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A DECOMPOSITION THEOREM FOR A CLASS OF CONTINUA FOR WHICH THE SET FUNCTION $\mathcal{T}$ IS CONTINUOUS

BY

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Dedicated to Professor James T. Rogers, Jr., on the occasion of his 65th birthday

Abstract. We prove a decomposition theorem for a class of continua for which F. B. Jones's set function $\mathcal{T}$ is continuous. This gives a partial answer to a question of D. Bellamy.

1. Introduction. F. Burton Jones defined the set function $\mathcal{T}$ in [6]. Since then many properties related to this function have been studied.

In 1970, David Bellamy [1] gave properties of continua for which the set function $\mathcal{T}$ is continuous. In [12] a class of decomposable nonlocally connected one-dimensional continua for which $\mathcal{T}$ is continuous was given, and in [13] the class of homogeneous continua for which $\mathcal{T}$ is continuous was characterized.

In 1980, Bellamy asked: *If $\mathcal{T}$ is continuous for the (Hausdorff) continuum S, is it true that the collection $\{\mathcal{T}(\{p\}) \mid p \in S\}$ is a continuous decomposition of S such that the quotient space is locally connected?* (see Problem 162 in the Houston Problem Book [5, p. 390]). We present a positive answer to this question assuming that the continuum S is also point $\mathcal{T}$-symmetric (Theorem 3.8). Theorems 3.4 and 3.7 are of independent interest.

2. Definitions. If Z is a topological space, then given $A \subset Z$ the interior of A is denoted by $\mathrm{Int}(A)$. We write $\mathrm{Int}_Z(A)$ if there is a possibility of confusion.

A *map* is a continuous function. A surjective map $f\colon X \twoheadrightarrow Y$ between topological spaces is *monotone* provided that $f^{-1}(y)$ is connected for every

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$y \in Y$. The surjective map f is *open* (*closed*, respectively) if $f(U)$ is open (closed, respectively) in Y for each open (closed, respectively) subset U of X. If $f\colon X \twoheadrightarrow Y$ and Z is a nonempty subset of X, then $f|_Z\colon Z \to Y$ denotes the restriction of f to Z. Given a space X, 1_X denotes the identity map on X.

Given a topological space Z, a *decomposition* of Z is a family $\mathcal{G}$ of nonempty and mutually disjoint subsets of Z such that $\bigcup \mathcal{G} = Z$. A decomposition $\mathcal{G}$ of a topological space Z is said to be *continuous* if the quotient map $q\colon Z \twoheadrightarrow Z/\mathcal{G}$ is both closed and open.

A *continuum* is a compact connected Hausdorff space. A *subcontinuum* of a space Z is a continuum contained in Z. A continuum is *decomposable* if it is the union of two proper subcontinua. A continuum is *indecomposable* if it is not decomposable.

Given a continuum X, we define the set function $\mathcal{T}$ as follows: if $A \subset X$ then

$$\mathcal{T}(A) = X \setminus \{x \in X \mid \text{there exists a subcontinuum } W \text{ of } X \text{ such that } x \in \operatorname{Int}(W) \subset W \subset X \setminus A\}.$$

We write $\mathcal{T}_X$ if there is a possibility of confusion. Let us observe that for any subset A of X, $\mathcal{T}(A)$ is a closed subset of X and $A \subset \mathcal{T}(A)$. A continuum X is *aposyndetic* provided that $\mathcal{T}(\{p\}) = \{p\}$ for every $p \in X$.

A continuum X is *$\mathcal{T}$-additive* provided that $\mathcal{T}(A \cup B) = \mathcal{T}(A) \cup \mathcal{T}(B)$ for each pair of nonempty closed subsets A and B of X. We say that X is *point $\mathcal{T}$-symmetric* if for any two points p and q of X, $p \in \mathcal{T}(\{q\})$ if and only if $q \in \mathcal{T}(\{p\})$. The set function $\mathcal{T}$ is *idempotent on* X provided that $\mathcal{T}^2(A) = \mathcal{T}(A)$ for each subset A of X, where $\mathcal{T}^2 = \mathcal{T} \circ \mathcal{T}$.

We say that $\mathcal{T}$ is *continuous for a continuum* X provided that $\mathcal{T}\colon 2^X \to 2^X$ is continuous, where 2^X is the hyperspace of nonempty closed subsets of X, topologized with the Vietoris topology (or the Hausdorff metric if X is metric) [14]. If $f\colon X \to Y$ is continuous, then so is $2^f\colon 2^X \to 2^Y$ given by $2^f(A) = f(A)$ [14, (1.168)].

