarrow \varphi_{B^1}(f_n \chi_E) \leq \varepsilon/4.$$  

Since $\{f_n\}$ is a $\overline{\mu}$-Cauchy sequence, there exists $n_1 \in \mathbb{N}$ such that $\overline{\mu}(E_{n,m}) < \delta$ for $n, m \geq n_1$. Taking $n, m \geq \max(n_0, n_1)$ we get

$$|\varphi_{B^1}(f_n) - \varphi_{B^1}(f_m)| = \left| \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f_n(t)| d\mu - \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f_m(t)| d\mu \right|$$

$$\leq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f_n(t) - f_m(t)| d\mu$$

$$\leq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f_n(t) - f_m(t)| d\mu$$

$$+ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f_n(t) - f_m(t)| d\mu$$

$$\leq \varphi_{B^1}(f_n \chi_{E_{n,m}}) + \varphi_{B^1}(f_m \chi_{E_{n,m}}) + \frac{\varepsilon}{2} \overline{\mu}(E_{n,m}^c)$$

$$\leq \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon.$$  

Now, for an arbitrary Orlicz function $\phi$, it is sufficient to show that $(\phi(f_n))_n$ is a $\overline{\mu}$-Cauchy sequence; the result follows then from the case $\phi(x) = |x|$.

By Lemma 7, we know that if $\sup_n \varphi_{B^\phi}(f_n) < \infty$ then for every $\theta > 0$, there exists $M > 0$ such that $\overline{\mu}(\{t \in \mathbb{R} : |f_n(t)| \geq M\}) < \theta$ for all $n$.

Put $G_n = \{t \in \mathbb{R} : |f_n(t)| \leq M\}$ and let $\varepsilon > 0$. Since $\phi$ is uniformly continuous on $[-M, M]$, there exists $\eta > 0$ such that

$$|\phi(t_1) - \phi(t_2)| \geq \varepsilon \Rightarrow |t_1 - t_2| > \eta.$$  

Now since for all $t \in G_n \cap G_m$, we have $f_n(t), f_m(t) \in [-M, M]$, it follows that

$$|\phi(f_n(t)) - \phi(f_m(t))| \geq \varepsilon \Rightarrow |f_n(t) - f_m(t)| > \eta,$$
whence, for any $\varepsilon, \theta > 0$,
\[
\bar{\mu}\{t \in \mathbb{R} : |\phi(f_n(t)) - \phi(f_m(t))| \geq \varepsilon\} \\
\leq \bar{\mu}\{t \in G_n \cap G_m : |\phi(f_n(t)) - \phi(f_m(t))| \geq \varepsilon\} \\
+ \bar{\mu}\{t \in (G_n \cap G_m)^c : |\phi(f_n(t)) - \phi(f_m(t))| \geq \varepsilon\} \\
\leq \bar{\mu}\{t \in G_n \cap G_m : |f_n(t) - f_m(t)| \geq \eta\} + 2\theta.
\]

Letting $n, m \to \infty$, we get
\[
\forall \varepsilon > 0, \forall \theta > 0, \quad \bar{\mu}\{t \in \mathbb{R} : |\phi(f_n(t)) - \phi(f_m(t))| \geq \varepsilon\} \leq 2\theta.
\]
Finally, since $\theta$ is arbitrary, we get the desired result. ■

**Lemma 9.** Let $\{f_n\} \subset B^\phi$-a.p. be a $\bar{\mu}$-Cauchy sequence equicontinuous in norm. Then $\{f_n\}$ is a modular Cauchy sequence. In particular, if $\phi \in \Delta_2$, the sequence $\{f_n\}$ is norm convergent to some $f \in B^\phi$-a.p.

