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ON THE BLOW-UP PHENOMENON FOR THE MASS-CRITICAL FOCUSING HARTREE EQUATION IN $\mathbb{R}^4$

BY

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Abstract. We characterize the dynamics of the finite time blow-up solutions with minimal mass for the focusing mass-critical Hartree equation with $H^1(\mathbb{R}^4)$ data and $L^2(\mathbb{R}^4)$ data, where we make use of the refined Gagliardo–Nirenberg inequality of convolution type and the profile decomposition. Moreover, we analyze the mass concentration phenomenon of such blow-up solutions.

1. Introduction. In this paper, we consider the Cauchy problem for the Hartree equation

$$\begin{cases} iu_t + \Delta u = f(u) & \text{in } \mathbb{R}^d \times \mathbb{R}, \\ u(0) = u_0(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here $f(u) = \lambda(V * |u|^2)u$, $V(x) = |x|^{-\gamma}, 0 < \gamma < d$, and $*$ denotes the convolution in $\mathbb{R}^d$. If $\lambda > 0$, we call the equation (1.1) *defocusing*; if $\lambda < 0$, we call it *focusing*. This equation describes the mean-field limit of many-body quantum systems; see, e.g., [6], [7] and [36]. An essential feature of the Hartree equation is that the convolution kernel $V(x)$ still retains the fine structure of micro two-body interactions of the quantum system. By contrast, NLS arises in further limiting regimes where two-body interactions are modeled by a single real parameter in terms of the scattering length. In particular, NLS cannot provide effective models for quantum systems with long-range interactions such as the physically important case of the Coulomb potential $V(x) \sim |x|^{-(d-2)}$ in $d \geq 3$, whose scattering length is infinite.

There are many works on the global well-posedness and scattering of equation (1.1). For the defocusing case with $2 < \gamma < \min(4, d)$, J. Ginibre and G. Velo [8] proved the global well-posedness and scattering results in the energy space. Later, K. Nakanishi [32] made use of a new Morawetz estimate to obtain similar results for more general functions $V(x)$. Recently, the present authors have proved the global well-posedness and scattering for

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the defocusing, energy-critical Hartree equation (see [28] and [29]). For the global well-posedness and scattering of the focusing, energy-critical Hartree equation we refer to [17] and [30].

In this paper, we mainly aim to characterize the dynamics of the finite time blow-up solutions with minimal mass for the focusing mass-critical Hartree equation with $H^1(\mathbb{R}^4)$ data and $L^2(\mathbb{R}^4)$ data.

Now we recall the related results about the focusing mass-critical Schrödinger equation

$$iu_t + \Delta u = -|u|^{4/d}u, \qquad u(0) = u_0, \tag{1.2}$$

where d is the spatial dimension. Equation (1.2) is called mass-critical due to scaling invariance. If $u_0 \in H^1$ has radial symmetry, the mass concentration phenomenon for the blow-up solution was observed near the blow-up time in [22]. Later on, the radial symmetry assumption was removed by M. Weinstein [35] and Nawa [33]. For a more detailed analysis of the blow-up dynamics of (1.2), see [20], [21], [24], [25], [26] and the references therein. If u_0 only lies in L^2, the situation seems quite different because we cannot use the energy conservation law. The pioneering work in this direction is due to J. Bourgain [3] for $d = 2$, who proved that there exists a blow-up time T^*,

$$\lim_{t\uparrow T^*} \sup_{\substack{\text{cubes } I\subset\mathbb{R}^2,\\ \text{side}(I)<(T^*-t)^{1/2}}} \Big(\int_I |u(t,x)|^2\,dx\Big)^{1/2} \ge c(\|u_0\|_{L^2_x}) > 0,$$

where $c(\|u_0\|_{L^2_x})$ is a constant depending on the mass of the initial data. A new proof can be found in S. Keraani [12] by means of the profile decomposition in [23]. Bourgain's result was extended to dimension $d = 1$ by R. Carles and S. Keraani [4] and to dimension $d \ge 3$ by P. Bégout and A. Vargas [2]. Recently, R. Killip, T. Tao and M. Visan [13] established global well-posedness and scattering for (1.2) with radial data in dimension two and mass strictly smaller than that of the ground state. Later R. Killip, M. Visan and X. Zhang [14] extended those results to $d \ge 3$. We dealt with the corresponding problem for the Hartree equation in [31].

This paper is devoted to the study of the blow-up behavior of the mass-critical Hartree equation in dimension four:

$$\begin{cases} iu_t + \Delta u = -(|x|^{-2} * |u|^2)u & \text{in } \mathbb{R}^4\times\mathbb{R},\\ u(0) = u_0(x) & \text{in } \mathbb{R}^4. \end{cases} \tag{1.3}$$

The corresponding free equation is

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^4\times\mathbb{R},\\ u(0) = u_0(x) & \text{in } \mathbb{R}^4. \end{cases} \tag{1.4}$$

Note that $\gamma = 2$ is the unique exponent which is mass-critical in the sense that the natural scaling

$$u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x)$$

leaves mass invariant. At the same time, $|x|^{-2}$ is just the physically important case of Coulomb potential for dimension $d = 4$. Moreover, equation (1.3) also has pseudo-conformal symmetry: If $u(t,x)$ solves (1.3), then so does

$$v(t,x) = \frac{1}{|T-t|^2}\,\overline{u}\left(\frac{1}{t-T}, \frac{x}{t-T}\right)e^{i|x|^2/4(t-T)}. \tag{1.5}$$

First we deal with equation (1.3) with data in $H^1(\mathbb{R}^4)$. For the solution $u(t) \in H^1$ of (1.3), the following quantities are conserved:

$$M(u(t)) = \|u(t)\|_{L^2_x} = \|u(0)\|_{L^2_x},$$
$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^4} |\nabla u|^2\,dx - \frac{1}{4}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{|u(x)|^2|u(y)|^2}{|x-y|^2}\,dx\,dy = E(u(0)).$$

According to the local well-posedness theory [5], [27], the solution $u(t) \in H^1(\mathbb{R}^4)$ of (1.3) blows up at finite time T if and only if

$$\lim_{t\to T} \|\nabla u(t)\|_{L^2} = \infty.$$

The blow-up theory is mainly connected with the notion of *ground state*, the unique radial positive solution of the elliptic equation

$$-\Delta Q + Q = (V * |Q|^2)Q. \tag{1.6}$$

The existence of the positive solution is proved by the concentration compactness principle at the beginning of Section 3, which is closely related to a refined Gagliardo–Nirenberg inequality of convolution type,

$$\|u\|_{L^V}^4 \le \frac{2}{\|Q\|_{L^2}^2}\,\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2, \tag{1.7}$$

where the definition of the L^V norm is given in (1.9) below. The radial symmetry of the positive solution can be obtained from [19]. By adapting Lieb's uniqueness proof in [18] for the ground states $\phi \in H^1$ of the Choquard–Pekar equation ($V(x) = |x|^{-1}$ in dimension $d = 3$), the analogous result for (1.6) can be obtained. See details in [15]. However, the uniqueness proof strongly depends on the specific features of equation (1.6). It is different from the corresponding results for semilinear elliptic equations in [16]. As our result (Theorem 1.1) depends on the uniqueness of the ground state of equation (1.6), it is the reason why we consider the case $d = 4$.

Together with the notion of the ground state Q, the invariance (1.5) yields an explicit blow-up solution such that $\|u\|_{L^2} = \|Q\|_{L^2}$. One can ask if there are other finite time blow-up solutions of (1.3) with minimal mass

$\|Q\|_{L^2}$ and how to characterize the dynamics of such blow-up solutions near the blow-up time.

Now, we can characterize the finite time blow-up solutions with minimal mass in $H^1(\mathbb{R}^4)$.

THEOREM 1.1. *Let $u_0 \in H^1(\mathbb{R}^4)$ be such that $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and u be a blow-up solution of* (1.3) *at finite time T. Then there exists $x_0 \in \mathbb{R}^4$ such that $e^{i|x-x_0|^2/4T}u_0 \in \mathcal{A}$, where*

$$\mathcal{A} = \{\rho^2 e^{i\theta} Q(\rho x + y) : y \in \mathbb{R}^4,\ \rho \in \mathbb{R}_*^+,\ \theta \in [0, 2\pi)\}.$$

THEOREM 1.2. *Let u be a solution of* (1.3) *which blows up at finite time $T > 0$ with initial data $u_0 \in H^1(\mathbb{R}^4)$, and $\lambda(t) > 0$ such that $\lambda(t)\|\nabla u\|_{L^2} \to \infty$ as $t \uparrow T$. Then there exists $x(t) \in \mathbb{R}^4$ such that*

$$\liminf_{t\uparrow T} \int\limits_{|x-x(t)|\le\lambda(t)} |u(t,x)|^2\,dx \ge \int\limits_{\mathbb{R}^4} |Q|^2\,dx.$$

The counterpart of Theorem 1.1 for the Schrödinger equation has been established by F. Merle in [21]. The counterpart of Theorem 1.2 was proved by M. Weinstein in [35]. T. Hmidi and S. Keraani gave a direct and simplified proof of the above results in [9]. The new ingredient for the Hartree equation is the refined Gagliardo–Nirenberg inequality (1.7) of convolution type, whose proof is based on the well-known concentration compactness method and thus one has to deal with the intertwining of convolution and orthogonality.

