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WEIGHTED NORM ESTIMATES AND L_p-SPECTRAL INDEPENDENCE OF LINEAR OPERATORS

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Abstract. We investigate the L_p-spectrum of linear operators defined consistently on $L_p(\Omega)$ for $p_0 \le p \le p_1$, where (Ω,μ) is an arbitrary σ-finite measure space and $1 \le p_0 < p_1 \le \infty$. We prove p-independence of the L_p-spectrum assuming weighted norm estimates. The assumptions are formulated in terms of a measurable semi-metric d on (Ω,μ); the balls with respect to this semi-metric are required to satisfy a subexponential volume growth condition. We show how previous results on L_p-spectral independence can be treated as special cases of our results and give examples—including strictly elliptic operators in Euclidean space and generators of semigroups that satisfy (generalized) Gaussian bounds—to indicate improvements.

1. Introduction and main result. Let (Ω,μ) be a σ-finite measure space and $1 \le p < q \le \infty$. Suppose that A is a linear operator that acts consistently in $L_r(\Omega)$ for $r \in [p,q]$, i.e.,

$$A\colon L_p(\Omega)\cap L_q(\Omega) \to L_p(\Omega)\cap L_q(\Omega)$$

is linear and extends, for all $r \in [p,q]$, to a bounded operator $A_r\colon L_r(\Omega) \to L_r(\Omega)$ which, in addition, is weak*-continuous if $r=\infty$. The "consistency" refers to the property

$$A_r f = A_s f \qquad (f \in L_r(\Omega)\cap L_s(\Omega),\ r,s \in [p,q]).$$

It is then a natural question whether the spectrum $\sigma(A_r)$ of A_r in $L_r(\Omega)$ depends on $r \in [p,q]$ or not. This question also makes sense for unbounded operators A_r in $L_r(\Omega)$, $r \in [p,q]$, if consistency is rephrased in terms of resolvents or semigroup operators (whenever the A_r are generators). We refer to Corollary 2 and Remark 3(iii) for details.

There are several kinds of assumptions that are sufficient for r-independence of $\sigma(A_r)$. We refer to the discussion in [14] and the references there. One method that has been widely used in the context of the L_p-theory of elliptic operators relies on the exploitation of certain bounds, especially of

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Gaussian type, that hold for the kernels of associated integral operators, namely semigroup or resolvent operators ([9], [1], [20], [7], [10], [13], [14]). If these operators are not integral operators, which is in general the case if $p > 1$ and $q < \infty$, then r-independence of the spectrum still holds under the assumption of certain weighted norm estimates ([15]–[18]). It is the latter kind of assumptions, i.e., weighted norm estimates, that we shall use in this paper.

The case studied in most of the cited papers is that Ω is an open subset of $\mathbb{R}^N$ and μ is Lebesgue measure. We shall deal here with an arbitrary σ-finite measure space (Ω, μ). The weighted norm estimates for the operator A will be formulated in terms of a measurable semi-metric d on Ω with suitable properties. The underlying philosophy is that, in order to get r-independence of the spectrum for a given operator A on (Ω, μ) by our result, one has *to find* a semi-metric d on Ω such that A and d satisfy our assumptions. With respect to the problem of r-independence of the spectrum as stated above, this seems to be the natural procedure. Even if the space Ω carries a natural metric d_0, as is the case for open subsets of $\mathbb{R}^N$ or Riemannian manifolds ([20]), a suitable semi-metric d may be quite different. We want to emphasize that, in the case of weighted norm estimates, the consideration of a general semi-metric space (Ω, d) instead of (subsets of) $\mathbb{R}^N$ means that we have to use a new approach in the proof that is rather different from the arguments used previously in this context.

In order to make the assumptions in our main result less restrictive we now slightly change the setting outlined above.

Let (Ω', μ) be a σ-finite measure space and suppose that d is a measurable semi-metric on Ω' satisfying

$$\gamma := \operatorname*{ess\,inf}_{z \in \Omega'} \mu(B(z, R)) > 0 \quad \text{for some } R > 0, \tag{1}$$

where $B(z, R)$ denotes the open ball with center z and radius R with respect to d. Further assume that

$$\operatorname*{ess\,sup}_{z \in \Omega'} \mu(B(z, r)) \le m(r) \quad (r > 0), \tag{2}$$

where $m\colon [0, \infty) \to [0, \infty)$ is a non-decreasing function such that $m(0) = 0$ and for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ satisfying

$$m(r) \le C_\varepsilon e^{\varepsilon r} \quad (r > 0). \tag{3}$$

Note that we do not demand $m(r) \to 0$ as $r \to 0$ (cf. [7, p. 177]), and that (2) and (3) mean that the volume growth is uniformly subexponential. We point out that uniformly subexponential volume growth has a different definition in the context of Riemannian manifolds since in general condition (1) does not hold in that context; see the end of Subsection 2.1.

Now let $\Omega \subset \Omega'$ be a measurable subset. We tacitly assume that functions defined on Ω are extended by 0 outside Ω when considered as functions on Ω'. In the following we consider operators on $L_p(\Omega)$, $1 \le p \le \infty$. The reason for introducing the space Ω' is that condition (1) is not assumed to hold for the balls of Ω but only for the (possibly larger) balls of Ω'. An important example for this situation is $\Omega' = \mathbb{R}^N$ and a subset $\Omega \subset \mathbb{R}^N$.

We fix an increasing sequence (Ω_n) of measurable subsets of Ω that have finite d-diameter and finite μ-volume such that $\Omega = \bigcup_n \Omega_n$. We denote by $L_{1,\mathrm{loc}}(\Omega)$ the set of all (equivalence classes of) measurable functions f on Ω with $\|f\|_{L_1(\Omega_n)} < \infty$ for all $n \in \mathbb{N}$, and by $L_{\infty,\mathrm{c}}(\Omega)$ the space of all (equivalence classes of) bounded measurable functions f on Ω for which there is an $n \in \mathbb{N}$ such that $f = 0$ a.e. on $\Omega \setminus \Omega_n$ (see also Remark 3(ii) for these spaces).

We define weight functions $\varrho_{\varepsilon,z}$ by

$$\varrho_{\varepsilon,z}(x) := \varrho_\varepsilon(z,x) := e^{-\varepsilon d(z,x)} \qquad (x, z \in \Omega',\ \varepsilon \in \mathbb{R}). \tag{4}$$

Note that the $\varrho_{\varepsilon,z}$ are multiplicators on $L_{1,\mathrm{loc}}(\Omega)$ and on $L_{\infty,\mathrm{c}}(\Omega)$.

For a linear operator $T\colon D(T) \subset L_p(\Omega) \to L_q(\Omega)$, $p, q \in [1,\infty]$, we denote its norm by $\|T\|_{p\to q}$. The following is our main result; the proof is given in Section 3.

THEOREM 1. *Assume that* (1) *to* (3) *hold. Let* $1 \le p < q \le \infty$ *and* $A\colon L_{\infty,\mathrm{c}}(\Omega) \to L_{1,\mathrm{loc}}(\Omega)$ *be a linear operator satisfying*

$$\|\varrho_{\varepsilon_0,z} A \varrho_{\varepsilon_0,z}^{-1}\|_{p\to q} \le C \qquad (z \in \Omega') \tag{5}$$

for some $C, \varepsilon_0 > 0$. *Then A extends to consistent bounded operators A_r on* $L_r(\Omega)$ $(r \in [p,q])$, *and the spectrum $\sigma(A_r)$ is independent of $r \in [p,q]$. In the case $q = \infty$, the operator $A\colon L_{\infty,\mathrm{c}}(\Omega) \to L_\infty(\Omega)$ is $\sigma(L_{\infty,\mathrm{c}}, L_1)$-weak*-continuous*; *$A_\infty$ denotes the unique weak*-continuous extension.*

In [12, Sec. 6] it was observed that, in the case of semigroup generators, p-independence of the spectrum of $R(\lambda, A_p)^n$ for some $n \in \mathbb{N}$ and all large λ implies p-independence of $\sigma(A_p)$. Following the lines of [12, proof of Thm. 1.7] we thus obtain the following application of Theorem 1.