Let X and Z be continua, and let $f\colon X \twoheadrightarrow Z$ be continuous. We say that f is *$\mathcal{T}_{XZ}$-continuous* provided that $\mathcal{T}_X f^{-1}(B) \subset f^{-1}\mathcal{T}_Z(B)$ for every subset B of Z.

3. A decomposition theorem. We prove a decomposition theorem for point $\mathcal{T}$-symmetric continua for which $\mathcal{T}$ is continuous (Theorem 3.8). We restrict ourselves to decomposable nonlocally connected continua for it is well known that $\mathcal{T}$ is a constant map on indecomposable continua [2, (f), p. 5], and the identity map on locally connected continua [2, (b), p. 5].

Before proving Theorem 3.8, we assume we have a continuous decomposition of a decomposable nonlocally connected continuum for which $\mathcal{T}$

is continuous and prove that the quotient space is locally connected and many of the elements of the decomposition are indecomposable continua (Theorem 3.4).

Let us note the following:

3.1. Remark. David Bellamy asked: *If the set function $\mathcal{T}$ is continuous for the Hausdorff continuum S, then is it true that S is $\mathcal{T}$-additive?* (see [5, Problem 161, p. 389]). Let us observe that, since for continua X for which $\mathcal{T}$ is continuous, being $\mathcal{T}$-additive is equivalent to being point $\mathcal{T}$-symmetric [1, Lemma 9], by Theorem 3.8, both questions of Bellamy are equivalent, i.e., Problems 161 and 162 in the Houston Problem Book [5, pp. 389–390] are equivalent.

We begin with the following simple lemma.

3.2. Lemma. *Let X be a continuum and let $z \in X$. If*

$$\mathcal{G} = \{\mathcal{T}(\{x\}) \mid x \in X\}$$

is a decomposition of X, and W is a subcontinuum of X such that $\mathcal{T}(\{z\}) \cap \mathrm{Int}(W) \neq \emptyset$, then $\mathcal{T}(\{z\}) \subset W$.

Proof. Note that if X is an indecomposable continuum, then $\mathcal{G} = \{X\}$, and the result follows. Hence, assume X is a decomposable continuum, and let W be a subcontinuum of X such that $\mathcal{T}(\{z\}) \cap \mathrm{Int}(W) \neq \emptyset$. Let $x \in \mathcal{T}(\{z\}) \cap \mathrm{Int}(W)$ and suppose that there exists $y \in \mathcal{T}(\{z\}) \setminus W$. Thus, $x \in \mathrm{Int}(W) \subset W \subset X \setminus \{y\}$, i.e., $x \notin \mathcal{T}(\{y\})$. Since $\mathcal{G}$ is a decomposition, $\mathcal{T}(\{x\}) = \mathcal{T}(\{z\}) = \mathcal{T}(\{y\})$, a contradiction. Therefore, $\mathcal{T}(\{z\}) \subset W$. ∎

The proof of the following result (used in the proof of Theorem 3.4) may be found in [8, 2.1]. The theorem was originally proved by E. Dyer.

3.3. Theorem. *Let X and Y be nondegenerate metric continua. If $f\colon X \twoheadrightarrow Y$ is a surjective, monotone and open map, then there exists a dense G_δ subset W of Y having the following property: for each $y \in W$, for each subcontinuum B of $f^{-1}(y)$, for each $x \in \mathrm{Int}_{f^{-1}(y)}(B)$ and for each neighborhood U of B in X, there exist a subcontinuum Z of X containing B and a neighborhood V of y in Y such that $x \in \mathrm{Int}_X(Z)$, $(f|_Z)^{-1}(V) \subset U$ and $f|_Z\colon Z \to Y$ is a monotone surjective map.*

3.4. Theorem. *Let X be a continuum for which $\mathcal{T}_X$ is continuous. If*

$$\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$$

is a continuous decomposition of X, then $X/\mathcal{G}$ is a locally connected continuum and $\mathcal{T}_X(2^X)$ is homeomorphic to $2^{X/\mathcal{G}}$. (In particular, if X is metric, then $\mathcal{T}_X(2^X)$ is homeomorphic to the Hilbert cube.) Moreover, all the elements of $\mathcal{G}$ are nowhere dense in X; and if X is metric, then there exists *a dense G_δ subset $\mathcal{W}$ of $X/\mathcal{G}$ such that if $q(z) \in \mathcal{W}$, then $\mathcal{T}_X(\{z\})$ is an indecomposable continuum, where $q\colon X \twoheadrightarrow X/\mathcal{G}$ is the quotient map.*