Next we consider the blow-up behavior of (1.3) with L^2 data. In [27], we showed that for any $u_0 \in L^2(\mathbb{R}^4)$, there exists a unique maximal solution u to (1.3), with

$$u \in C((-T_*, T^*), L^2(\mathbb{R}^4)) \cap L^3_{\text{loc}}((-T_*, T^*), L^3(\mathbb{R}^4)),$$

and we have the following alternative: either $T_* = T^* = \infty$ or

$$\min\{T_*, T^*\} < \infty \quad \text{and} \quad \|u\|_{L^3_t((-T_*,T^*),L^3_x)} = \infty.$$

Moreover, there exists $\delta > 0$ such that if

$$\|u_0\|_{L^2} < \delta, \tag{1.8}$$

then the initial value problem (1.3) has a unique global solution $u(t,x) \in L^3_{t,x}(\mathbb{R} \times \mathbb{R}^4)$. We define δ_0 as the supremum of δ in (1.8) such that the global existence for the Cauchy problem (1.3) holds, with $u \in (C\cap L^\infty)(\mathbb{R}, L^2(\mathbb{R}^4)) \cap L^3(\mathbb{R}\times\mathbb{R}^4)$. Then in the ball $B_{\delta_0} := \{u_0 : \|u_0\|_{L^2} < \delta_0\}$, (1.3) admits a complete scattering theory with respect to the associated linear problem. Similar to the focusing mass-critical Schrödinger equation, we also conjecture that δ_0 should be $\|Q\|_{L^2}$ for the Hartree equation. We have verified the conjecture for radial data in [31]. For general data, it remains open.

Definition 1.1. Let $u_0 \in L^2(\mathbb{R}^4)$. A solution of (1.3) is said to be a *blow-up solution* for $t > 0$ if either $T^* < \infty$, or

$$T^* = \infty \quad \text{and} \quad \|u\|_{L^3_t((0,\infty),\, L^3_x)} = \infty,$$

and similarly for $t < 0$.

Now we are in a position to state the existence of blow-up solutions in both time directions with minimal mass in $L^2(\mathbb{R}^4)$.

Theorem 1.3. *There exists an initial data $u_0 \in L^2(\mathbb{R}^4)$ with $\|u_0\|_{L^2} = \delta_0$ for which the solution of* (1.3) *blows up for both $t > 0$ and $t < 0$.*

As a direct consequence of the above theorem and the pseudo-conformal transform (1.5), we obtain the existence of finite time blow-up solutions with minimal mass in $L^2(\mathbb{R}^4)$.

Corollary 1.1. *There exists an initial data $u_0 \in L^2(\mathbb{R}^4)$ with $\|u_0\|_{L^2} = \delta_0$, for which the solution of* (1.3) *blows up at finite time $T^* > 0$.*

Theorem 1.4. *Let u be a blow-up solution of* (1.3) *at finite time $T^* > 0$ such that $\|u_0\|_{L^2} < \sqrt{2}\,\delta_0$. Let $t_n \uparrow T^*$ as $n \to \infty$, and let $\lambda(t) > 0$ be such that*

$$\frac{\sqrt{T^* - t}}{\lambda(t)} \to 0 \qquad \text{as } t \uparrow T^*.$$

Then there exist a subsequence of $\{t_n\}_{n=1}^{\infty}$ (still denoted by $\{t_n\}$) and $x(t) \in \mathbb{R}^4$ with the following properties.

(i) *There exists $\psi \in L^2(\mathbb{R}^4)$ with $\|\psi\|_{L^2} \geq \delta_0$ such that the solution U of* (1.3) *with initial data ψ blows up for both $t > 0$ and $t < 0$.*

(ii) *There exists a sequence $\{\rho_n, \xi_n, x_n\}_{n=1}^{\infty} \subset \mathbb{R}^*_+ \times \mathbb{R}^4 \times \mathbb{R}^4$ such that*

$$\rho_n^2 e^{ix\cdot\xi_n} u(t_n, \rho_n x + x_n) \rightharpoonup \psi \qquad \text{weakly in } L^2.$$

Furthermore,

$$\lim_{n\to\infty} \frac{\rho_n}{\sqrt{T^* - t_n}} \leq \frac{1}{\sqrt{T^{**}}}$$

where T^{**} denotes the lifespan of U.

(iii) *We have*

$$\liminf_{t\uparrow T^*} \int_{|x-x(t)|\leq\lambda(t)} |u(x,t)|^2\, dx \geq \delta_0^2.$$

Corollary 1.2. *Let u be a blow-up solution with minimal mass of* (1.3) *at finite time $T^* > 0$. Let $t_n \uparrow T^*$ as $n \to \infty$. Then there exists a subsequence of $\{t_n\}_{n=1}^{\infty}$ (still denoted by $\{t_n\}_{n=1}^{\infty}$) and $x(t) \in \mathbb{R}^4$ with the following properties:*

(i) *There exists $\psi \in L^2(\mathbb{R}^4)$ with $\|\psi\|_{L^2} \geq \delta_0$ such that the solution U of* (1.3) *with initial data ψ blows up for both $t > 0$ and $t < 0$.*

(ii) *There exists a sequence* $\{\rho_n, \xi_n, x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^* \times \mathbb{R}^4 \times \mathbb{R}^4$ *such that*

$$\rho_n^2 e^{ix\cdot\xi_n} u(t_n, \rho_n x + x_n) \to \psi \qquad \text{strongly in } L^2.$$

Furthermore,

$$\lim_{n\to\infty} \frac{\rho_n}{\sqrt{T^* - t_n}} \le \frac{1}{\sqrt{T^{**}}}$$

where T^{**} *denotes the lifespan of* U.

(iii) *We have*

$$\liminf_{t\uparrow T^*} \int\limits_{|x-x(t)|\le\lambda(t)} |u(x,t)|^2\,dx \ge \delta_0^2.$$

Similar results for the nonlinear Schrödinger equation have appeared in F. Merle and L. Vega [23] and S. Keraani [12]. Since the nonlinearity is non-local for the Hartree equation, we have to introduce a suitable decomposition in physical space to exploit the orthogonality.

We will often use the notations $a \lesssim b$ and $a = O(b)$ to mean that there exists some constant C such that $a \le Cb$. The derivative operator ∇ refers to the derivatives with respect to space variables only. We also occasionally use subscripts to denote the spatial derivatives and use the summation convention over repeated indices.

For $1 \le p \le \infty$, we define the dual exponent p' by $1/p + 1/p' = 1$. For any time interval I, we use $L_t^q L_x^r(I \times \mathbb{R}^4)$ to denote the spacetime Lebesgue norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^4)} := \Big(\int\limits_I \|u\|_{L^r(\mathbb{R}^4)}^q\,dt\Big)^{1/q}$$

with the usual modifications when $q = \infty$. When $q = r$, we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$.

We say that a pair (q, r) is *admissible* if

$$\frac{2}{q} = 4\Big(\frac{1}{2} - \frac{1}{r}\Big), \qquad 2 \le q \le \infty.$$

For a spacetime slab $I \times \mathbb{R}^4$, we define the *Strichartz norms*

$$\|u\|_{\dot S^0(I)} := \sup_{(q,r)\text{ admissible}} \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^4)}, \qquad \|u\|_{\dot S^1(I)} := \|\nabla u\|_{\dot S^0(I)}.$$

We also define $\dot{\mathcal{N}}^0$ to be the Banach dual space of $\dot S^0$.

Throughout this paper, we write

$$\|u\|_{L^V} := \Big(\iint |u(x)|^2 V(x-y)|u(y)|^2\,dx\,dy\Big)^{1/4}. \tag{1.9}$$

The rest of this paper is organized as follows: In Section 2, we recall the preliminary estimates such as Strichartz estimates and the virial identity. In Section 3, we prove Theorems 1.1 and 1.2. Section 4 is devoted to the proof of Theorems 1.3 and 1.4.

2. Preliminaries. We now recall some useful estimates. First, we have the following *Strichartz inequalities*:

LEMMA 2.1 ([5], [10]). *Let u be an $\dot{S}^0(I)$ solution to the Schrödinger equation in* (1.1). *Then*

$$\|u\|_{\dot{S}^0} \lesssim \|u(t_0)\|_{L^2(\mathbb{R}^4)} + \|f(u)\|_{L_t^{q'} L_x^{r'}(I\times\mathbb{R}^4)}$$

for any $t_0 \in I$ and any admissible pair (q,r). The implicit constant is independent of the choice of the interval I.

By definition, it immediately follows that for any function u on $I \times \mathbb{R}^4$,

$$\|u\|_{L_t^\infty L_x^2} + \|u\|_{L^3_{t,x}} \lesssim \|u\|_{\dot{S}^0},$$

where all spacetime norms are taken on $I \times \mathbb{R}^4$.