**Proof.** Set $E_{n,m} = \{t \in \mathbb{R} : |f_n(t) - f_m(t)| > \varepsilon/2\}$. The sequence $\{f_n\}$ being equicontinuous in norm, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have
\[
\bar{\mu}(E) < \delta \implies \varrho_B(2f_n|E) \leq \varepsilon/2.
\]
Since $\{f_n\}$ satisfies the $\bar{\mu}$-Cauchy condition, there exists $n_1 \in \mathbb{N}^*$ such that
\[
n, m \geq n_1 \implies \bar{\mu}(E_{n,m}) < \delta.
\]
Taking $n, m \geq \max(n_0, n_1)$ we get
\[
\varrho_B(f_n - f_m) \leq \varrho_B((f_n - f_m)\chi_{E_{n,m}}) + \varrho_B((f_n - f_m)\chi_{(E_{n,m})^c}) \\
\leq \frac{1}{2}[\varrho_B(2f_n\chi_{E_{n,m}}) + \varrho_B(2f_m\chi_{E_{n,m}})] + \frac{\varepsilon}{2}\bar{\mu}((E_{n,m})^c) \\
\leq \frac{1}{2}\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = \varepsilon.
\]

**Lemma 10.** Let $f \in E^\phi([0, 1])$, where $E^\phi([0, 1])$ is the Orlicz class
\[
E^\phi([0, 1]) = \{f \text{ measurable : } \varrho_{\phi}(\lambda f) < \infty, \forall \lambda > 0\},
\]
and let $\varrho_{\phi}$ be the usual Orlicz modular. Then:

1. If $\tilde{f}$ is the 1-periodic extension of $f$ to the whole $\mathbb{R}$, then $\tilde{f} \in B^\phi$-a.p.
2. The injection $i : E^\phi([0, 1]) \to B^\phi$-a.p., $i(f) = \tilde{f}$, is an isometry with respect to the modular and for the respective Orlicz norms.

**Proof.** (1) Let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{n} A_i \subset [0, \alpha]$, $0 < \alpha < 1$. Let $m \in \mathbb{N}$. Since $\sum_{i=1}^{n} \phi(ma_i)\mu(A_i) < \infty$, it follows from (1.2) that there exists $P_m \in \mathcal{P}$ (the set of generalized trigonometric polynomials) for which
\[
\varrho_B\left(\frac{m}{4}(\tilde{f} - P_m)\right) \leq \frac{1}{m},
\]
where $\tilde{f}$ is the 1-periodic extension of $f$. 

Let $\lambda > 0$ and $m_0 \in \mathbb{N}$ be such that $\lambda \leq m_0/4$. Then
\[ \varrho_{B^\phi}(\lambda(f - P_m)) \leq \varrho_{B^\phi}\left(\frac{m}{4}(f - P_m)\right) \leq \frac{1}{m}, \quad \forall m \geq m_0. \]
This means that $\lim_{m \to \infty} \|f - P_m\|_{B^\phi} = 0$, i.e. $\tilde{f} \in B^\phi$-a.p.

Consider now the general case of $f \in E^\phi([0,1])$. It is known (see [1]) that the step functions are dense in $E^\phi([0,1])$, hence given $\varepsilon > 0$, there is a $g_{\varepsilon} = \sum_{i=1}^{n} a_i \chi_{A_i}$ for which $\|g_{\varepsilon} - f\|_{\phi} \leq \varepsilon/4$. Since $f$ is absolutely continuous, we may choose $\delta > 0$ such that $\mu(A) \leq \delta \Rightarrow \|f_{\chi_A}\|_{\phi} \leq \varepsilon/4$. We take $\alpha > 0$ with $1 - \alpha \leq \delta$ and put $A_i^\alpha = A_i \cap [0,\alpha]$, $i = 1, n$. Then the function $g_{\varepsilon}^\alpha = \sum_{i=1}^{n} a_i \chi_{A_i^\alpha}$ belongs to $E^\phi([0,1])$. If $\tilde{f}$ and $\tilde{g}_{\varepsilon}^\alpha$ are the respective 1-periodic extensions, then
\[
\|\tilde{f} - \tilde{g}_{\varepsilon}^\alpha\|_{B^\phi} = \|f - g_{\varepsilon}^\alpha\|_{\phi} \leq \|(f - g_{\varepsilon}^\alpha)\chi_{[0,\alpha]}\|_{\phi} + \|(f - g_{\varepsilon}^\alpha)\chi_{[\alpha,1]}\|_{\phi} \\
\leq \|f - g_{\varepsilon}\|_{\phi} + \|f\chi_{[\alpha,1]}\|_{\phi} \leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2.
\]
Now, since $\tilde{g}_{\varepsilon}^\alpha \in B^\phi$-a.p., there exists $P_{\varepsilon} \in \mathcal{P}$ for which $\|\tilde{g}_{\varepsilon}^\alpha - P_{\varepsilon}\|_{B^\phi} \leq \varepsilon/2$. Finally,
\[
\|\tilde{f} - P_{\varepsilon}\|_{B^\phi} \leq \|\tilde{f} - \tilde{g}_{\varepsilon}^\alpha\|_{B^\phi} + \|\tilde{g}_{\varepsilon}^\alpha - P_{\varepsilon}\|_{B^\phi} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,
\]
i.e. $\tilde{f} \in B^\phi$-a.p.