Proof. Note that if X is either indecomposable or locally connected, then $\mathcal{G} = \{X\}$ or $\mathcal{T}_X = 1_{2^X}$, respectively, and the theorem follows. Thus, assume X is decomposable and not locally connected.

Since $\mathcal{G}$ is a continuous decomposition of X, $X/\mathcal{G}$ is a continuum [10, Theorem 1, p. 64]. Let $q\colon X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Note that for every $x \in X$, $\mathcal{T}_X(\{x\})$ is a continuum [2, Theorem 4]. Hence, q is a monotone map. Since $\mathcal{G}$ is a continuous decomposition, q is an open map. Let $\chi \in X/\mathcal{G}$. Then, by [2, Theorem 1(e)], $q^{-1}\mathcal{T}_{X/\mathcal{G}}(\{\chi\}) = \mathcal{T}_X(q^{-1}(\chi))$. Let $x \in X$ be such that $\mathcal{T}_X(\{x\}) = q^{-1}(\chi)$. Recall that since $\mathcal{T}_X$ is continuous, $\mathcal{T}_X$ is idempotent [1, Lemma 3]. Thus, $\mathcal{T}_X(q^{-1}(\chi)) = \mathcal{T}_X^2(\{x\}) = \mathcal{T}_X(\{x\})$. Hence, $\mathcal{T}_{X/\mathcal{G}}(\{\chi\}) = qq^{-1}\mathcal{T}_{X/\mathcal{G}}(\{\chi\}) = q\mathcal{T}_X(\{x\}) = \{\chi\}$. Therefore, $X/\mathcal{G}$ is an aposyndetic continuum.

Note that $q^{-1}\mathcal{T}_{X/\mathcal{G}}(\Gamma) = \mathcal{T}_X(q^{-1}(\Gamma))$ for each subset Γ of $X/\mathcal{G}$ [2, Theorem 1 (e)]. Hence, q is a $\mathcal{T}_{XX/\mathcal{G}}$-continuous surjective open map. Since $\mathcal{T}_X$ is continuous, $\mathcal{T}_{X/\mathcal{G}}$ is continuous [1, Theorem 4]. It is known that aposyndetic continua Y for which $\mathcal{T}_Y$ is continuous are locally connected [11, 3.2.16]. Therefore, $X/\mathcal{G}$ is a locally connected continuum.

To see that $\mathcal{T}_X(2^X)$ is homeomorphic to $2^{X/\mathcal{G}}$, let $g\colon 2^{X/\mathcal{G}} \to 2^X$ be given by $g(\Gamma) = q^{-1}(\Gamma)$. By [9, Theorem 2, p. 165], g is continuous. Note that $2^q \circ g = 1_{2^{X/\mathcal{G}}}$. In particular, $g\colon 2^{X/\mathcal{G}} \twoheadrightarrow g(2^{X/\mathcal{G}})$ is a homeomorphism ($2^{X/\mathcal{G}}$ is compact by [10, Theorem 1, p. 45] and 2^X is Hausdorff by [9, Theorem 3, p. 168]). We show that $\mathcal{T}_X(2^X) = g(2^{X/\mathcal{G}})$.

Let $\Gamma \in 2^{X/\mathcal{G}}$. Then $\mathcal{T}_X(g(\Gamma)) = \mathcal{T}_X(q^{-1}(\Gamma)) = q^{-1}\mathcal{T}_{X/\mathcal{G}}(\Gamma) = q^{-1}(\Gamma) = g(\Gamma)$; the second equality is true by [2, Theorem 1(e)], and the second last equality is valid by [2, (b), p. 5]. Thus, $g(\Gamma) \in \mathcal{T}_X(2^X)$ and $g(2^{X/\mathcal{G}}) \subset \mathcal{T}_X(2^X)$.