LEMMA 2.2. *Let $f(u)(t,x) = \pm u(V * |u|^2)(t,x)$, where $V(x) = |x|^{-2}$. For any time interval I and $t_0 \in I$, we have*

$$\Big\| \int_{t_0}^{t} e^{i(t-s)\Delta} f(u)(s,x)ds \Big\|_{\dot{S}^0(I)} \lesssim \|u\|^3_{L^3_{t,x}}.$$

Proof. By the Strichartz estimate, the Hardy–Littlewood–Sobolev inequality and the Hölder inequality, we have

$$\begin{aligned}\Big\| \int_{t_0}^{t} e^{i(t-s)\Delta} f(u)(s,x)\, ds \Big\|_{\dot{S}^0(I)} &\lesssim \|f(u)(t,x)\|_{L_t^1 L_x^2} \lesssim \|V * |u|^2\|_{L_t^{3/2} L_x^6} \|u\|_{L^3_{t,x}} \\ &\lesssim \|u\|^3_{L^3_{t,x}}.\end{aligned}$$

In addition, we can obtain the virial identity appearing in the proof of the localized Morawetz estimates [28]. Indeed, let $V_0^a(t) = \int a(x)|u(t,x)|^2\, dx$, where $a(x)$ is real-valued and u is the solution of (1.1) with $f(u) = -(|x|^{-\gamma} * |u|^2)u$. Then we get

$$M_0^a(t) =: \partial_t V_0^a(t) = 2\Im \int a_j u_j \overline{u}\, dx$$

and

$$\begin{aligned}\partial_t M_0^a(t) =& -2\Im \int a_{jj} u_t \overline{u}\, dx - 4\Im \int a_j \overline{u}_j u_t\, dx \\ =& -\int \Delta\Delta a |u|^2\, dx + 4\Re \int a_{jk} \overline{u}_j u_k\, dx \\ & - \iint (\nabla a(x) - \nabla a(y)) \nabla V(x-y) |u(y)|^2 |u(x)|^2\, dx\, dy.\end{aligned} \tag{2.1}$$

LEMMA 2.3. *If we choose $a(x) = |x|^2$, then*

$$\partial_t M_0^a(t) = 8 \int |\nabla u|^2\, dx - 2\gamma \iint V(x-y)|u(y)|^2 |u(x)|^2\, dx\, dy. \tag{2.2}$$

LEMMA 2.4. *If* $a(x) = |x|^2$ *and* $\gamma = 2$, *we have*

$$\partial_t^2 V_0^a(t) = 16E(u(0)). \tag{2.3}$$

If $E(u(0)) < 0$, *then the nonnegative function* $V_0^a(t)$ *is concave, so the maximal interval of existence is finite. This implies that the solution of* (1.3) *has to blow up in both directions.*

3. The blow-up dynamics of the focusing mass-critical Hartree equation with H^1 data. Let $V(x) = |x|^{-2}$. We study the minimizing functional

$$J := \min\{J(u) : u \in H^1(\mathbb{R}^4)\}, \quad \text{where} \quad J(u) := \frac{\|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2}{\|u\|_{L^V}^4}.$$

First, we have

LEMMA 3.1. *If* W *is a minimizer of* $J(u)$, *then*

$$\Delta W + \alpha(|x|^{-2} * |W|^2)W = \beta W, \tag{3.1}$$

where $\alpha = 2J/\|W\|_{L^2}^2$ *and* $\beta = \|\nabla W\|_{L^2}^2/\|W\|_{L^2}^2$.

REMARK 3.1. If W is a minimizer of $J(u)$, then $|W|$ is also a minimizer. Hence, we can assume that W is positive. In fact, we have

$$-|\nabla W| \le \nabla|W| \le |\nabla W|$$

in the sense of distributions. In particular, $|W| \in H^1$ and $J(|W|) \le J(W)$.

Proof of Lemma 3.1. The minimizing function W is in $H^1(\mathbb{R}^4)$ and satisfies the Euler–Lagrange equation

$$\frac{d}{d\varepsilon} J(W + \varepsilon v)\Big|_{\varepsilon=0} = 0.$$

Equivalently, we have

$$\|\nabla W\|_{L^2}^2 \|W\|_{L^V}^4 \int 2\Re(W\bar{v})\,dx + \|W\|_{L^2}^2 \|W\|_{L^V}^4 \int 2\Re(\nabla W \nabla \bar{v})\,dx$$
$$- \|\nabla W\|_{L^2}^2 \|W\|_{L^2}^2 \Big(\int (V * 2\Re(W\bar{v}))|W|^2\,dx + \int (V * |W|^2) 2\Re(W\bar{v})\,dx \Big) = 0.$$

Since

$$\int (V * 2\Re(W\bar{v}))|W|^2\,dx = \int (V * |W|^2) 2\Re(W\bar{v})\,dx,$$

we have

$$\Delta W + \frac{2J}{\|W\|_{L^2}^2}(V * |W|^2)W = \frac{\|\nabla W\|_{L^2}^2}{\|W\|_{L^2}^2} W.$$

PROPOSITION 3.1. *J is attained at a function u with the following properties:*

$$u(x) = aQ(\lambda x + b) \quad \text{for some } a \in \mathbb{C}^*,\ \lambda > 0, \text{ and any } b \in \mathbb{R}^4,$$

where Q satisfies (1.6). *Moreover,*

$$J = \|Q\|_{L^2}^2/2.$$

We prove this proposition by the following profile decomposition.

LEMMA 3.2 (Profile decomposition [9]). *For a bounded sequence $\{u_n\}_{n=1}^{\infty} \subset H^1(\mathbb{R}^4)$, there is a subsequence of $\{u_n\}_{n=1}^{\infty}$ (still denoted by $\{u_n\}$) and a sequence $\{U^{(j)}\}_{j\geq 1}$ in $H^1(\mathbb{R}^4)$ and for any $j \geq 1$, a family $\{x_n^j\}$ such that:*

(i) *If $j \neq k$, then $|x_n^j - x_n^k| \to \infty$ as $n \to \infty$.*
(ii) *For every $l \geq 1$,*

$$u_n(x) = \sum_{j=1}^{l} U^{(j)}(x - x_n^j) + r_n^l(x), \tag{3.2}$$

where, for any $p \in (2, 4)$,

$$\limsup_{n\to\infty} \|r_n^l\|_{L^p(\mathbb{R}^4)} \to 0 \qquad \text{as } l \to \infty. \tag{3.3}$$

(iii) *We have*

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|U^{(j)}\|_{L^2}^2 + \|r_n^l\|_{L^2}^2 + o_n(1), \tag{3.4}$$

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla U^{(j)}\|_{L^2}^2 + \|\nabla r_n^l\|_{L^2}^2 + o_n(1). \tag{3.5}$$

Proof of Proposition 3.1. Choose a sequence $\{u_n\}_{n=1}^{\infty} \subset H^1(\mathbb{R}^4)$ such that $J(u_n) \to J$. Suppose $\|u_n\|_{L^2} = 1$ and $\|u_n\|_{L^V} = 1$. Then

$$J(u_n) = \int |\nabla u_n|^2 \, dx \to J.$$

Note that $\{u_n\}_{n=1}^{\infty}$ is bounded in H^1, so by Lemma 3.2, we have (3.2)–(3.5). From (3.4) and (3.5), we have

$$\sum_{j=1}^{l} \|U^{(j)}\|_{L^2}^2 \leq 1, \qquad \sum_{j=1}^{l} \|\nabla U^{(j)}\|_{L^2}^2 \leq J. \tag{3.6}$$

Moreover, by the Hölder and Young inequalities, we have

$$\|r_n^l\|_{L^V}^4 \leq \|r_n^l\|_{L^{8/3}}^4.$$

From (3.3), $\limsup_{n\to\infty} \|r_n^l\|_{L^{8/3}} \xrightarrow{l\to\infty} 0$. It follows that

$$\limsup_{n\to\infty} \|r_n^l\|_{L^V} \xrightarrow{l\to\infty} 0.$$

Moreover,

$$\iint \frac{|\sum_{j=1}^{l} U^{(j)}(x-x_n^j)|^2 |\sum_{j=1}^{l} U^{(j)}(y-x_n^j)|^2}{|x-y|^2}\,dx\,dy$$

$$\leq \sum_{j=1}^{l} \iint \frac{|U^{(j)}(x-x_n^j)|^2 |U^{(j)}(y-x_n^j)|^2}{|x-y|^2}\,dx\,dy \tag{3.7}$$

$$+ \sum_{j=1}^{l} \sum_{k\neq j} \iint \frac{|U^{(j)}(x-x_n^j)|\,|U^{(k)}(x-x_n^k)|(\sum_{i=1}^{l} |U^{(i)}(y-x_n^i)|)^2}{|x-y|^2}\,dx\,dy \tag{3.8}$$

$$+ \sum_{j=1}^{l} \sum_{k\neq j} \iint \frac{|U^{(j)}(y-x_n^j)|\,|U^{(k)}(y-x_n^k)|(\sum_{i=1}^{l} |U^{(i)}(x-x_n^i)|)^2}{|x-y|^2}\,dx\,dy \tag{3.9}$$

$$+ \sum_{j=1}^{l} \sum_{k\neq j} \iint \frac{|U^{(j)}(x-x_n^j)|^2 |U^{(k)}(y-x_n^k)|^2}{|x-y|^2}\,dx\,dy. \tag{3.10}$$