COROLLARY 2. *Assume that* (1) *to* (3) *hold. Let* $1 \le p < q \le \infty$. *Let $T_r(t)$ be consistent C_0-semigroups on $L_r(\Omega)$ with generators A_r* $(p \le r \le q)$, *and A_∞ the weak* generator of the* (*weak*-continuous*) *semigroup $T_\infty(t)$ if $q = \infty$. Suppose that*

$$\|\varrho_{\varepsilon_0,z} T_p(t) \varrho_{\varepsilon_0,z}^{-1}\|_{p\to q} \le c t^{-\alpha} e^{\omega t} \qquad (z \in \Omega',\ t > 0) \tag{6}$$

for some $c, \alpha > 0$, $\omega \in \mathbb{R}$. Then the spectrum $\sigma(A_r)$ is independent of $r \in [p,q]$.

REMARK 3. (i) In the case $p = 1$, $q = \infty$, Theorem 1 and Corollary 2 also hold if condition (1) is replaced by the weaker assumption

(1′) there exists $R > 0$ such that $\mu(B(z, R)) > 0$ for a.e. $z \in \Omega'$;

see the paragraphs introducing Lemma 9 and Proposition 10.

(ii) Observe that the spaces $L_{1,\mathrm{loc}}(\Omega)$ and $L_{\infty,\mathrm{c}}(\Omega)$ depend on the choice of the sequence (Ω_n). The particular choice of (Ω_n) may depend on the problem under consideration. For instance, one may want to choose the sets Ω_n also bounded with respect to another measurable (semi-)metric d_0 on Ω'. To simply consider the space of all functions which are integrable over every d-bounded set instead of $L_{1,\mathrm{loc}}(\Omega)$ and the space of all bounded functions with support essentially contained in a d-bounded set instead of $L_{\infty,\mathrm{c}}(\Omega)$ may not always be the most convenient choice. See Subsection 2.3 for an example.

(iii) By the spectral mapping theorem for the resolvent (see, e.g., [5, Lemma 2.11]) we have independence of the spectrum $\sigma(A_r)$ for $r \in [p, q]$ if the A_r are unbounded operators in $L_r(\Omega)$ such that, for some $\lambda \in \bigcap_{r\in[p,q]} \varrho(A_r)$, the resolvent operators $R := R(\lambda, A_r)$ are consistent and (5) holds for $A = R\restriction_{L_{\infty,\mathrm{c}}(\Omega)}$. See Subsections 2.2 and 2.3 for applications.

(iv) If the measure space (Ω, μ) is finite, then it is known that in (5) no weights are needed for the assertion of Theorem 1 (see [1, Prop. 1.1]). This also follows from our result if we take $d = 0$ as a semi-metric on $\Omega' = \Omega$. Observe that (1) holds since $B(z, R) = \Omega$ for all $R > 0$.

(v) The space (Ω, μ) must be assumed to be σ-finite since otherwise condition (2) cannot hold: We show that condition (2) implies an upper bound for the volume of *all* balls in Ω'. Let $x \in \Omega'$. If $\mu(B(x, r)) > 0$ then (2) implies that there exists $z \in B(x, r)$ satisfying $\mu(B(z, 2r)) \le m(2r)$. It follows that $\mu(B(x, r)) \le m(2r)$ for all $x \in \Omega'$. (A simple example shows that the latter bound cannot be improved.)

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2. Comments and examples. In this section we comment on the existing literature on L_p-spectral independence and present some examples to indicate the improvements.

The case studied most is (Ω, μ) where $\Omega \subset \mathbb{R}^N$ is an open subset and μ is Lebesgue measure. If Ω is endowed with the Euclidean distance d then conditions (2) and (3) hold. If we choose a suitable superset Ω' of Ω, also (1) holds (take, e.g., a δ-neighbourhood of Ω, or simply $\mathbb{R}^N$).

In this setting, p-independence of the L_p-spectrum has been proved for the generator A of consistent C_0-semigroups in $L_p(\Omega, \mu)$ under the assumption that the semigroup operators are given by kernels which satisfy Gaussian bounds. Making use of Corollary 2, we will study this case in Subsection 2.1 below in a more general setting. The first proof of L_p-spectral independence making use of Gaussian bounds was carried out in [9] for Schrödinger operators with negative part of the potential in the Kato class. In the general case of validity of Gaussian bounds, L_p-spectral independence has been proved in [1] under the assumption that $\varrho(A_2)$ is connected, for selfadjoint A_2 in [7] and under certain commutator estimates in [10]. In [13] those additional assumptions were removed. It is known that Gaussian bounds hold for large classes of uniformly elliptic operators (see, e.g., [2]–[4], [6], [19], to mention but a few).

Weighted norm estimates may be used to prove L_p-spectral independence if the semigroup does not act on all L_p-spaces, $1 \leq p < \infty$. In [17] this was done for Schrödinger operators with form bounded negative part of the potential. In [8] weighted norm estimates were used to prove L_p-spectral independence for higher order elliptic operators. The ideas from [17] were put in a more general context in [16], assuming so-called generalized Gaussian bounds which involve weight functions $x \mapsto \exp(\xi \cdot x)$, $\xi \in \mathbb{R}^N$. Those weight functions correspond to the Euclidean metric in $\mathbb{R}^N$. In [18] more general weight functions were used in order to study selfadjoint elliptic operators with unbounded coefficients. There the weights are $x \mapsto \exp(\xi\varphi(x))$ where φ is a so-called L_1-regular function on $\mathbb{R}^N$. This corresponds to the semi-metric $d(x, y) := |\varphi(x) - \varphi(y)|$ on $\mathbb{R}^N$. In [15], L_p-spectral independence was proved for closed (not necessarily selfadjoint) operators assuming a weighted norm estimate for a single resolvent. In Subsection 2.2 we discuss properties of L_1-regular functions and show that Theorem 1 of the present paper extends [15, Thm. 1].

We want to emphasize that, until now, all proofs of L_p-spectral independence assuming weighted norm estimates relied on the "box method" where $\mathbb{R}^N$ is split up into congruent cubes Q_j and one works in the spaces $l_r(L_p(Q_j))$. In contrast, our proof of Theorem 1 below does *not* use the box method but relies on Lemma 9 as a substitute. Indeed, working in a general measure space that carries a semi-metric it is not clear what one should use instead of the partition into cubes of equal size. In Subsection 2.3 below we give an example of a strictly elliptic operator which illustrates the limitations of cube partitions even in Euclidean space.

Before proceeding with the examples, we introduce some notation. Let (Ω', μ) and d be as in the introduction and assume that (1) to (3) hold. For a measurable function g on $\Omega' \times \Omega'$ and $r \in [1, \infty]$ we define

$$n_r(g) := \max\Big(\operatorname*{ess\,sup}_{x \in \Omega'} \|g(x, \cdot)\|_{L_r(\Omega')}, \operatorname*{ess\,sup}_{y \in \Omega'} \|g(\cdot, y)\|_{L_r(\Omega')}\Big) \in [0, \infty].$$

If $n_r(g)$ is finite then g defines a bounded integral operator $I_g \colon L_p(\Omega') \to$

$L_q(\Omega')$ for all $1 \le p \le q \le \infty$ satisfying $r^{-1} + p^{-1} = 1 + q^{-1}$. In fact,

$$\|I_g \colon L_p(\Omega') \to L_q(\Omega')\| \le n_r(g), \tag{7}$$

which can be proved by an application of Fubini's theorem and Riesz–Thorin interpolation. In the case $r = 1$ the operator I_g is bounded in every space $L_p(\Omega')$ $(1 \le p \le \infty)$.

For ϱ_ε as in (4) we have $n_\infty(\varrho_\varepsilon) \le 1$, and by assumption (3) we obtain

$$n_p(\varrho_\varepsilon) \le \Big(\int_0^\infty e^{-\varepsilon p r}\, dm(r)\Big)^{1/p} < \infty \qquad (1 \le p < \infty). \tag{8}$$

Here we have used the fact that, for any non-increasing function $\varphi \colon [0,\infty) \to [0,\infty)$ and almost all $x \in \Omega'$, one can estimate

$$\int_{\Omega'} \varphi(d(x,y))\, d\mu(y) \le \int_0^\infty \varphi(r)\, dm(r)$$

by assumption (2); cf. [7, p. 179].