Let $K \in \mathcal{T}_X(2^X)$. Then there exists $A \in 2^X$ such that $\mathcal{T}_X(A) = K$. We prove that $K = g(q(A))$. Note that $g(q(A)) = q^{-1}(q(A)) = \bigcup\{q^{-1}(q(a)) \mid a \in A\} = \bigcup\{\mathcal{T}_X(\{a\}) \mid a \in A\}$. Since $\mathcal{G}$ is a decomposition, X is point $\mathcal{T}_X$-symmetric. Hence, X is $\mathcal{T}_X$-additive [1, Lemma 9]. Since X is $\mathcal{T}_X$-additive, $\bigcup\{\mathcal{T}_X(\{a\}) \mid a \in A\} = \mathcal{T}_X(A)$ [3, Theorem B]. Thus, $g(q(A)) = \mathcal{T}_X(A) = K$, $K \in g(2^{X/\mathcal{G}})$ and $\mathcal{T}_X(2^X) \subset g(2^{X/\mathcal{G}})$.

Therefore, $\mathcal{T}_X(2^X) = g(2^{X/\mathcal{G}})$. Since $g(2^{X/\mathcal{G}})$ is homeomorphic to $2^{X/\mathcal{G}}$, we see that $\mathcal{T}_X(2^X)$ is homeomorphic to $2^{X/\mathcal{G}}$. If X is metric, then $X/\mathcal{G}$ is a metric continuum [15, 3.10]. Since, in this case, $2^{X/\mathcal{G}}$ is homeomorphic to the Hilbert cube [14, (1.97)], $\mathcal{T}_X(2^X)$ is homeomorphic to the Hilbert cube.

Since q is an open map, all the elements of $\mathcal{G}$ are clearly nowhere dense in X.

Suppose X is metric. Since the quotient map q is surjective, monotone and open, let $\mathcal{W}$ be the dense G_δ subset of $X/\mathcal{G}$ given by Theorem 3.3. Let $\chi \in \mathcal{W}$ and let $z \in X$ be such that $q(z) = \chi$. Suppose $\mathcal{T}_X(\{z\})$ is decomposable. Then there exist two subcontinua H and K of $\mathcal{T}_X(\{z\})$ such that $\mathcal{T}_X(\{z\}) = H \cup K$. Let $x \in H \setminus K$ and let U be an open subset of X such that $H \subset U$ and $K \setminus U \neq \emptyset$. By Theorem 3.3, there exist a subcontinuum Z of X containing H and a neighborhood $\mathcal{V}$ of χ in $X/\mathcal{G}$ such that $x \in \mathrm{Int}_X(Z)$ and $(f|_Z)^{-1}(\mathcal{V}) \subset U$. Since $x \in \mathcal{T}_X(\{z\}) \cap \mathrm{Int}_X(Z)$, by Lemma 3.2, $\mathcal{T}_X(\{z\}) \subset Z$. Observe that this implies that $\mathcal{T}_X(\{z\}) \subset (f|_Z)^{-1}(\mathcal{V}) \subset U$, a contradiction. Therefore, $\mathcal{T}_X(\{z\})$ is indecomposable. ∎

In order to prove the decomposition theorem, we present some needed results and the following definition:

Let X be a continuum, and let $z \in X$. We say that $\mathcal{T}(\{z\})$ *has property BL* provided that $\mathcal{T}(\{z\}) \subset \mathcal{T}(\{x\})$ for each $x \in \mathcal{T}(\{z\})$.

3.5. Lemma. *Let X be a decomposable continuum for which $\mathcal{T}$ is idempotent, and let $z \in X$. If $\mathcal{T}(\{z\})$ has property BL, then $\mathcal{T}(\{x\}) = \mathcal{T}(\{z\})$ for every $x \in \mathcal{T}(\{z\})$. In particular, $\mathcal{T}(\{x\})$ has property BL.*

Proof. Let $z \in X$ be such that $\mathcal{T}(\{z\})$ has property BL, and let $x \in \mathcal{T}(\{z\})$. Since $\mathcal{T}$ is idempotent and $x \in \mathcal{T}(\{z\})$, we have $\mathcal{T}(\{x\}) \subset \mathcal{T}^2(\{z\}) = \mathcal{T}(\{z\})$. Hence, as $\mathcal{T}(\{z\})$ has property BL, we see that $\mathcal{T}(\{z\}) \subset \mathcal{T}(\{x\})$. Therefore, $\mathcal{T}(\{x\}) = \mathcal{T}(\{z\})$. ∎