Without loss of generality we can assume that all $U^{(j)}$'s are continuous and compactly supported. Then

$$(3.7) = \sum_{j=1}^{l} \iint \frac{|U^{(j)}(x)|^2 |U^{(j)}(y)|^2}{|x-y|^2}\,dx\,dy,$$

and by orthogonality, we have

$$(3.8) \leq \sum_{i=1}^{l} \sum_{j=1}^{l} \sum_{k\neq j} \|U^{(i)}(y-x_n^i)\|_{L^{8/3}}^2 \|U^{(j)}(\cdot-x_n^j)U^{(k)}(\cdot-x_n^k)\|_{L^{4/3}} \to 0$$

as $n\to\infty$. (3.9) can be similarly estimated. Finally,

$$\begin{aligned}(3.10) &= \sum_{j=1}^{l} \sum_{k\neq j} \iint \frac{|U^{(j)}(x)|^2 |U^{(k)}(y)|^2}{|x-y-x_n^j+x_n^k|^2}\,dx\,dy \\ &\leq \sum_{j=1}^{l} \sum_{k\neq j} \frac{C}{|x_n^j-x_n^k|^2} \|U^{(j)}\|_{L^2}^2 \|U^{(k)}\|_{L^2}^2 \to 0, \qquad n\to\infty.\end{aligned}$$

Therefore, we conclude

$$\Big\|\sum_{j=1}^{l} U^{(j)}(x-x_n^j)\Big\|_{L^V}^4 \to \sum_{j=1}^{l} \|U^{(j)}\|_{L^V}^4 \quad \text{as } n\to\infty.$$

Thus, we have

$$\lim_{l\to\infty} \sum_{j=1}^{l} \|U^{(j)}\|_{L^V}^4 = 1.$$

By the definition of J, we have

$$J\|U^j\|_{L^V}^4 \le \|U^{(j)}\|_{L^2}^2\|\nabla U^{(j)}\|_{L^2}^2.$$

So we get

$$J\sum_{j=1}^{l}\|U^j\|_{L^V}^4 \le \sum_{j=1}^{l}\|U^{(j)}\|_{L^2}^2\|\nabla U^{(j)}\|_{L^2}^2.$$

On the other hand,

$$\sum_{j=1}^{l}\|U^{(j)}\|_{L^2}^2\|\nabla U^{(j)}\|_{L^2}^2 \le \sum_{j=1}^{l}\|U^{(j)}\|_{L^2}^2\sum_{j=1}^{l}\|\nabla U^{(j)}\|_{L^2}^2 \le J.$$

Thus we conclude that only one term $U^{(j_0)}$ is nonzero, i.e.

$$\|U^{(j_0)}\|_{L^2}=1,\quad \|U^{(j_0)}\|_{L^V}=1,\quad \|\nabla U^{(j_0)}\|_{L^2}^2=J. \tag{3.11}$$

This shows that $U^{(j_0)}$ is a minimizer of $J(u)$. From (3.11), we have

$$\Delta U^{(j_0)}+2J(|x|^{-2}*|U^{(j_0)}|^2)U^{(j_0)}=JU^{(j_0)}.$$

By Remark 3.1, we can assume that U^{j_0} is positive. Let $U^{(j_0)}=aQ(\lambda x+b)$, where Q is the positive solution of (1.6). An easy computation gives that $\lambda^2=2a^2=J$.

Next we compute the best constant J in terms of Q. Multiplying (1.6) by Q and integrating both sides of the resulting equation, we have

$$-\int|\nabla Q|^2\,dx+\int(V*|Q|^2)|Q|^2\,dx=\int|Q|^2\,dx. \tag{3.12}$$

Since

$$\int(x\cdot\nabla Q)Q\,dx=-2\int|Q|^2\,dx,$$

$$\int x\cdot\nabla Q\Delta Q\,dx=-\sum_{i,j}\int(\delta_{ij}\partial_iQ\partial_jQ+x_i\partial_i\partial_jQ\partial_jQ)=\|\nabla Q\|_{L^2}^2,$$

and

$$\int x\cdot\nabla Q(V*|Q|^2)Q\,dx=\frac{1}{2}\int x\cdot\nabla Q^2(V*|Q|^2)\,dx$$

$$=\frac{1}{2}\int x\cdot\nabla((V*|Q|^2)Q^2)\,dx-\frac{1}{2}\int x\cdot(\nabla V*Q^2)Q^2\,dx$$

$$=-2\int(V*|Q|^2)Q^2\,dx+\iint\frac{x\cdot(x-y)}{|x-y|^4}Q(x)^2Q(y)^2\,dx\,dy=-\frac{3}{2}\|Q\|_{L^V}^4,$$

we have

$$\|\nabla Q\|_{L^2}^2-\frac{3}{2}\|Q\|_{L^V}^4=-2\|Q\|_{L^2}^2.$$

Together with (3.12), this yields $\|\nabla Q\|_{L^2}^2 = \|Q\|_{L^2}^2$. So,

$$J = \|\nabla U^{(j_0)}\|_{L^2}^2 = \|Q\|_{L^2}^2/2.$$

So far, we have obtained the existence of a positive solution of (1.6). In addition, Theorem 3 of [15] together with Theorem 1.2 of [19] implies that this positive solution is also radially symmetric and unique in $H^1(\mathbb{R}^4)$. Note that the uniqueness proof strongly depends on the specific features of equation (1.6). In fact, the uniqueness of the ground state Q of (1.6) has not be resolved completely for the general potential $V(x)$, and is stated as an open problem in [6].

We first make use of the ground state Q to give a sufficient condition for the global existence of (1.3), which together with (1.5) implies that $\|Q\|_{L^2}$ is the minimal mass of blow-up solutions.

THEOREM 3.1. *If* $u_0 \in H^1(\mathbb{R}^4)$ *and* $\|u_0\|_{L^2} < \|Q\|_{L^2}$, *then the solution* $u(t)$ *of* (1.3) *is global in time.*

Proof. By the local well-posedness theory, it suffices to prove that for every $t \in \mathbb{R}$, we have

$$\|\nabla u(t)\|_{L^2} < \infty.$$

Now from Proposition 3.1 and the conservation of mass, we have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\int |\nabla u(t)|^2\,dx - \frac{1}{4}\int (V * |u(t)|^2)|u(t)|^2\,dx \\ &\geq \frac{1}{2}\,\|\nabla u(t)\|_{L^2}^2 - \frac{1}{4}\,\frac{2}{\|Q\|_{L^2}^2}\,\|u(t)\|_{L^2}^2\|\nabla u(t)\|_{L^2}^2 \\ &= \frac{1}{2}\,\|\nabla u(t)\|_{L^2}^2\left(1 - \frac{\|u_0\|_{L^2}^2}{\|Q\|_{L^2}^2}\right). \end{aligned} \tag{3.13}$$

Since $\|u_0\|_{L^2} < \|Q\|_{L^2}$, we have the uniform bound of $\|\nabla u(t)\|_{L^2}^2$. This proves the global existence.

Before we prove Theorem 1.1, we state a proposition in two equivalent forms.

PROPOSITION 3.2 (Static version). *If* $u \in H^1(\mathbb{R}^4)$ *is such that* $\|u\|_{L^2} = \|Q\|_{L^2}$ *and* $E(u) = 0$, *then*

$$u(x) = e^{i\theta}\lambda^2 Q(\lambda x + b) \quad \textit{for some } \theta \in \mathbb{R},\ \lambda > 0,\ b \in \mathbb{R}^4.$$

Proof. Since $E(u) = 0$, we have $\|\nabla u\|_{L^2}^2 = \frac{1}{2}\|u\|_{L^V}^4$. So we get

$$J(u) = \frac{\|Q\|_{L^2}^2\|\nabla u\|_{L^2}^2}{\|u\|_{L^V}^4} = \frac{1}{2}\,\|Q\|_{L^2}^2 = J.$$

By Proposition 3.1 and the uniqueness of the ground state Q, u is of the form $u(x) = aQ(\lambda x + b)$. The condition $\|u\|_{L^2} = \|Q\|_{L^2}$ ensures that $|a| = \lambda^2$. So $u(x) = e^{i\theta}\lambda^2 Q(\lambda x + b)$.

Proposition 3.3 (Dynamic version). *Let* $\{u_n\}_{n=1}^{\infty}$ *be a sequence in* $H^1(\mathbb{R}^4)$ *such that* $\|u_n\|_{L^2} = \|Q\|_{L^2}$, $E(u_n) \le M$ *and* $\|\nabla u_n\|_{L^2} \to \infty$. *Define*

$$\lambda_n := \frac{\|\nabla u_n\|_{L^2}}{\|\nabla Q\|_{L^2}}.$$

Then there exists a subsequence (still denoted by $\{u_n\}$*), a sequence* $\{y_n\} \subset \mathbb{R}^4$ *and a real number* θ *such that*

$$e^{i\theta}\lambda_n^{-2}u_n(\lambda_n^{-1}x + y_n) \to Q(x) \quad \textit{strongly in } H^1. \tag{3.14}$$

Proof. Let

$$\tilde{u}_n(x) = \frac{1}{\lambda_n^2}\, u_n\left(\frac{x}{\lambda_n}\right).$$

Then $\|\tilde{u}_n\|_{L^2} = \|Q\|_{L^2}$ and $\|\nabla\tilde{u}_n\|_{L^2} = \|\nabla Q\|_{L^2}$. Moreover,

$$E(\tilde{u}_n) = E(u_n)/\lambda_n^2 \to 0 \quad \text{as } n \to \infty.$$

So we have

$$J(\tilde{u}_n) = \|Q\|_{L^2}^2 \frac{\|\nabla\tilde{u}_n\|_{L^2}^2}{\|\tilde{u}_n\|_{L^V}^4} = \|Q\|_{L^2}^2 \frac{\|\nabla\tilde{u}_n\|_{L^2}^2}{2\|\nabla\tilde{u}_n\|_{L^2}^2 - 4E(\tilde{u}_n)} \;\to\; \frac{\|Q\|_{L^2}^2}{2} = J$$

as $n \to \infty$. Therefore, by Lemma 3.2, we can choose a subsequence $\{\tilde{u}_n\}$ and $\{x_n\} \subset \mathbb{R}^4$ such that $\tilde{u}_n(x + x_n) \to aQ(\lambda x + b)$ in H^1. The conditions $\|\tilde{u}_n\|_{L^2} = \|Q\|_{L^2}$ and $\|\nabla\tilde{u}_n\|_{L^2} = \|\nabla Q\|_{L^2}$ imply $|a| = \lambda = 1$, so we have (3.14) for $y_n = \lambda_n^{-1}(x_n - b)$.