2.1. *Gaussian bounds in metric spaces.* In this subsection we discuss a result of Davies [7] which stimulated us to study the problem of p-independence in the context of (semi-)metric spaces. The result is as follows:

Let (Ω, d) be a separable metric space, and μ a Borel measure on Ω satisfying

$$\mu(B(x,r)) \le \begin{cases} c_0 r^N & \text{if } 0 \le r \le 1, \\ c_0 r^M & \text{if } 1 \le r < \infty \end{cases}$$

for all $x \in \Omega$ and $r > 0$, for some constants $c_0 > 0$ and $0 < N \le M < \infty$. Let H be a non-negative selfadjoint operator in $L_2(\Omega)$. Assume that the generated semigroup e^{-tH} on $L_2(\Omega)$ is given by an integral kernel K satisfying the Gaussian upper bound

$$|K(t,x,y)| \le c_1 t^{-N/2} e^{-c_2\, d(x,y)^2/t} \qquad (t > 0,\ x, y \in \Omega) \tag{9}$$

for some positive constants c_1, c_2.

Then the semigroup e^{-tH} on $L_1(\Omega) \cap L_\infty(\Omega)$ extends to consistent C_0-semigroups $T_p(t)$ on $L_p(\Omega)$ for all $1 \le p < \infty$. Let $-H_p$ denote the generator of $T_p(t)$. Then the spectrum of H_p is independent of $p \in [1, \infty)$.

We now show that this result of Davies can be deduced from Corollary 2, thus removing the assumption that the semigroup is symmetric.

Proposition 4. *Assume that* $(1')$, (2) *and* (3) *hold. Let* $T(t)$ *be a* C_0-*semigroup on* $L_2(\Omega)$ *given by an integral kernel* K *satisfying the Gaussian upper bound* (9) *for some positive constants* c_1, c_2. *Assume that* $T(t)$ *extends to consistent semigroups* $T_p(t)$ *on* $L_p(\Omega)$ *for* $1 \le p \le \infty$ (*strongly continuous for* $1 \le p < \infty$ *and weak*-continuous for* $p = \infty$). *Let* A_p *denote the* *generator of $T_p(t)$ for $1 \le p < \infty$, and let A_∞ denote the weak* generator of $T_\infty(t)$. Then the spectrum of A_p is independent of $p \in [1,\infty]$.*

If $m(r) \le c_0 r^N$ $(0 < r \le 1)$ for some $c_0 > 0$ then a straightforward computation shows that $n_1(t^{-N/2}\exp(-c_2 d^2/t)) \le M e^{t/c_2}$ for all $t > 0$, for some $M > 0$. Thus, by (7), $T(t)$ extends to a family of consistent semigroups $T_p(t)$ on $L_p(\Omega)$. As in [7] one shows that these semigroups are strongly continuous for $p < \infty$ (for $p > 1$ this is clear by interpolation).

Proof of Proposition 4. By the inequality $r \le t/4c_2 + c_2 r^2/t$, the kernel $k_{z,t}$ of $\varrho_{1,z} T(t) \varrho_{1,z}^{-1}$ satisfies

$$k_{z,t}(x,y) = e^{d(z,y)-d(z,x)} K(t,x,y) \le e^{d(x,y)} c_1 t^{-N/2} e^{-c_2\, d(x,y)^2/t} \le c_1 t^{-N/2} e^{t/4c_2}$$

for all $x, y \in \Omega$. We thus find that (6) is satisfied for $\Omega' = \Omega$, $\varepsilon_0 = 1$, $p = 1$ and $q = \infty$, so the assertion follows from Corollary 2 (see also Remark 3(i)). ∎

With slight modifications, the same can be shown if the semigroup kernel satisfies a Gaussian upper bound of order m or, more generally, an upper bound like

$$(10) \qquad |K(t,x,y)| \le c_1 t^{-M/2} \exp(-c_2 d(x,y)^\gamma / t^{\gamma-1}) \qquad (t > 0,\ x, y \in \Omega)$$

for some $c_1, c_2 > 0$, $M > 0$ and $\gamma > 1$. Bounds of this type have been proved in [4], [12] for semigroups generated by certain elliptic operators on open subsets $\Omega \subset \mathbb{R}^N$. In [4], the operator was assumed to be uniformly elliptic, satisfying Neumann or Robin boundary conditions, and (10) was shown with $\gamma = 2$ and $M \ge N$ depending on some weak regularity assumption on Ω. In [12], the bound (10) was shown with $M = N$ and some $\gamma \in (1,2)$ for elliptic operators with unbounded coefficients in the principal part, satisfying Dirichlet boundary conditions.

In [20], K. T. Sturm showed that the spectrum of uniformly elliptic operators on an N-dimensional complete Riemannian manifold with Ricci curvature bounded below is p-independent if the volume grows uniformly subexponentially in the following sense: Instead of (2) the stronger bound

$$\mu(B(x,r)) \le m(r)\mu(B(x,1)) \quad (x \in M,\ r > 0)$$

holds with m satisfying (3). This bound is stronger since $\sup_{x\in M} \mu(B(x,1)) < \infty$ by Bishop's comparison principle, but $\inf_{x \in M} \mu(B(x,1)) > 0$ does not hold in general, i.e., our condition (1) is not satisfied.

2.2. *Weighted norm estimates and L_1-regular functions.* In this subsection we present a result of [15] on L_p-spectral independence for closed operators acting in $L_p(\Omega)$ for some open set $\Omega \subset \mathbb{R}^N$. Choosing an appropriate semi-metric d on $\mathbb{R}^N$ we show that this result is also a consequence of our main theorem. We need to introduce the following notion due to Semenov ([18]): A function $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ is called *L_1-regular* if it is (uniformly) Lipschitz continuous and

$$\sup_{k\in\mathbb{Z}^N} \sum_{j\in\mathbb{Z}^N} e^{-|\varphi(k)-\varphi(j)|} < \infty.$$

We use Theorem 1 to reprove

THEOREM ([15, Theorem 1]). *Given $1 \le p < q < \infty$ let T_p and T_q be closed operators in $L_p(\Omega)$ and $L_q(\Omega)$, respectively. If there exist $\varepsilon > 0$, $C < \infty$, $\lambda_0 \in \varrho(T_p) \cap \varrho(T_q)$ and an L_1-regular function $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ such that*

(i) *$R(\lambda_0, T_p)$ and $R(\lambda_0, T_q)$ are consistent,*
(ii) *$\|e^{\xi\varphi}R(\lambda_0, T_p)e^{-\xi\varphi}\restriction_{L_{\infty,c}(\Omega)}\|_{p\to q} \le C$ for all $\xi \in \mathbb{R}^N$ with $|\xi| \le \varepsilon$,*

then $\sigma(T_p) = \sigma(T_q)$, and $R(\lambda, T_p)$ and $R(\lambda, T_q)$ are consistent for all $\lambda \in \varrho(T_p) = \varrho(T_q)$.

In [15], this theorem was applied to (not necessarily selfadjoint) strictly elliptic operators in divergence form with lower order terms. We show that, for the semi-metric on $\Omega' := \mathbb{R}^N$ defined by $d(x,y) := |\varphi(x)-\varphi(y)|_\infty$ and for $\mu := |\cdot|$ the Lebesgue measure, assumptions (1) to (3) and the weighted norm estimate (5) hold. The lower volume estimate (1) is a direct consequence of the Lipschitz continuity of φ: Let $L > 0$ be such that $|\varphi(x) - \varphi(y)|_\infty \le L|x-y|_\infty$ for all $x, y \in \mathbb{R}^N$. Then $B(x,r) \supset x + [-r/L, r/L]^N$ and hence $|B(x,r)| \ge (2/L)^N r^N$ for all $r > 0$.

The following lemma in particular shows that (2) and (3) hold.