3.6. Corollary. *Let X be a decomposable continuum for which $\mathcal{T}$ is idempotent. If z_1 and z_2 are two points of X such that $\mathcal{T}(\{z_1\})$ and $\mathcal{T}(\{z_2\})$ have property BL, then either $\mathcal{T}(\{z_1\}) = \mathcal{T}(\{z_2\})$ or $\mathcal{T}(\{z_1\}) \cap \mathcal{T}(\{z_2\}) = \emptyset$.*

Proof. Let z_1 and z_2 be two points of X such that $\mathcal{T}(\{z_1\})$ and $\mathcal{T}(\{z_2\})$ have property BL, and suppose that $\mathcal{T}(\{z_1\}) \cap \mathcal{T}(\{z_2\}) \neq \emptyset$. Let $z_3 \in \mathcal{T}(\{z_1\}) \cap \mathcal{T}(\{z_2\})$. Then, by Lemma 3.5, $\mathcal{T}(\{z_1\}) = \mathcal{T}(\{z_3\}) = \mathcal{T}(\{z_2\})$. ∎

The proof of the following theorem is based on a technique of Bellamy and Lum [4, Lemma 5].

3.7. Theorem. *Let X be a continuum for which $\mathcal{T}$ is idempotent. Then for each $x \in X$, there exists $z \in \mathcal{T}(\{x\})$ such that $\mathcal{T}(\{z\})$ has property BL.*

Proof. First, observe that if X is either indecomposable or locally connected, then $\mathcal{G} = \{X\}$ or $\mathcal{G} = \{\{x\} \mid x \in X\}$, respectively, and the theorem follows. Thus, assume X is decomposable and not locally connected.

Let $x \in X$. Note that, by [11, 3.1.53],

$$\mathcal{T}(\{x\}) = \bigcup \{\mathcal{T}(\{w\}) \mid w \in \mathcal{T}(\{x\})\}.$$

Let $\mathcal{G}_x = \{\mathcal{T}(\{w\}) \mid w \in \mathcal{T}(\{x\})\}$. Partially order $\mathcal{G}_x$ by inclusion. Let

$\{\mathcal{T}(\{w_\lambda\})\}_{\lambda\in\Lambda}$ be a chain of elements of $\mathcal{G}_x$. We show that this chain has a lower bound in $\mathcal{G}_x$.

As $\{\mathcal{T}(\{w_\lambda\})\}_{\lambda\in\Lambda}$ is a chain of continua ([2, Theorem 4]), $\bigcap_{\lambda\in\Lambda}\mathcal{T}(\{w_\lambda\})$ is a nonempty subcontinuum of $\mathcal{T}(\{x\})$. Let $w_0 \in \bigcap_{\lambda\in\Lambda}\mathcal{T}(\{w_\lambda\})$. Since $\mathcal{T}$ is idempotent,

$$\mathcal{T}(\{w_0\}) \subset \bigcap_{\lambda\in\Lambda}\mathcal{T}(\{w_\lambda\}) \subset \mathcal{T}(\{x\}).$$

Hence, by Zorn's lemma, there exists $z \in \mathcal{T}(\{x\})$ such that $\mathcal{T}(\{z\})$ is a minimal element, i.e., each $w \in \mathcal{T}(\{z\})$ satisfies $\mathcal{T}(\{z\}) \subset \mathcal{T}(\{w\})$. Therefore, $\mathcal{T}(\{z\})$ has property BL. ■

The following theorem gives a partial answer to a question of David Bellamy; see Problem 162 of the Houston Problem Book [5, p. 390].

3.8. THEOREM. *Let X be a point $\mathcal{T}_X$-symmetric continuum for which $\mathcal{T}_X$ is continuous. Then*

$$\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$$

is a continuous decomposition of X such that the quotient space $X/\mathcal{G}$ is a locally connected continuum and $\mathcal{T}_X(2^X)$ is homeomorphic to $2^{X/\mathcal{G}}$. (In particular, if X is metric, then $\mathcal{T}_X(2^X)$ is homeomorphic to the Hilbert cube.) Moreover, all the elements of $\mathcal{G}$ are nowhere dense in X; and if X is metric, then there exists a dense G_δ subset $\mathcal{W}$ of $X/\mathcal{G}$ such that if $q(z) \in \mathcal{W}$, then $\mathcal{T}_X(\{z\})$ is an indecomposable continuun, where $q\colon X \twoheadrightarrow X/\mathcal{G}$ is the quotient map.