In order to prove Theorem 1.1, we also need the following lemma. The proof relies heavily on the techniques of V. Banica [1].

Lemma 3.3. *Suppose* $u \in H^1(\mathbb{R}^4)$ *and* $\|u\|_{L^2} = \|Q\|_{L^2}$. *Then for all real functions* $w \in C^1$ *with bounded* ∇w, *we have*

$$\Big|\int_{\mathbb{R}^4} \nabla w(x)\Im(u\nabla u)(x)\,dx\Big| \le \sqrt{2}\,E(u)^{1/2}\Big(\int |u|^2|\nabla w|^2\,dx\Big)^{1/2}.$$

Proof. Since

$$\|ue^{isw(x)}\|_{L^2} = \|u\|_{L^2} = \|Q\|_{L^2}$$

for any $s \in \mathbb{R}$, by (3.13) we know that $E(ue^{isw(x)}) \ge 0$. So, for any s,

$$\frac{1}{2}\int_{\mathbb{R}^4} |\nabla u + isu\nabla w|^2\,dx - \frac{1}{4}\int_{\mathbb{R}^4} (V * |u|^2)|u|^2\,dx \ge 0.$$

Hence

$$E(u) + s\int_{\mathbb{R}^4} \nabla w\Im(u\nabla u)\,dx + \frac{s^2}{2}\int_{\mathbb{R}^4} |u|^2|\nabla w|^2\,dx \ge 0.$$

As this holds for any s, the discriminant is nonpositive. Hence we get the result.

Now we turn to the proof of Theorems 1.1 and 1.2, which is borrowed from [9].

Proof of Theorem 1.1. Suppose $u(t,x)$ is the solution of (1.3) which blows up at T and let $t_n \uparrow T$. Let $u_n = u(t_n)$. By Proposition 3.3,

$$e^{i\theta}\lambda_n^{-2}u_n(\lambda_n^{-1}x + y_n) \to Q(x) \quad \text{strongly in } H^1.$$

From this we get

$$|u(t_n,x)|^2\,dx - \|Q\|_{L^2}^2\delta_{x=y_n} \rightharpoonup 0 \tag{3.15}$$

where $y_n \to 0$ (up to translation) or $y_n \to \infty$.

Now let $\phi \in C_0^\infty(\mathbb{R}^4)$ be a nonnegative radial function such that

$$\phi(x) = |x|^2 \quad \text{if } |x| < 1 \quad \text{and} \quad |\nabla\phi|^2 \le C\phi(x).$$

For every $p \in \mathbb{N}^*$ we define

$$\phi_p(x) = p^2\phi(x/p) \quad \text{and} \quad g_p(t) = \int \phi_p(x)|u(t,x)|^2\,dx.$$

By Lemma 3.3, for every $t \in [0,T)$, we have

$$|\dot{g}_p(t)| = 2\Big|\int_{\mathbb{R}^4} \nabla\phi_p(x)\Im(u\nabla u)(x)\,dx\Big| \le 2\sqrt{2}E(u_0)^{1/2}\Big(\int|u|^2|\nabla\phi_p(x)|^2\,dx\Big)^{1/2}$$
$$\le CE(u_0)^{1/2}\Big(\int|u|^2\phi_p(x)\,dx\Big)^{1/2} \le C(u_0)\sqrt{g_p(t)}.$$

Integrating with respect to t, we get

$$|\sqrt{g_p(t)} - \sqrt{g_p(t_n)}| \le C(u_0)|t_n - t|.$$

If $y_n \to 0$, then $g_p(t_n) \to \|Q\|_{L^2}^2\phi_p(0) = 0$ by (3.15); if $|y_n| \to \infty$, also $g_p(t_n) \to 0$ since ϕ_p is compactly supported. So, if we let $n \to \infty$, we have

$$g_p(t) \le C(u_0)(T-t)^2.$$

Now fix $t \in [0,T)$ and let $p \to \infty$. Then by (2.3) we get

$$8t^2E(e^{i|x|^2/4t}u_0) = \int|x|^2|u(t,x)|^2\,dx \le C(u_0)(T-t)^2. \tag{3.16}$$

Hence $|y_n|^2\|Q\|_{L^2}^2 \le C(u_0)T^2$. Thus y_n cannot go to infinity. This implies that $\{y_n\}$ converges to 0. Letting t go to T, from (3.16) we get $E(e^{i|x|^2/4T}u_0) = 0$. Note also that $\|e^{i|x|^2/4T}u_0\|_{L^2} = \|Q\|_{L^2}$. By Proposition 3.2, we conclude that $e^{i|x|^2/4T}u_0 \in \mathcal{A}$.

Proof of Theorem 1.2. We define

$$\rho(t) = \|\nabla Q\|_{L^2}/\|\nabla u\|_{L^2} \quad \text{and} \quad v(t,x) = \rho^2u(t,\rho x).$$

Let $t_n \uparrow T$, and set $v_n(x) = v(t_n,x)$. Then by mass conservation and the definition of $\rho(t)$, we have

$$\|v_n\|_{L^2} = \|u_0\|_{L^2} \quad \text{and} \quad \|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}.$$

Since u blows up at time T, we have $\rho(t_n) \to 0$ as $t_n \to T$. Hence

$$E(v_n) = \rho_n^2 E(u_0) \to 0 \quad \text{ as } n \to \infty.$$

In particular,

$$\|v_n\|_{L^V}^4 \to 2\|\nabla Q\|_{L^2}^2 \quad \text{ as } n \to \infty.$$

According to Lemma 3.2, the sequence $\{v_n\}_{n=1}^{\infty}$ can be written, up to a subsequence, as

$$v_n(x) = \sum_{j=1}^{l} U^{(j)}(x - x_n^j) + r_n^l(x)$$

so that (3.3)–(3.5) hold. This implies, in particular, that

$$2\|\nabla Q\|_{L^2}^2 \le \limsup_{n\to\infty} \|v_n\|_{L^V}^4 = \limsup_{n\to\infty} \Big\| \sum_{j=1}^{\infty} U^j(\cdot - x_n^j) \Big\|_{L^V}^4.$$

As in the proof of Proposition 3.1, the pairwise orthogonality of the family $\{x^j\}_{j=1}^{\infty}$, together with (1.6) and (3.5), gives

$$\begin{aligned}
2\|\nabla Q\|_{L^2}^2 &\le \sum_{j=1}^{\infty} \|U^j\|_{L^V}^4 \le \sum_{j=1}^{\infty} \frac{2}{\|Q\|_{L^2}^2} \|U^j\|_{L^2}^2 \|\nabla U^j\|_{L^2}^2 \\
&\le \frac{2}{\|Q\|_{L^2}^2} \sup_{j\ge1} \|U^j\|_{L^2}^2 \sum_{j=1}^{\infty} \|\nabla U^j\|_{L^2}^2 \le \frac{2}{\|Q\|_{L^2}^2} \|\nabla v_n\|_{L^2}^2 \sup_{j\ge1} \|U^j\|_{L^2}^2 \\
&= \frac{2}{\|Q\|_{L^2}^2} \|\nabla Q\|_{L^2}^2 \sup_{j\ge1} \|U^j\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\sup_{j\ge1} \|U^j\|_{L^2}^2 \ge \|Q\|_{L^2}^2.$$

Since $\sum \|U^j\|_{L^2}^2$ converges, the supremum above is attained. In particular, there exists j_0 such that

$$\|U^{j_0}\|_{L^2}^2 \ge \|Q\|_{L^2}^2.$$

On the other hand, a change of variables gives

$$v_n(x + x_n^{j_0}) = U^{j_0}(x) + \sum_{\substack{1\le j\le l \\ j\ne j_0}} U^j(x + x_n^{j_0} - x_n^j) + \tilde{r}_n^l(x),$$

where $\tilde{r}_n^l(x) = r_n^l(x + x_n^{j_0})$. The pairwise orthogonality of the family $\{x^j\}_{j=1}^{\infty}$ implies $U^j(\cdot + x_n^{j_0} - x_n^j) \rightharpoonup 0$ weakly for every $j \ne j_0$. Hence we get

$$r_n(\cdot + x_n^{j_0}) \rightharpoonup U^{j_0} + \tilde{r}^l,$$

where $\tilde{r}^l$ denotes the weak limit of $\{\tilde{r}_n^l\}_{n=1}^\infty$. However,

$$\|\tilde{r}^l\|_{L^V} \leq \limsup_{n\to\infty} \|\tilde{r}_n^l\|_{L^V} = \limsup_{n\to\infty} \|r_n^l\|_{L^V} \xrightarrow{l\to\infty} 0.$$