LEMMA 5. *Let $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz continuous. Define the semi-metric d on $\mathbb{R}^N$ as above. Then the following are equivalent*:

(i) *φ is L_1-regular,*
(ii) *$n_1(e^{-\varepsilon d}) < \infty$ for all $\varepsilon > 0$,*
(iii) *there exists $C > 0$ such that $|B(x,r)| \le C(1+r)^N$ for all $r > 0$.*

Proof. (iii)$\Rightarrow$(ii) is shown in (8).

(ii)$\Rightarrow$(i). For $j \in \mathbb{Z}^N$ let $Q_j := j + [-1/2, 1/2]^N$. Let L be as above. Then for $k \in \mathbb{Z}^N$ we have

$$\sum_{j\in\mathbb{Z}^N} e^{-|\varphi(k)-\varphi(j)|} \le \sum_{j\in\mathbb{Z}^N} e^{L/2} \int_{Q_j} e^{-|\varphi(k)-\varphi(y)|}\,dy = e^{L/2} \int_{\mathbb{R}^N} e^{-d(k,y)}\,dy.$$

Since φ is Lipschitz continuous the function $x \mapsto \int_{\mathbb{R}^N} e^{-d(x,y)}\,dy$ is continuous, and we conclude that

$$\sup_{k\in\mathbb{Z}^N} \sum_{j\in\mathbb{Z}^N} e^{-|\varphi(k)-\varphi(j)|} \le e^{L/2} \sup_{k\in\mathbb{R}^N} \int_{\mathbb{R}^N} e^{-d(k,y)}\,dy = e^{L/2} n_1(e^{-d}) < \infty.$$

(i)⇒(iii) (cf. [15, Appendix A]). Let $r > 0$ and let $n \in \mathbb{N}$ with $n - 1 < r \le n$. Then

$$\begin{aligned} B(x,r) &= \{y \in \mathbb{R}^N;\ \varphi(y) \in \varphi(x) + (-r,r)^N\} \\ &\subset \bigcup \{\varphi^{-1}(\varphi(x) + j + [0,1]^N);\ j \in \{-n,\dots,n-1\}^N\}. \end{aligned}$$

It follows that $|B(x,r)| \le (2n)^N \sup_{z\in\mathbb{R}^N} |\varphi^{-1}(z + [0,1]^N)|$. Since $(2n)^N \le 2^N(1+r)^N$ it remains to show that the supremum is finite.

To this end, let $z \in \mathbb{R}^N$ and $Q := z + [0,1]^N$. Let $x_0, y \in \varphi^{-1}(Q)$ and choose $k, j \in \mathbb{Z}^N$ with $x_0 \in Q_k$ and $y \in Q_j$. Then $d(x_0,y) \le 1$ and $d(x_0,k), d(y,j) \le L/2$, hence $d(j,k) \le L+1$. Therefore $\varphi^{-1}(Q) \subset \bigcup\{Q_j;\ j \in \mathbb{Z}^N, d(j,k) \le L+1\}$ and

$$|\varphi^{-1}(Q)| \le \#\{j \in \mathbb{Z}^N;\ d(j,k) \le L+1\} \le \sum_{j\in\mathbb{Z}^N} e^{L+1-d(j,k)}.$$

By the L_1-regularity of φ this shows that $|\varphi^{-1}(Q)|$ can be estimated from above independently of the cube Q, and the proof is finished. ∎

To conclude, we show that the weighted norm estimate (ii) of the above theorem with the unbounded weights $e^{\xi\varphi}$ implies estimate (5) with the bounded weights $\varrho_{\varepsilon,z} = e^{-\varepsilon d(\cdot,z)}$. To this end let $E := \{\pm\varepsilon e_j;\ j = 1,\dots,N\}$ where e_j are the standard unit vectors of $\mathbb{R}^N$. Fix $z \in \mathbb{R}^N$ and let $\varrho_\xi := e^{\xi(\varphi-\varphi(z))}$ for $\xi \in E$. Then

$$\varrho_{\varepsilon,z}^{-1} = e^{\varepsilon|\varphi-\varphi(z)|_\infty} = \max_{\xi\in E} \varrho_\xi^{-1}.$$

For $f \in L_{\infty,\mathrm{c}}(\Omega)$ and the dual operator A' of A (cf. Lemma 7 of Section 3 below) we obtain

$$\begin{aligned} \|\varrho_{\varepsilon,z}^{-1} A' \varrho_{\varepsilon,z} f\|_{p'} &\le \sum_{\xi\in E} \|\varrho_\xi^{-1} A' \varrho_\xi \varrho_\xi^{-1} \varrho_{\varepsilon,z} f\|_{p'} \\ &\le \sum_{\xi\in E} \|\varrho_\xi^{-1} A' \varrho_\xi\|_{q'\to p'} \|\varrho_\xi^{-1} \varrho_{\varepsilon,z} f\|_{q'}. \end{aligned}$$

Noting $\varrho_\xi^{-1}\varrho_{\varepsilon,z} \le 1$, we conclude, by Lemma 7(ii) and the definition of ϱ_ξ, that

$$\begin{aligned} \|\varrho_{\varepsilon,z} A \varrho_{\varepsilon,z}^{-1}\|_{p\to q} = \|\varrho_{\varepsilon,z}^{-1} A' \varrho_{\varepsilon,z}\|_{q'\to p'} &\le \sum_{\xi\in E} \|\varrho_\xi^{-1} A' \varrho_\xi\|_{q'\to p'} \\ &\le 2N \max_{\xi\in E} \|e^{\xi\varphi} A e^{-\xi\varphi}\|_{p\to q}. \end{aligned}$$

Therefore (5) is satisfied and we can apply Theorem 1, taking into account Remark 3(iii).

2.3. *Elliptic operators in Euclidean space.* We now want to discuss in some detail the fact that even in Euclidean space it is very helpful to have the possibility to resort to a totally different semi-metric and also not to be restricted to “cube partitions”.

Let $\Omega \subset \mathbb{R}^N$ be an open set, and μ the Lebesgue measure on Ω. Let $a_{jk} \in L_{1,\mathrm{loc}}(\Omega)$ be real-valued with $a_{jk} = a_{kj}$ for all $1 \le j, k \le N$ and

$$\sum_{j,k=1}^{N} a_{jk}\xi_j\xi_k \ge \alpha|\xi|^2 \quad \text{a.e. } (\xi \in \mathbb{R}^N)$$

for some $\alpha > 0$. We define the differential operator $-\nabla \cdot (a\nabla)$ with zero Dirichlet boundary conditions by the form method. Let τ be the following sesquilinear form in L^2:

$$\tau(u,v) := \int_\Omega \sum_{j,k=1}^{N} a_{jk}(x)\partial_k u(x)\partial_j \overline{v}(x)\,dx, \qquad D(\tau) := C_c^\infty(\Omega).$$

It is well-known that the form τ is closable and that $\overline{\tau}$ is a Dirichlet form. Let H be the selfadjoint operator in L_2 associated with $\overline{\tau}$. Then $e^{-tH}\restriction_{L_{\infty,\mathrm{c}}(\Omega)}$ extends to a C_0-semigroup e^{-tH_p} on L_p for all $1 \le p < \infty$. Here $L_{\infty,\mathrm{c}}(\Omega)$ means the space corresponding to the sequence $\Omega_n := \{x \in \Omega; |x| \le n, \operatorname{dist}(x, \partial\Omega) \ge 1/n\}$, where dist denotes the Euclidean distance. This is the appropriate choice for later application of [15, Thm. 8].

If H is uniformly elliptic, i.e., the coefficients a_{jk} are bounded, then the semigroup e^{-tH} satisfies an upper Gaussian estimate, and $\sigma(H_p)$ is independent of $p \in [1,\infty)$ ([1, Example 5.2]).

In [18], Semenov studied elliptic operators that are not necessarily uniformly elliptic. In this case the semigroup may not satisfy an upper Gaussian estimate (with respect to the Euclidean metric), but under certain conditions p-independence of the spectrum still holds. He showed the same if the operator is perturbed by a real-valued potential with form small negative part. Then the semigroup does not necessarily exist in all L_p-spaces.