Proof. Note that if X is either indecomposable or locally connected, then $\mathcal{G} = \{X\}$ or $\mathcal{T}_X = 1_{2^X}$, respectively, and the theorem follows. Thus, assume X is decomposable and not locally connected.

Let x be a point in X. By Theorem 3.7, there exists $z \in \mathcal{T}_X(\{x\})$ such that $\mathcal{T}_X(\{z\})$ has property BL (recall that since $\mathcal{T}_X$ is continuous, $\mathcal{T}_X$ is idempotent [1, Lemma 3]). Since X is point $\mathcal{T}_X$-symmetric and $z \in \mathcal{T}_X(\{x\})$, we have $x \in \mathcal{T}_X(\{z\})$. Thus, since $\mathcal{T}_X(\{z\})$ has property BL, we see that $\mathcal{T}_X(\{x\}) = \mathcal{T}_X(\{z\})$, by Lemma 3.5. Thus, $\mathcal{T}_X(\{x\})$ has property BL. Therefore, by Corollary 3.6,

$$\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$$

is a decomposition of X. Since $\mathcal{T}_X$ is continuous, $\mathcal{G}$ is a continuous decomposition. Now, the theorem follows from Theorem 3.4. ■

Regarding nonaposyndetic homogeneous metric continua, we have the following:

3.9. THEOREM. *If X is a metric homogeneous continuum, then $\mathcal{T}_X(2^X) \subset g(2^{X/\mathcal{G}})$. Moreover, $\mathcal{T}_X(2^X) = g(2^{X/\mathcal{G}})$ if and only if $\mathcal{T}_X$ is continuous.*

Proof. Note that if X is indecomposable, then $\mathcal{G} = \{X\}$. Hence, $X/\mathcal{G}$ is a one-point set and the assertion follows. Also, if X is aposyndetic, then $\mathcal{G} = \{\{x\} \mid x \in X\}$ and $X/\mathcal{G}$ is homeomorphic to X. Thus, the assertion follows as well.

Let X be a nonaposyndetic metric homogeneous continuum. By Jones's aposyndetic decomposition theorem (see [7]), $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is a continuous decomposition of X. Hence, the quotient map is monotone and open. Thus, $g\colon 2^{X/\mathcal{G}} \to 2^X$ given by $g(\Gamma) = q^{-1}(\Gamma)$ is continuous [9, Theorem 2, p. 165]. To show $\mathcal{T}_X(2^X) \subset g(2^{X/\mathcal{G}})$, let $K \in \mathcal{T}_X(2^X)$. Then there exists $Z \in 2^X$ such that $\mathcal{T}_X(Z) = K$. By [13, 3.5], $\mathcal{T}_X(Z) = q^{-1}\mathcal{T}_{X/\mathcal{G}}q(Z) = g(\mathcal{T}_{X/\mathcal{G}}q(Z))$, i.e., $K = g(\mathcal{T}_{X/\mathcal{G}}q(Z))$. Therefore, $\mathcal{T}_X(2^X) \subset g(2^{X/\mathcal{G}})$.

Now we prove that $\mathcal{T}_X(2^X) = g(2^{X/\mathcal{G}})$ if and only if $\mathcal{T}_X$ is continuous for X. If $\mathcal{T}_X$ is continuous for X, then $\mathcal{T}_X(2^X) = g(2^{X/\mathcal{G}})$ by Theorem 3.4. Next, suppose $\mathcal{T}_X(2^X) = g(2^{X/\mathcal{G}})$. Let $\Gamma \in 2^{X/\mathcal{G}}$. Then, by [2, Theorem 1(c)], $\mathcal{T}_{X/\mathcal{G}}(\Gamma) = q\mathcal{T}_X q^{-1}(\Gamma) = q\mathcal{T}_X g(\Gamma)$. Since $g(\Gamma) \in \mathcal{T}_X(2^X)$ and $\mathcal{T}_X$ is idempotent [13, 3.3], we have $\mathcal{T}_X g(\Gamma) = g(\Gamma)$. Hence, $\mathcal{T}_{X/\mathcal{G}}(\Gamma) = \Gamma$. Thus, $X/\mathcal{G}$ is locally connected [2, (b), p. 5] and, by [13, 3.6], $\mathcal{T}_X$ is continuous. ∎

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