(2) It is clear that $i : E^\phi([0,1]) \to B^\phi$-a.p. is a modular isometry. The fact that it is also an isometry for the Orlicz norms follows immediately since
\[
\|f\|_{\phi} = \inf_{k > 0} \left\{ \frac{1}{k} (1 + \varrho_{\phi}(kf)) \right\} = \inf_{k > 0} \left\{ \frac{1}{k} (1 + \varrho_{B^\phi}(k\tilde{f})) \right\} = \|\tilde{f}\|_{B^\phi}. \tag*{\blacksquare}
\]
We can now state our main result.

**Theorem 1.** The space $(B^\phi$-a.p., $\| \cdot \|_{B^\phi}$) is $k$-convex iff $\phi \in \Delta_2 \cap \nabla_2$ and $\phi$ is strictly convex.

**Proof.** **Necessity.** As known for general Banach spaces, $k$-convexity implies strict convexity and reflexivity. From [9], reflexivity of $B^\phi$-a.p. implies that $\phi \in \Delta_2 \cap \nabla_2$. It remains to show that $\phi$ is strictly convex. Indeed, strict convexity of $\phi$ is necessary for strict convexity of the Orlicz class $E^\phi([0,1])$ (cf. [1]) and using Proposition 10, we deduce that it is also necessary for strict convexity of $B^\phi$-a.p.

For the sufficiency, let $\{f_n\} \subset B^\phi$-a.p. with $\|f_n\|_{B^\phi} = 1$ and $\|f_n + f_m\|_{B^\phi} \to 2$ as $n, m \to \infty$. Given any $\varepsilon > 0$, take $n_0$ and $\delta$ as in Lemma 6. Since $f_{n_0} \in B^\phi$-a.p. there is a $\delta' > 0$ such that $\bar{\mu}(E) < \delta' \Rightarrow \varrho_{B^\phi}(f_{n_0}\chi_E) \leq \delta$ and then by Lemma 6 we obtain $\varrho_{B^\phi}(f_m\chi_E) \leq \varepsilon$ for all $m \geq n_0$.

On the other hand, since $\|f_n + f_m\|_{B^\phi} \to 2$ as $n, m \to \infty$, from Lemma 2 it follows that $\{k_n f_n\}$ is a $\bar{\mu}$-Cauchy sequence. Now, we will show that it is also modular equicontinuous.
Given any \( \varepsilon > 0 \), from Remark 2 there is \( \delta > 0 \) such that \( \varrho_{B^\phi}(f_n \chi_E) \leq \delta \Rightarrow \|k_n f_n \chi_E\|_{B^\phi} \leq \varepsilon \) and then from the arguments presented above we also have the implication \( \overline{\mu}(E) < \delta' \Rightarrow \|k_n f_n \chi_E\|_{B^\phi} \leq \varepsilon, \forall n \geq n_0 \) for some \( \delta' \).

This means that the sequence \( \{k_n f_n\}_n \) is norm equicontinuous.

Moreover, from Lemma 8, \( \{\varrho_{B^\phi}(k_n f_n)\}_{n \geq 1} \) is a Cauchy sequence in \( \mathbb{R} \), whence it converges to some \( l \in \mathbb{R} \).

Now, using (1.1), we may write \( \|f_n\|_{B^\phi} = (1/k_n)(1 + \varrho_{B^\phi}(k_n f_n)) \) and letting \( n \to \infty \) we get \( \lim_{n \to \infty} k_n = 1 + l \).

Finally, from Lemma 9, the sequence \( (k_n f_n) \) is modular Cauchy and again by the \( \Delta_2 \)-condition it is a norm Cauchy sequence, i.e. it converges in norm to some \( g \in B^\phi \)-a.p.

Consequently, \( \{f_n\} \) is norm convergent to \( g/(1 + l) \).

REFERENCES