In [15], the conditions from [18] were slighty relaxed, and an additional first order perturbation was included. This required a new method of proof since the resulting operator need no longer be selfadjoint.

We are now going to present an example in dimension $N = 2$ that is not covered by the above mentioned results but is by our Theorem 1: Let $\Omega := \Omega' := \{x \in \mathbb{R}^2;\, x_1 > 0,\, 0 < x_2 < x_1^{-2}\} \cup \{x \in \mathbb{R}^2;\, x_1 \le 0,\, 0 < x_2 < 2\}$. Note that Ω has infinite measure, but the unbounded subset $\Omega_0 := \Omega \cap \{x;\, x_1 \ge 0, x_2 \le 2\}$ has finite measure.

Let a_{jk} be as above and assume that $a_{22}(x) \le c_0 x_2$ for $x_2 > 2$ and $a_{11}(x) \le c_0$ for $x_1 < 0$. The formula

$$d(x,y) := |\sqrt{x_2 \vee 2} - \sqrt{y_2 \vee 2}| + |x_1 \wedge 0 - y_1 \wedge 0| \qquad (x, y \in \Omega)$$

(where $\vee$ denotes the maximum, and $\wedge$ the minimum) defines a semi-metric on Ω. We now consider open balls of radius r with respect to this semi-metric. Obviously such a ball contains Ω_0 if and only if it contains a point of Ω_0. A direct calculation shows that $4r \le \mu(B(x,r)) \le 4r + 2\sqrt{2}$ for all $x \in \Omega$, $r > 0$. Hence conditions (1) to (3) are fulfilled for the metric d if we choose $m(r) := 4r + 2\sqrt{2}$ $(r > 0)$ and $m(0) := 0$.

Observe that in this example the space of all essentially bounded functions with support in a d-bounded set is strictly larger than the space $L_{\infty,\mathrm{c}}(\Omega)$ we have chosen, since it contains the characteristic function of Ω_0 (cf. Remark 3(ii)). On the other hand, the space of all functions that are integrable on all d-bounded sets is strictly smaller than $L_{1,\mathrm{loc}}(\Omega)$.

In order to show the weighted estimate (5) for the resolvents of H, we want to apply [15, Thm. 8]. For $0 < \varepsilon \le 1$, $z \in \Omega$ and $\varrho := \varrho_{\varepsilon,z}$ we obtain

$$\begin{aligned}\varrho^{-2} \sum_{j,k=1}^{2} a_{jk} \partial_k \varrho \partial_j \varrho &= \sum_{j,k=1}^{2} a_{jk} \varepsilon^2 \partial_k d(\cdot, z) \partial_j d(\cdot, z) \\ &= \varepsilon^2 (a_{11} (\partial_1 d(\cdot, z))^2 \chi_{\{x;\, x_1 < 0\}} + a_{22} (\partial_2 d(\cdot, z))^2 \chi_{\{x,\, x_2 > 2\}}) \le c_0.\end{aligned}$$

Further, $\varrho \wedge n$ and $\varrho^{-1} \wedge n$ are Lipschitz continuous for all $n \in \mathbb{N}$, so the conditions of [15, Thm. 8] are satisfied. We conclude that for $1 < p < q < \infty$ there exist $\lambda \in \mathbb{R}$ and $C > 0$ such that

$$\|\varrho_{\varepsilon,z} (\lambda + H)^{-1} \varrho_{\varepsilon,z}^{-1} \restriction_{L_{\infty,\mathrm{c}}(\Omega)}\|_{p \to q} \le C \quad (0 < \varepsilon \le 1,\ z \in \Omega).$$

(In fact the same is true for $p = 1$, but this is not covered by the above mentioned theorem.) By Remark 3(iii), it follows that the spectrum of H_p is independent of $p \in (1, \infty)$.

Applying [15, Thm. 2] directly to this situation would lead to more restrictive conditions on the coefficients a_{jk}: On the one hand, the linear growth of a_{22} would not be allowed, on the other hand there would be an additional condition on a_{11} on the subset Ω_0. This is clear from the discussion on L_1-regular functions in the previous subsection.

3. Proof of the main result. We start with a series of preparatory results. For this whole section we fix $1 \le p \le q \le \infty$.

In order to prove the inclusion $\varrho(A_r) \subset \varrho(A_s)$ for $r, s \in [p, q]$, one has to show that for $\lambda \in \varrho(A_r)$ the operator $R(\lambda, A_r)\restriction_{L_{\infty,\mathrm{c}}(\Omega)}$ extends to a bounded operator on $L_s(\Omega)$. This is expressed in the following elementary lemma which is stated in the general context of topological spaces (cf. [15, Prop. 4] or [1, Prop. 2.3]).

Let E, F, G be Hausdorff spaces with $E, F \hookrightarrow G$ such that $E \cap F$ is dense in both E and F. Let $D \subset E \cap F$ be a subset that is dense with respect to the initial topology coming from the embeddings $E \cap F \hookrightarrow E$ and $E \cap F \hookrightarrow F$.

LEMMA 6. *Let* $A_E \colon E \to E$ *and* $A_F \colon F \to F$ *be continuous mappings satisfying* $A_E{\restriction_D} = A_F{\restriction_D}$. *Assume that* A_E *is continuously invertible and that* $A_E^{-1}{\restriction_D}$ *extends to a continuous mapping* $R \colon F \to F$. *Then* A_F *is continuously invertible*, *and* $A_F^{-1} = R$.

Proof. Since D is dense in $E \cap F$ and $E, F \hookrightarrow G$, we have $A_E{\restriction_{E\cap F}} = A_F{\restriction_{E\cap F}}$ and $A_E^{-1}{\restriction_{E\cap F}} = R{\restriction_{E\cap F}}$. Hence $RA_F = A_F R = I$ on $E \cap F$. The density of $E \cap F$ in F yields the claim. ∎

To apply this lemma we further need the following. Consider the dual system $\langle L_{\infty,\mathrm{c}}(\Omega), L_{1,\mathrm{loc}}(\Omega)\rangle$ and endow $L_{\infty,\mathrm{c}}(\Omega)$ and $L_{1,\mathrm{loc}}(\Omega)$ with the corresponding weak topologies. If $A \colon L_{\infty,\mathrm{c}}(\Omega) \to L_{1,\mathrm{loc}}(\Omega)$ is a weakly continuous operator then A has a dual operator $A' \colon L_{\infty,\mathrm{c}}(\Omega) \to L_{1,\mathrm{loc}}(\Omega)$, i.e. $\int Af \cdot g = \int f \cdot A'g$ for all $f, g \in L_{\infty,\mathrm{c}}(\Omega)$.

LEMMA 7. *Let* $A \colon L_{\infty,\mathrm{c}}(\Omega) \to L_{1,\mathrm{loc}}(\Omega)$ *be a linear operator with* $\|A\|_{p\to q} < \infty$.

(i) *If* $p < \infty$ *then* A *is weakly continuous.*
(ii) *If* A *is weakly continuous then for the dual operator* A' *one has* $\|A'\|_{q'\to p'} = \|A\|_{p\to q}$.
(iii) *If* A *is weakly continuous and* $p = q = \infty$ *then* A *has a unique weak*-continuous extension* $A_\infty \colon L_\infty(\Omega) \to L_\infty(\Omega)$.

Proof. (i) Let $A_p \colon L_p(\Omega) \to L_q(\Omega)$ be the continuous extension of A. Then A_p is weakly continuous. Since $L_p(\Omega)$ is reflexive, A_p is weak*-continuous. This implies (i).

(ii) follows from $\|A\|_{p\to q} = \sup\{|\int Af \cdot g|;\ f, g \in L_{\infty,\mathrm{c}}(\Omega),\ \|f\|_p = \|g\|_{q'} = 1\}$.