By uniqueness of the weak limit, we get $\tilde{r}^l = 0$ for every $l \neq j_0$ so that $r_n(\cdot + x_n^{j_0}) \rightharpoonup U^{j_0}$ in H^1, that is,

$$\rho_n^2 u(t_n, \rho_n \cdot + x_n^{j_0}) \rightharpoonup U^{j_0} \in H^1 \quad \text{weakly.}$$

Thus for every $A > 0$,

$$\liminf_{n\to\infty} \int_{|x|\leq A} \rho_n^4 |u(t_n, \rho_n x + x_n)|^2 \, dx \geq \int_{|x|\leq A} |U^{j_0}|^2 \, dx.$$

In view of the assumption $\lambda(t_n)/\rho_n \to \infty$, this gives immediately

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq\lambda(t_n)} |u(t_n, x)|^2 \, dx \geq \int_{|x|\leq A} |U^{j_0}|^2 \, dx$$

for every $A > 0$, which means that

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq\lambda(t_n)} |u(t_n, x)|^2 \, dx \geq \int |U^{j_0}|^2 \, dx \geq \int |Q|^2 \, dx.$$

Since the sequence $\{t_n\}_{n=1}^\infty$ is arbitrary, we infer

$$\liminf_{t\to T} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq\lambda(t)} |u(t, x)|^2 \, dx \geq \int |Q|^2 \, dx.$$

But for every $t \in [0, T)$, the function $y \mapsto \int_{|x-y|\leq\lambda(t)} |u(t,x)|^2 \, dx$ is continuous and goes to 0 at infinity. As a result, we get

$$\sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq\lambda(t)} |u(t, x)|^2 \, dx = \int_{|x-x(t)|\leq\lambda(t)} |u(t, x)|^2 \, dx$$

for some $x(t) \in \mathbb{R}^4$, and Theorem 1.2 is proved.

4. The blow-up dynamics of the focusing mass-critical Hartree equation with L^2 data. In this section we prove Theorems 1.3 and 1.4.

DEFINITION 4.1. For every sequence $\mathbf{\Gamma}_n = \{\rho_n, t_n, \xi_n, x_n\}_{n=1}^\infty \subset \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4$, we define the isometric operator $\mathbf{\Gamma}_n$ on $L^3_{t,x}(\mathbb{R} \times \mathbb{R}^4)$ by

$$\mathbf{\Gamma}_n(f)(t, x) = \rho_n^2 e^{ix\cdot\xi_n} e^{-it|\xi_n|^2} f(\rho_n^2 t + t_n, \rho_n(x - t\xi_n) + x_n).$$

Two sequences $\mathbf{\Gamma}_n^j = \{\rho_n^j, t_n^j, \xi_n^j, x_n^j\}_{n=1}^\infty$ and $\mathbf{\Gamma}_n^k = \{\rho_n^k, t_n^k, \xi_n^k, x_n^k\}_{n=1}^\infty$ are said to be *orthogonal* if

$$\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} \to \infty$$

or

$$\rho_n^j = \rho_n^k \quad and \quad \frac{|\xi_n^j - \xi_n^k|}{\rho_n^j} + |t_n^j - t_n^k| + \left| \frac{\xi_n^j - \xi_n^k}{\rho_n^j} t_n^j + x_n^j - x_n^k \right| \to \infty.$$

LEMMA 4.1 (Linear profile decomposition [2]). *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^4)$. Then there exists a subsequence of $\{\varphi_n\}_{n=1}^\infty$ (still denoted by $\{\varphi_n\}_{n=1}^\infty$) with the following properties: there exists a family $\{V^j\}_{j=1}^\infty$ of solutions of* (1.4) *and a family of pairwise orthogonal sequences $\mathbf{\Gamma}^j = \{\rho_n^j, t_n^j, \xi_n^j, x_n^j\}_{n=1}^\infty$ such that for every $(t,x) \in \mathbb{R}\times\mathbb{R}^4$,*

$$e^{it\Delta}\varphi_n(x) = \sum_{j=1}^{l} \mathbf{\Gamma}_n^j V^j(t,x) + w_n^l(t,x) \tag{4.1}$$

with

$$\limsup_{n\to\infty} \|w_n^l\|_{L^3(\mathbb{R}\times\mathbb{R}^4)} \to 0 \qquad as\ l \to \infty. \tag{4.2}$$

Moreover, for every $l \geq 1$,

$$\|\varphi_n\|_{L^2}^2 = \sum_{j=1}^{l} \|V^j\|_{L^2}^2 + \|w_n^l\|_{L^2}^2 + o_n(1). \tag{4.3}$$

DEFINITION 4.2. Let $\mathbf{\Gamma}_n = \{\rho_n, t_n, \xi_n, x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4$ such that $\{t_n\}_{n=1}^\infty$ has a limit in $[-\infty, \infty]$ as $n \to \infty$. Let V be a solution of the linear Schrödinger equation (1.4). We say that U is the *nonlinear profile associated* to $\{V, \mathbf{\Gamma}_n\}_{n=1}^\infty$ if U is the unique maximal solution of (1.3) satisfying

$$\|(U - V)(t_n, \cdot)\|_{L^2(\mathbb{R}^4)} \to 0 \qquad \text{as } n \to \infty.$$

In order to prove Theorems 1.3 and 1.4, we first state a key theorem, which is similar to that in [11] and [12].

THEOREM 4.1 (Nonlinear profile decomposition). *Let $\{\varphi_n\}_{n=1}^\infty$ be a bounded family in $L^2(\mathbb{R}^4)$ and $\{u_n\}_{n=1}^\infty$ the corresponding family of solutions to* (1.3) *with initial data $\{\varphi_n\}_{n=1}^\infty$. Let $\{V^j, \mathbf{\Gamma}_n^j\}_{j=1}^\infty$ be the family of linear profiles associated to $\{\varphi_n\}_{j=1}^\infty$ via Lemma 4.1 and $\{U^j\}_{j=1}^\infty$ the family of nonlinear profiles associated to $\{V^j, \mathbf{\Gamma}_n^j\}_{j=1}^\infty$ via Definition 4.2. Let $\{I_n\}_{n=1}^\infty$ be a family of intervals containing the origin 0. Then the following statements are equivalent:*

(i) *For every $j \geq 1$,*

$$\lim_{n\to\infty} \|\mathbf{\Gamma}_n^j U^j\|_{L^3_{t,x}[I_n]} < \infty.$$

(ii) *We have*

$$\lim_{n\to\infty} \|u_n\|_{L^3_{t,x}[I_n]} < \infty.$$

Moreover, if (i) *or* (ii) *holds, then*

$$u_n = \sum_{j=1}^{l} \mathbf{\Gamma}_n^j U^j + w_n^l + r_n^l, \tag{4.4}$$

where w_n^l *is as in* (4.2) *and*

$$\lim_{n\to\infty} (\|r_n^l\|_{L^3_{t,x}[I_n]} + \sup_{t\in I_n} \|r_n^l\|_{L^2}) \to 0 \quad \text{as } l \to \infty. \tag{4.5}$$

Proof. Step 1. We prove (4.4) and (4.5) provided that (i) or (ii) holds. Let

$$r_n^l = u_n - \sum_{j=1}^{l} U_n^j - w_n^l, \quad \text{where} \quad U_n^j := \mathbf{\Gamma}_n^j U^j,$$

and let $V_n^j := \mathbf{\Gamma}_n^j V^j$. Then r_n^l satisfies the equation

$$\begin{cases} i\partial_t r_n^l + \Delta r_n^l = f_n^l, \\ r_n^l(0) = \sum_{j=1}^{l} (V_n^j - U_n^j)(0, x), \end{cases} \tag{4.6}$$

where

$$f_n^l := p(W_n^l + w_n^l + r_n^l) - \sum_{j=1}^{l} p(U_n^j),$$

$$p(z) := -(|x|^{-2} * |z|^2)z, \qquad W_n^l := \sum_{j=1}^{l} U_n^j.$$

It suffices to prove that

$$\lim_{n\to\infty} (\|r_n^l\|_{L^3_{t,x}[I_n]} + \sup_{t\in I_n} \|r_n^l\|_{L^2}) \xrightarrow{l\to\infty} 0. \tag{4.7}$$