(iii) By (ii) we have $\|A'\|_{1\to 1} < \infty$. Let $\tilde{A} \colon L_1(\Omega) \to L_1(\Omega)$ denote the continuous extension of A'. Then $A_\infty := \tilde{A}'$ is the unique weak*-continuous extension of A. ∎

COROLLARY 8. *Let* $A_r \colon L_r(\Omega) \to L_r(\Omega)$ *be consistent bounded operators* ($p \le r \le q$), *and* A_∞ *weak*-continuous if* $q = \infty$. *If* $\lambda \in \varrho(A_r)$ *and* $\|R(\lambda, A_r){\restriction_{L_{\infty,\mathrm{c}}(\Omega)}}\|_{s\to s} < \infty$ *for some* $r, s \in [p, q]$, *then* $\lambda \in \varrho(A_s)$.

Proof. For $r, s < \infty$ the result follows directly from Lemma 6 with $G = L_{1,\mathrm{loc}}(\Omega)$ and $D = L_{\infty,\mathrm{c}}(\Omega)$.

We endow $L_\infty(\Omega)$ with the weak*-topology. Then, for $t < \infty$, $L_{\infty,\mathrm{c}}(\Omega)$ is dense in $L_t(\Omega) \cap L_\infty(\Omega)$, and $L_t(\Omega) \cap L_\infty(\Omega)$ is dense in $L_t(\Omega)$ and $L_\infty(\Omega)$.

For the case $r = \infty$ it remains to note that $\lambda - A_\infty$ has a bounded inverse if and only if it has a weak*-continuous inverse. In the case $r < s = \infty$ the assumptions imply that $R(\lambda, A_r){\restriction_{L_{\infty,\mathrm{c}}(\Omega)}}$ is L_r- and L_∞-bounded. Hence, by Lemma 7(i) and (iii), $R(\lambda, A_r)\restriction_{L_{\infty,\mathrm{c}}(\Omega)}$ has a weak*-continuous extension $R\colon L_\infty(\Omega) \to L_\infty(\Omega)$, and again we are done by Lemma 6. ∎

Now, using the weights $\varrho_{\varepsilon,z}$, we construct a norm equivalent to the L_q-norm. Here is the only point where assumption (1) is needed. Observe that in the case $q = \infty$ it suffices to assume (1′).

LEMMA 9. *For all $f \in L_q(\Omega)$ and $\varepsilon > 0$ we have*

$$\|f\|_{q,\varepsilon} := \big\|z \mapsto \|\varrho_{\varepsilon,z} f\|_{L_q(\Omega)}\big\|_{L_q(\Omega')} \geq c_{q,\varepsilon}^{-1}\|f\|_q,$$

where $c_{q,\varepsilon} := \gamma^{-1/q} e^{\varepsilon R}$.

Proof. By Fubini's theorem, $\|f\|_{q,\varepsilon} = \big\|x \mapsto \|\varrho_{\varepsilon,x}\|_{L_q(\Omega')} f(x)\big\|_{L_q(\Omega)}$. Assumption (1) implies $\|\varrho_{\varepsilon,x}\|_{L_q(\Omega')} \geq \|e^{-\varepsilon R}\chi_{B(x,R)}\|_{L_q(\Omega')} \geq e^{-\varepsilon R}\gamma^{1/q}$ for almost all $x \in \Omega$. This gives the desired conclusion. ∎

The following consequence of (7) will be used throughout:

$$\big\|\Omega' \ni z \mapsto \|g(z,\cdot)f\|_p\big\|_q \leq n_t(g)\|f\|_s \qquad (p \leq s, t \leq q,\ p^{-1} + q^{-1} = s^{-1} + t^{-1}), \tag{11}$$

where g is a measurable function on $\Omega' \times \Omega'$, and f a measurable function on Ω. In particular, for $p = q$, $g = \varrho_\varepsilon$ we have $\|f\|_{q,\varepsilon} \leq n_q(\varrho_\varepsilon)\|f\|_q$, so $\|\cdot\|_{q,\varepsilon}$ is indeed equivalent to $\|\cdot\|_q$.

For a linear operator $A\colon L_{\infty,\mathrm{c}}(\Omega) \to L_{1,\mathrm{loc}}(\Omega)$ and $\varepsilon \in \mathbb{R}$ we will use the following notation:

$$\begin{aligned}\|A\|_{p\to q,\varepsilon} &:= \sup_{z\in\Omega'} \|\varrho_{\varepsilon,z} A \varrho_{\varepsilon,z}^{-1}\|_{p\to q}\\ &= \inf\{c > 0;\ \forall f \in L_{\infty,\mathrm{c}}(\Omega),\ z \in \Omega' : \|\varrho_{\varepsilon,z} Af\|_q \leq c\|\varrho_{\varepsilon,z} f\|_p\} \in [0,\infty].\end{aligned}$$

With this notation assumption (5) reads $\|A\|_{p\to q,\varepsilon_0} \leq C$.

Using Lemma 9 and the estimate (11) only, we see that assumption (5) implies that the operator A is L_q-bounded: For $f \in L_{\infty,\mathrm{c}}(\Omega)$,

$$\begin{aligned}\|Af\|_q \leq c_{q,\varepsilon_0}\|Af\|_{q,\varepsilon_0} &\leq c_{q,\varepsilon_0}\big\|z \mapsto \|A\|_{p\to q,\varepsilon_0}\|\varrho_{\varepsilon_0,z} f\|_p\big\|_q\\ &\leq c_{q,\varepsilon_0} n_p(\varrho_{\varepsilon_0})\|A\|_{p\to q,\varepsilon_0}\|f\|_q.\end{aligned}$$

But we are going to establish much more general estimates. For $\varepsilon > 0$ and $M \geq 1$ we define a class of weight functions

$$P(\varepsilon, M) := \{\varrho\colon \Omega' \to (0,\infty) \text{ measurable};\ \forall u, x \in \Omega' : \varrho(x)/\varrho(u) \leq M\varrho_{-\varepsilon,u}(x)\}.$$

For $\varrho \in P(\varepsilon, M)$, $\varepsilon' \in \mathbb{R}$ and $u \in \Omega'$ we have

$$\varrho_{\varepsilon',u}\varrho\varrho(u)^{-1} \leq M\varrho_{\varepsilon'-\varepsilon,u} \quad \text{and} \quad \varrho_{\varepsilon',u}\varrho(u)\varrho^{-1} \leq M\varrho_{\varepsilon'-\varepsilon,u}. \tag{12}$$

Note that due to the triangle inequality $\varrho_{\varepsilon,z} \in P(|\varepsilon|, 1)$ for $\varepsilon \in \mathbb{R}$ and $z \in \Omega'$.

We now estimate several operator norms in terms of $\|A\|_{p\to q,\varepsilon_0}$ (cf. [17, Prop. 3.2]). Again observe that in the case $p=1$, $q=\infty$ it suffices to assume (1′).

PROPOSITION 10.

(i) *Let $p\le s\le q$ and $t^{-1}:=p^{-1}+q^{-1}-s^{-1}$. For $0\le\varepsilon<\varepsilon_0$, $M\ge 1$ and $\varrho\in P(\varepsilon,M)$ we have*

$$\|\varrho A\varrho^{-1}\|_{s\to q}\le M^2 c_{q,\varepsilon_0+\varepsilon} n_t(\varrho_{\varepsilon_0-\varepsilon})\|A\|_{p\to q,\varepsilon_0}.$$

(ii) *There exist constants $C_{\varepsilon_0,\varepsilon}<\infty$, bounded as $\varepsilon\to 0$, such that*

$$\|A\|_{s\to t,\varepsilon}\le C_{\varepsilon_0,\varepsilon}\|A\|_{p\to q,\varepsilon_0}$$

for all $p\le s\le t\le q$ and $|\varepsilon|<\varepsilon_0$.

Proof. We begin with (i). Let $\varepsilon':=\varepsilon_0+\varepsilon$ and $f\in L_{\infty,\mathrm{c}}(\Omega)$. By Lemma 9 we have

$$\|\varrho A\varrho^{-1}f\|_q\le c_{q,\varepsilon'}\big\|u\mapsto\|\varrho_{\varepsilon',u}\varrho\varrho(u)^{-1}A\varrho(u)\varrho^{-1}f\|_q\big\|_q.$$

Using (12) and the definition of $\|A\|_{p\to q,\varepsilon_0}$ we get

$$\begin{aligned}\|\varrho_{\varepsilon',u}\varrho\varrho(u)^{-1}A\varrho(u)\varrho^{-1}f\|_q&\le M\|\varrho_{\varepsilon_0,u}A\varrho(u)\varrho^{-1}f\|_q\\&\le M\|A\|_{p\to q,\varepsilon_0}\|\varrho_{\varepsilon_0,u}\varrho(u)\varrho^{-1}f\|_p\le M^2\|A\|_{p\to q,\varepsilon_0}\|\varrho_{\varepsilon_0-\varepsilon,u}f\|_p.\end{aligned}$$

By (11) we conclude that

$$\begin{aligned}\|\varrho A\varrho^{-1}f\|_q&\le c_{q,\varepsilon'}M^2\|A\|_{p\to q,\varepsilon_0}\big\|u\mapsto\|\varrho_{\varepsilon_0-\varepsilon,u}f\|_p\big\|_q\\&\le M^2c_{q,\varepsilon'}\|A\|_{p\to q,\varepsilon_0}n_t(\varrho_{\varepsilon_0-\varepsilon})\|f\|_s.\end{aligned}$$

This proves (i).

For $t=q$, part (ii) is just a special case of (i) since $\varrho_{\varepsilon,z}\in P(|\varepsilon|,1)$ for $z\in\Omega'$. Note that the constant $C_{\varepsilon_0,\varepsilon}$ does not depend on p,q,s,t since $c_{q,\varepsilon'}\le c_{1,\varepsilon'}\vee c_{\infty,\varepsilon'}$ and $n_q(\varrho_{\varepsilon_0-\varepsilon})\le n_1(\varrho_{\varepsilon_0-\varepsilon})\vee n_\infty(\varrho_{\varepsilon_0-\varepsilon})$. Further, note that the constants stay bounded as $\varepsilon\to 0$.

We now prove (ii) for $s=t=p$. Note that the case $p=\infty$ is treated above, so we assume $p<\infty$. By Lemma 7(i) we know that $\varrho_{\varepsilon_0,z}A\varrho_{\varepsilon_0,z}^{-1}:L_{\infty,\mathrm{c}}(\Omega)\to L_{1,\mathrm{loc}}(\Omega)$ is weakly continuous for all $z\in\Omega'$. It easily follows that A is weakly continuous and that $(\varrho_{\varepsilon,z}A\varrho_{\varepsilon,z}^{-1})'=\varrho_{\varepsilon,z}^{-1}A'\varrho_{\varepsilon,z}$ for all $\varepsilon\in\mathbb{R}$ and $z\in\Omega'$. Hence by Lemma 7(ii) and by (i) we obtain

$$\|A'\|_{q'\to p',\varepsilon_1}=\|A\|_{p\to q,-\varepsilon_1}\le c_{q,\varepsilon_0+\varepsilon_1}n_q(\varrho_{\varepsilon_0-\varepsilon_1})\|A\|_{p\to q,\varepsilon_0}$$

for $\varepsilon_0>\varepsilon_1>0$. For $|\varepsilon|<\varepsilon_0$, choosing $|\varepsilon|<\varepsilon_1<\varepsilon_0$ and using (i) again for the dual situation we conclude that

$$\begin{aligned}\|A\|_{p\to p,\varepsilon}=\|A'\|_{p'\to p',-\varepsilon}&\le c_{p',\varepsilon_1+|\varepsilon|}n_{q'}(\varrho_{\varepsilon_1-|\varepsilon|})\|A'\|_{q'\to p',\varepsilon_1}\\&\le c_{p',\varepsilon_1+|\varepsilon|}n_{q'}(\varrho_{\varepsilon_1-|\varepsilon|})c_{q,\varepsilon_0+\varepsilon_1}n_q(\varrho_{\varepsilon_0-\varepsilon_1})\|A\|_{p\to q,\varepsilon_0}.\end{aligned}$$

Now Riesz–Thorin interpolation between the cases treated above proves the remaining cases. ∎

REMARK 11. Let $A\colon L_{\infty,\mathrm{c}}(\Omega) \to L_{1,\mathrm{loc}}(\Omega)$ be weakly continuous with $\|A\|_{p\to q,\varepsilon_0} < \infty$ for some $\varepsilon_0 > 0$. By Lemma 7(i) the latter implies weak continuity if (as in the assumptions of Theorem 1) $p < \infty$.

(i) Proposition 10(ii) shows that A extends to a bounded operator A_r on $L_r(\Omega)$ for $r \in [p,q]$, $r < \infty$. In the case $q = \infty$, Lemma 7(iii) implies that A has a unique weak*-continuous extension $A_\infty\colon L_\infty(\Omega) \to L_\infty(\Omega)$. The operators A_r are consistent since $L_{\infty,\mathrm{c}}(\Omega)$ is dense in $L_r(\Omega) \cap L_s(\Omega)$ if $L_\infty(\Omega)$ is endowed with the weak*-topology.

(ii) If $\varrho \in P(\varepsilon, M)$ for some $\varepsilon < \varepsilon_0$ and $M \ge 1$ then $\varrho A \varrho^{-1}$ extends to a bounded operator $A_\varrho\colon L_q(\Omega) \to L_q(\Omega)$ by Proposition 10(i). In the case $q = \infty$, again $A_\varrho\colon L_\infty(\Omega) \to L_\infty(\Omega)$ is the unique weak*-continuous extension. In case $\varrho = \varrho_{\varepsilon,z}$ for some $|\varepsilon| < \varepsilon_0, z \in \Omega'$ we write $A_{\varepsilon,z}$ for $A_{\varrho_{\varepsilon,z}}$.

(iii) If an operator A is defined on $D(A) \supset L_{\infty,\mathrm{c}}(\Omega)$, we also write $\|A\|_{p\to q,\varepsilon}$ instead of $\|A{\restriction}_{L_{\infty,\mathrm{c}}(\Omega)}\|_{p\to q,\varepsilon}$. If $A\colon L_p(\Omega) \to L_q(\Omega)$ is bounded (weak*-continuous in case $p = \infty$), then $\|\varrho_{\varepsilon,z} A f\|_q \le \|A\|_{p\to q,\varepsilon} \|\varrho_{\varepsilon,z} f\|_p$ for all $f \in L_p(\Omega)$. Therefore, if $B\colon L_q(\Omega) \to L_r(\Omega)$ is a bounded operator (weak*-continuous in case $q = \infty$), then $\|BA\|_{p\to r,\varepsilon} \le \|A\|_{p\to q,\varepsilon} \|B\|_{q\to r,\varepsilon}$.

The crucial part in the proof of Theorem 1 is the following estimate which implies convergence of weighted operators (cf. [16, Lemma 3.2.3] or [15, Prop. 5(iii)]).

PROPOSITION 12. *There exist* $\delta_{\varepsilon_0,\varepsilon} > 0$ *with* $\delta_{\varepsilon_0,\varepsilon} \to 0$ $(\varepsilon \to 0)$ *such that*

$$\|\varrho A \varrho^{-1} - A\|_{s\to t} \le \delta_{\varepsilon_0,\varepsilon} \|A\|_{p\to q,\varepsilon_0}$$

for all $p \le s \le t \le q$, $0 \le \varepsilon < \varepsilon_0$ *and* $\varrho \in P(\varepsilon, 1)$.