By the Strichartz estimates and the Young inequality, we have

$$\|r_n^l\|_{L^3_{t,x}[I_n]} + \sup_{t\in I_n} \|r_n^l\|_{L^2} \lesssim \Big\| p(W_n^l + w_n^l + r_n^l) - \sum_{j=1}^{l} p(U_n^j) \Big\|_{\dot{\mathcal{N}}^0[I_n]} + \|r_n^l(0,\cdot)\|_{L^2}$$

$$\lesssim \; - \Big\| p(W_n^l) - \sum_{j=1}^{l} p(U_n^j) \Big\|_{\dot{\mathcal{N}}^0[I_n]} \tag{4.8}$$

$$+ \|p(W_n^l + w_n^l) - p(W_n^l)\|_{L_t^1 L_x^2[I_n]} \tag{4.9}$$

$$+ \|p(W_n^l + w_n^l + r_n^l) - p(W_n^l + w_n^l)\|_{L_t^1 L_x^2[I_n]} \tag{4.10}$$

$$+ \|r_n^l(0,\cdot)\|_{L^2}.$$

We will estimate the three terms. First, we estimate (4.8) from above by

$$\sum_{j_1=1}^{l}\sum_{j_2\neq j_1}\|(|x|^{-2}*|U_n^{j_1}|^2)U_n^{j_2}\|_{L^{3/2}_{t,x}[I_n]} \tag{4.11}$$

$$+\sum_{j_1=1}^{l}\sum_{j_2\neq j_1}\sum_{j_3=1}^{l}\|(|x|^{-2}*(U_n^{j_1}U_n^{j_2}))U_n^{j_3}\|_{L^1_tL^2_x[I_n]}. \tag{4.12}$$

Without loss of generality we can assume that both U^{j_1} and U^{j_2} have compact support in t and x. Let $V(x)=|x|^{-2}$. Then

$$\begin{aligned}
&\iint |(V*|U_n^{j_1}|^2)U_n^{j_2}|^{3/2}\,dx\,dt\\
&=\iint\Big|\int(\rho_n^{j_1})^4|U^{j_1}((\rho_n^{j_1})^2t+t_n^{j_1},\rho_n^{j_1}(x-y-t\xi_n^{j_1})+x_n^{j_1})|^2V(y)\,dy\\
&\qquad\qquad\times(\rho_n^{j_2})^2U^{j_2}((\rho_n^{j_2})^2t+t_n^{j_2},\rho_n^{j_2}(x-t\xi_n^{j_2})+x_n^{j_2})\Big|^{3/2}dx\,dt\\
&=\left(\frac{\rho_n^{j_2}}{\rho_n^{j_1}}\right)^3\iint\Big|\int|U^{j_1}(\tilde t,\tilde x-\tilde y)|^2V(\tilde y)\,d\tilde y\,U^{j_2}\bigg(\left(\frac{\rho_n^{j_2}}{\rho_n^{j_1}}\right)^2\tilde t-\left(\frac{\rho_n^{j_2}}{\rho_n^{j_1}}\right)^2t_n^{j_1}+t_n^{j_2},\\
&\qquad\frac{\rho_n^{j_2}}{\rho_n^{j_1}}\tilde x+\frac{\rho_n^{j_2}(\xi_n^1-\xi_n^2)}{(\rho_n^{j_1})^2}\tilde t-\frac{\rho_n^{j_2}(\xi_n^{j_1}-\xi_n^{j_2})}{(\rho_n^{j_1})^2}t_n^{j_1}-\frac{\rho_n^{j_2}x_n^{j_1}}{\rho_n^{j_1}}+x_n^{j_2}\bigg)\Big|^{3/2}d\tilde x\,d\tilde t.
\end{aligned}$$

If $\rho_n^{j_2}/\rho_n^{j_1}+\rho_n^{j_1}/\rho_n^{j_2}\to\infty$ or $|t_n^{j_1}-t_n^{j_2}|\to\infty$, by the compact support assumption on t, we conclude that the quantity (4.11) converges to 0 as $n\to\infty$. Otherwise, by orthogonality we have

$$\frac{|\xi_n^{j_1}-\xi_n^{j_2}|}{\rho_n^{j_1}}+\left|\frac{\xi_n^{j_1}-\xi_n^{j_2}}{\rho_n^{j_1}}t_n^{j_1}+x_n^{j_1}-x_n^{j_2}\right|\to\infty. \tag{4.13}$$

Without loss of generality, we may assume that $\rho_n^{j_2}/\rho_n^{j_1}\to 1$. Then the complicated expression of the function U^{j_2} of $\tilde t$ and $\tilde x$ can be simplified to

$$U^{j_2}\bigg(\tilde t-t_n^{j_1}+t_n^{j_2},\frac{\xi_n^{j_1}-\xi_n^{j_2}}{\rho_n^{j_1}}\tilde t+\tilde x-x_n^{j_1}+x_n^{j_2}-\frac{\xi_n^{j_1}-\xi_n^{j_2}}{\rho_n^{j_1}}t_n^{j_1}\bigg).$$

Meanwhile, we have

$$\begin{aligned}
\int|U^{j_1}(\tilde t,\tilde x-\tilde y)|^2V(\tilde y)\,d\tilde y\le&\int_{|\tilde y|\le 1}|U^{j_1}(\tilde t,\tilde x-\tilde y)|^2V(\tilde y)\,d\tilde y\\
&+\sum_{j=0}^{\infty}\int_{2^j\le|\tilde y|\le 2^{j+1}}|U^{j_1}(\tilde t,\tilde x-\tilde y)|^2V(\tilde y)\,d\tilde y.
\end{aligned}$$

Note that U^{j_1} is compactly supported in x, so for any fixed j,

$$\int_{2^j\le|\tilde y|\le 2^{j+1}}|U^{j_1}(\tilde t,\cdot-\tilde y)|^2V(\tilde y)\,d\tilde y$$

is also compactly supported. Thus (4.13) implies that for any $j_1 \neq j_2$,

$$\lim_{n\to\infty} \iint \Big| \int_{2^j \le |\tilde y| \le 2^{j+1}} |U^{j_1}(\tilde t, \cdot - \tilde y)|^2 V(\tilde y)\, d\tilde y\, U^{j_2}\Big(\tilde t - t_n^{j_1} + t_n^{j_2}, \frac{\xi_n^{j_1} - \xi_n^{j_2}}{\rho_n^{j_1}}\, \tilde t + \tilde x - x_n^{j_1} + x_n^{j_2} - \frac{\xi_n^{j_1} - \xi_n^{j_2}}{\rho_n^{j_1}}\, t_n^{j_1}\Big)\Big|^{3/2} d\tilde x\, d\tilde t = 0.$$

Therefore, the quantity (4.11) converges to 0 as $n \to \infty$.

On the other hand,

$$\|(|x|^{-2} * (U_n^{j_1} U_n^{j_2}))U_n^{j_3}\|_{L_t^1 L_x^2[I_n]} \le C \|U_n^{j_1} U_n^{j_2}\|_{L_{t,x}^{3/2}} \|U_n^{j_3}\|_{L_{t,x}^3}.$$

By orthogonality, $\|U_n^{j_1} U_n^{j_2}\|_{L_{t,x}^{3/2}} \to 0$ as $n \to \infty$. Because $\|U_n^{j_3}\|_{L_{t,x}^3}$ is bounded, we see that the quantity (4.12) also converges to 0 as $n \to \infty$.

Next, we prove that

$$\lim_{l\to\infty} \big(\lim_{n\to\infty} \|W_n^l + w_n^l\|_{L_{t,x}^3[I_n]}\big) \le C.$$

From (4.3), we have

$$\|w_n^l\|_{L_{t,x}^3[I_n]} \le C \|w_n^l(0)\|_{L^2} \le C \|\varphi_n\|_{L^2}.$$

It suffices to verify

$$\lim_{l\to\infty} \big(\lim_{n\to\infty} \|W_n^l\|_{L_{t,x}^3[I_n]}\big) \le C. \tag{4.14}$$

From the orthogonality of $\mathbf{\Gamma}_n^j$, as in [11], we can see that for every $l \ge 1$,

$$\|W_n^l\|_{L_{t,x}^3[I_n]}^3 = \Big\| \sum_{j=1}^{l} U_n^j \Big\|_{L_{t,x}^3[I_n]}^3 \to \sum_{j=1}^{l} \|U_n^j\|_{L_{t,x}^3[I_n]}^3 \quad \text{as } n \to \infty.$$

Meanwhile by (4.3), the series $\sum \|V^j\|_{L^2}^2$ converges. Thus for every $\epsilon > 0$, there exists $l(\epsilon)$ such that

$$\|V^j\|_{L^2} \le \epsilon, \quad \forall j > l(\epsilon).$$

The theory of small data asserts that, for ϵ sufficiently small, U^j is global and $\|U^j\|_{L_{t,x}^3} \lesssim \|V^j\|_{L^2}$, which yields

$$\sum_{j > l(\epsilon)} \|U^j\|_{L_{t,x}^3}^3 < \infty.$$

So we have to deal only with a finite number of nonlinear profiles $\{U^j\}_{1\le j \le l(\epsilon)}$. But in view of the pairwise orthogonality of $\{\mathbf{\Gamma}_n^j\}_{j=1}^{\infty}$, one has

$$\lim_{n\to\infty} \Big\| \sum_{j=1}^{l(\epsilon)} U_n^j \Big\|_{L_{t,x}^3[I_n]} \le \sum_{j=1}^{l(\epsilon)} \lim_{n\to\infty} \|U_n^j\|_{L_{t,x}^3[I_n]} < \infty,$$

and thus (4.14) follows.