Proof. First note that $\|A\|_{s\to t,\varepsilon_1} \le C_{\varepsilon_0,\varepsilon_1} \|A\|_{p\to q,\varepsilon_0}$ for all $0 < \varepsilon_1 < \varepsilon_0$ by Proposition 10(ii). Hence it suffices to treat the case $s = p$, $t = q$. Using Lemma 9, for $f \in L_{\infty,\mathrm{c}}(\Omega)$ we have

$$\|(\varrho A \varrho^{-1} - A) f\|_q \le c_{q,\varepsilon_0+\varepsilon} \big\| u \mapsto \|\varrho_{\varepsilon_0+\varepsilon,u} (\varrho A \varrho^{-1} - A) f\|_q \big\|_q. \tag{13}$$

We now write

$$\varrho A \varrho^{-1} - A = \varrho \varrho(u)^{-1} A (\varrho(u) \varrho^{-1} - 1) + (\varrho \varrho(u)^{-1} - 1) A,$$

insert this into (13), use the triangle inequality and estimate the two resulting terms separately. For the second we have, using (12) and (11),

$$\big\| u \mapsto \|\varrho_{\varepsilon_0+\varepsilon,u} (\varrho \varrho(u)^{-1} - 1) A f\|_q \big\|_q \le n_q(\varrho_{\varepsilon_0+\varepsilon}(\varrho_{-\varepsilon} - 1)) \|A\|_{p\to q} \|f\|_p.$$

Using (12) and (11) again, the first term can be estimated by

$$\begin{aligned}\Big\|u\mapsto \|\varrho_{\varepsilon_0+\varepsilon,u}\varrho\varrho(u)^{-1}A(\varrho(u)\varrho^{-1}-1)f\|_q\Big\|_q &\le \Big\|u\mapsto \|\varrho_{\varepsilon_0,u}A(\varrho(u)\varrho^{-1}-1)f\|_q\Big\|_q\\ &\le \|A\|_{p\to q,\varepsilon_0}\Big\|u\mapsto \|\varrho_{\varepsilon_0,u}(\varrho_{-\varepsilon,u}-1)f\|_p\Big\|_q\\ &\le \|A\|_{p\to q,\varepsilon_0}n_q(\varrho_{\varepsilon_0}(\varrho_{-\varepsilon}-1))\|f\|_p.\end{aligned}$$

The claim follows after noting that $\|A\|_{p\to q}\le C_{\varepsilon_0,0}\|A\|_{p\to q,\varepsilon_0}$ and

$$n_q(\varrho_{\varepsilon_0}(\varrho_{-\varepsilon}-1))\le n_q(\varrho_{\varepsilon_0/2})\sup_{r\ge 0}e^{-\varepsilon_0 r/2}(e^{\varepsilon r}-1)\to 0\qquad(\varepsilon\to 0),$$

where we used $\varrho_{\varepsilon_0}=\varrho^2_{\varepsilon_0/2}$. ∎

The following consequence of Proposition 12 will be used in the proof of Theorem 1.

COROLLARY 13. *Let A be an operator on $L_q(\Omega)$ satisfying $\|A\|_{q\to q,\varepsilon_0}<\infty$ for some $\varepsilon_0>0$. In the case $q=\infty$ assume that A is weak*-continuous. Then for all $\lambda\in\varrho(A)$ there exists $\varepsilon>0$ such that $\lambda\in\varrho(A_{\varepsilon,z})$ for all $z\in\Omega'$, and $\|R(\lambda,A)\|_{q\to q,\varepsilon}<\infty$.*

Proof. Let $\lambda\in\varrho(A)$. By assumption we have $\|\lambda-A\|_{q\to q,\varepsilon_0}<\infty$. Applying Proposition 12 and recalling that inversion is continuous in the open set of invertible elements in $\mathcal{L}(L_q(\Omega))$, we deduce for some $\varepsilon>0$ that $\lambda-A_{\varepsilon,z}$ is invertible for all $z\in\Omega'$ (that is, the first claim), and that

$$\sup_{z\in\Omega'}\|R(\lambda,A_{\varepsilon,z})\|_{q\to q}<\infty. \tag{14}$$

To prove $\|R(\lambda,A)\|_{q\to q,\varepsilon}<\infty$ observe that for all $z\in\Omega'$ the function $\varrho^{-1}_{\varepsilon,z}\colon D(\varrho^{-1}_{\varepsilon,z},L_q)\to L_q$ is surjective since $\varrho_{\varepsilon,z}$ is bounded. Hence $R(\lambda,A_{\varepsilon,z})\restriction_{D(\varrho^{-1}_{\varepsilon,z},L_q)}=\varrho_{\varepsilon,z}R(\lambda,A)\varrho^{-1}_{\varepsilon,z}$, and we are done by (14). ∎

Proof of Theorem 1. In Remark 11(i) it is shown that A extends to consistent bounded operators A_r on $L_r(\Omega)$ ($p\le r\le q$, A_∞ weak*-continuous if $q=\infty$). So we need to prove the inclusion $\varrho(A_r)\subset\varrho(A_s)$ for all $r,s\in[p,q]$. Let $\lambda\in\varrho(A_r)$.

First we treat the case $\lambda\ne 0$. Then we can rewrite the resolvent $R(\lambda)$ of A_r as follows:

$$R(\lambda)=\lambda^{-1}I+\lambda^{-2}A_r+\lambda^{-2}A_rR(\lambda)A_r.$$

We have to show that $\lambda\in\varrho(A_s)$, which by Corollary 8 amounts to showing L_s-boundedness of $R(\lambda)\restriction_{L_{\infty,c}(\Omega)}$.

It is clear that $\lambda^{-1}I+\lambda^{-2}A$ is L_s-bounded; we will show that $A_rR(\lambda)A$ is L_s-bounded. According to Corollary 13 we have $\|R(\lambda)\|_{r\to r,\varepsilon}<\infty$ for some $0<\varepsilon<\varepsilon_0$. Moreover, $\|A\|_{p\to r,\varepsilon}+\|A\|_{r\to q,\varepsilon}<\infty$ by Proposition 10(ii).

Remark 11(iii) implies $\|A_r R(\lambda)A\|_{p\to q,\varepsilon} < \infty$. Another application of Proposition 10(ii) gives L_s-boundedness of $A_r R(\lambda)A$.

In the case $\lambda = 0$ we simply write $R(\lambda) = A_r R(\lambda)^3 A_r$. Hence by the above we have $\|R(\lambda)\|_{p\to q,\varepsilon} < \infty$, and again $\|R(\lambda)\|_{s\to s} < \infty$ by Proposition 10(ii). ■

REMARK 14. (i) Let A_p be a bounded operator on $L_p(\Omega)$ with $0 \in \varrho(A_p)$ and $\|A_p\|_{p\to q} < \infty$. Then $\|I\colon L_p(\Omega) \to L_q(\Omega)\| \le \|A_p^{-1}\|_{p\to p}\|A_p\|_{p\to q} < \infty$. Therefore (Ω,μ) cannot contain a sequence (M_n) of subsets satisfying $M_n \supset M_{n+1}$ $(n \in \mathbb{N})$ and $0 < \mu(M_n) \to 0$ $(n \to \infty)$.

Then $\Omega = \bigcup_{n=0}^{\infty} M_n$ where $\mu(M_0) = 0$, M_n are pairwise disjoint atoms of (Ω,μ) $(n \ge 1)$ and $\inf_{n\ge 1}\mu(M_n) > 0$. Therefore, for all $s < \infty$, the space $L_s(\Omega)$ is isometrically isomorphic to the weighted space of sequences $\{(x_n);\ \sum_n |x_n|^s \mu(M_n) < \infty\}$, and $L_\infty(\Omega)$ is isometrically isomorphic to l_∞. In this case we have $\|A_p\restriction_{L_1(\Omega)\cap L_p(\Omega)}\|_{1\to\infty} < \infty$.

(ii) Let $\varphi\colon \Omega' \to (0,\infty)$ be a *subexponential weight function*, i.e., for all $\varepsilon > 0$ there exists $M > 0$ such that $\varrho(x)/\varrho(y) \le M\exp(\varepsilon d(x,y))$ for all $x, y \in \Omega'$. Using Lemma 6, Proposition 10(i) and Corollary 13, one easily proves the following version of Theorem 1 for the weighted space $L_p(\varphi) := \{f;\ \varphi f \in L_p(\Omega)\}$, where $1 \le p < \infty$.

Assume that (1), (2) and (3) hold. Let A be a bounded operator on $L_p(\Omega)$ satisfying $\|A\|_{p\to p,\varepsilon_0} < \infty$ for some $\varepsilon_0 > 0$. Then $A\restriction_{L_p\cap L_p(\varphi)}$ extends to a bounded operator A_φ on $L_p(\varphi)$, and $\sigma(A_\varphi) = \sigma(A)$.

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