Now, we estimate (4.9):

$$\begin{aligned}
&\|p(W_n^l + w_n^l) - p(W_n^l)\|_{L_t^1 L_x^2[I_n]} \\
&\quad \lesssim \|(|x|^{-2} * |W_n^l + w_n^l|^2) w_n^l\|_{L_t^1 L_x^2[I_n]} + \|(|x|^{-2} * (W_n^l w_n^l)) w_n^l\|_{L_t^1 L_x^2[I_n]} \\
&\qquad + \|(|x|^{-2} * |w_n^l|^2) W_n^l\|_{L_t^1 L_x^2[I_n]} \\
&\quad \lesssim \|W_n^l\|^2_{L^3_{t,x}[I_n]} \|w_n^l\|_{L^3_{t,x}[I_n]} + \|w_n^l\|^2_{L^3_{t,x}[I_n]} (\|W_n^l\|_{L^3_{t,x}[I_n]} + \|w_n^l\|_{L^3_{t,x}[I_n]}) \\
&\quad = o_n(1).
\end{aligned}$$

The last equality is due to (4.14) and the fact that $\|w_n^l\|_{L^3_{t,x}[I_n]} \to 0$ as $l \to \infty$.

(4.10) can be estimated similarly:

$$\begin{aligned}
(4.10) \lesssim\ &\|W_n^l + w_n^l\|^2_{L^3_{t,x}[I_n]} \|r_n^l\|_{L^3_{t,x}[I_n]} + \|W_n^l + w_n^l\|_{L^3_{t,x}[I_n]} \|r_n^l\|^2_{L^3_{t,x}[I_n]} \\
&+ \|r_n^l\|^3_{L^3_{t,x}[I_n]}.
\end{aligned}$$

Now we can prove (4.7). Collecting all the previous facts, we have

$$\begin{aligned}
&\sup_{t \in I_n} \|r_n^l\|_{L^2} + \|r_n^l\|_{L^3_{t,x}[I_n]} \\
&\le C(\|W_n^l + w_n^l\|_{L^3_{t,x}[I_n]} \|r_n^l\|_{L^3_{t,x}[I_n]} + \|r_n^l\|^3_{L^3_{t,x}[I_n]} + \|r_n^l\|^2_{L^3_{t,x}[I_n]} + \|r_n^l(0,\cdot)\|_{L^2}) \\
&\quad + o_n(1).
\end{aligned} \tag{4.15}$$

As in [12], for every $\varepsilon > 0$ we can divide $I_n^+ = I_n \cap \mathbb{R}_+$ into finitely many n-dependent intervals, namely,

$$I_n^+ = [0, a_n^1] \cup [a_n^1, a_n^2] \cup \cdots \cup [a_n^{p-1}, a_n^p),$$

with each interval denoted by I_n^i $(i = 1, \ldots, p)$, so that for every $1 \le i \le p$ and every $l \ge 1$,

$$\limsup_{n \to \infty} \|W_n^l + w_n^l\|_{L^3_{t,x}(I_n^i \times \mathbb{R}^4)} \le \varepsilon.$$

The $I_n^- = I_n \cap \mathbb{R}_-$ can be similarly dealt with. Applying (4.15) on I_n^1, it follows that

$$\begin{aligned}
&\sup_{t \in I_n^1} \|r_n^l\|_{L^2} + \|r_n^l\|_{L^3_{t,x}[I_n^1]} \\
&\qquad \lesssim \epsilon \|r_n^l\|_{L^3_{t,x}[I_n^1]} + \|r_n^l\|^3_{L^3_{t,x}[I_n^1]} + \|r_n^l\|^2_{L^3_{t,x}[I_n^1]} + \|r_n^l(0,\cdot)\|_{L^2} + o_n(1).
\end{aligned}$$

By choosing ϵ sufficiently small, we obtain

$$\sup_{t \in I_n^1} \|r_n^l\|_{L^2} + \|r_n^l\|_{L^3_{t,x}[I_n^1]} \lesssim \|r_n^l(0,\cdot)\|_{L^2} + \sum_{\alpha=2}^{3} \|r_n^l\|^{\alpha}_{L^3_{t,x}[I_n^1]} + o(1).$$

Observe that, by the definition of the nonlinear profile U_n^j, we have

$$\lim_{n\to\infty} \|r_n^l(0,\cdot)\|_{L^2} = 0$$

for every $l \geq 1$. This fact and a standard bootstrap argument show easily that

$$\lim_{n\to\infty} (\sup_{t\in I_n^1} \|r_n^l\|_{L^2} + \|r_n^l\|_{L^3_{t,x}[I_n^1]}) \xrightarrow{l\to\infty} 0.$$

This gives in particular

$$\lim_{n\to\infty} \|r_n^l(a_n^1,\cdot)\|_{L^2} \xrightarrow{l\to\infty} 0$$

and allows us to repeat the same argument for I_n^2. We iterate the same process for every $1 \leq i \leq p$. Since $I = I_n^1 \cup I_n^2 \cup \cdots \cup I_n^p$ and p is finite independently of n and l, we get

$$\lim_{n\to\infty} (\|r_n^l\|_{L^3_{t,x}[I_n]} + \sup_{t\in I_n} \|r_n^l\|_{L^2}) \to 0$$

as $l \to \infty$, which is (4.7).

Step 2. Now we prove the equivalence of (i) and (ii).

(i)⇒(ii). Suppose that for all j, $\lim_{n\to\infty} \|\mathbf{\Gamma}_n^j U^j\|_{L^3_{t,x}[I_n]} < \infty$. Then

$$\|u_n\|_{L^3_{t,x}[I_n]} \leq \sum_{j=1}^{l} \|U_n^j\|_{L^3_{t,x}[I_n]} + \|r_n^l\|_{L^3_{t,x}[I_n]} + \|w_n^l\|_{L^3_{t,x}[I_n]}.$$

From (4.2), we have

$$\limsup_{n\to\infty} \|w_n^l\|_{L^3_{t,x}[I_n]} \xrightarrow{l\to\infty} 0 \quad \text{and} \quad \lim_{n\to\infty} \|r_n^l\|_{L^3_{t,x}[I_n]} \xrightarrow{l\to\infty} 0.$$

It immediately follows that

$$\lim_{n\to\infty} \|u_n\|_{L^3_{t,x}[I_n]} < \infty.$$

(ii)⇒(i). If (i) does not hold, there exists a family of $\tilde{I}_n \subset I_n$ with 0 included such that

$$\sum_{j=1}^{\infty} \lim_{n\to\infty} \|U_n^j\|^3_{L^3_{t,x}[\tilde{I}_n]} > M$$

for arbitrarily large M and

$$\|u_n\|_{L^3_{t,x}[\tilde{I}_n]} < \infty.$$

By orthogonality, we have

$$\lim_{n\to\infty} \|u_n\|^3_{L^3_{t,x}[\tilde{I}_n]} \geq \sum_{j=1}^{\infty} \lim_{n\to\infty} \|U_n^j\|^3_{L^3_{t,x}[\tilde{I}_n]} > M.$$

This leads to

$$\lim_{n\to\infty} \|u_n\|^3_{L^3_{t,x}[I_n]} \geq \lim_{n\to\infty} \|u_n\|^3_{L^3_{t,x}[\tilde{I}_n]} > M,$$

which implies that

$$\lim_{n\to\infty} \|u_n\|_{L^3_{t,x}[I_n]} = \infty.$$

This contradicts (ii) and completes the proof of Theorem 4.1.

Proof of Theorem 1.3. We choose $\{u_{0,n}\}$ such that $\|u_{0,n}\|_{L^2} \downarrow \delta_0$, and let u_n be the solution of (1.3) with data $u_{0,n}$. By the definition of δ_0, we can assume that the interval of existence for u_n is finite. By the time translation symmetry and scaling, we may assume that $\{u_n\}_{n=1}^{\infty}$ is well defined on $[0,1]$, and

$$\lim_{n\to\infty} \|u_n\|_{L^3_t([0,1],L^3_x)} = \infty.$$

Let $\{U^j, V^j, \rho^j_n, s^j_n, \xi^j_n, x^j_n\}$ be the family of linear and nonlinear profiles associated to $\{u_n\}_{n=1}^{\infty}$ via Lemma 4.1 and Theorem 4.1. Then the equivalence in Theorem 4.1 implies that there exists a j_0 such that U^{j_0} blows up. On one hand, by the definition of B_{δ_0},

$$\|V^{j_0}\|_{L^2} \geq \delta_0.$$

On the other hand, we have

$$\sum_{j\geq 0} \|V^{j_0}\|^2_{L^2} \leq \lim_{n\to\infty} \|u_{0,n}\|^2_{L^2} = \delta_0^2.$$

Thus by mass conservation and the definition of nonlinear profile, we have

$$\|U^{j_0}\|_{L^2} = \|V^{j_0}\|_{L^2} \leq \delta_0.$$

Therefore,

$$\|U^{j_0}\|_{L^2} = \delta_0,$$

because U^{j_0} is the solution of (1.3) satisfying $U(s^{j_0}, x) = V(s^{j_0}, x)$, where $s^{j_0} = \lim_{n\to\infty} s^{j_0}_n$. If s^{j_0} is finite, then U^{j_0} is the blow-up solution with minimal mass. If $s^{j_0} = \infty$, we can use the pseudo-conformal transformation to get a blow-up solution with minimal mass. This shows the existence of initial data such that the solution of (1.3) blows up in finite time for $t > 0$. In the proof of Theorem 1.4 we will show that there exists an initial data $u_0 \in L^2(\mathbb{R}^4)$ with $\|u_0\|_{L^2} = \delta_0$ such that the solution u of (1.3) blows up for both $t > 0$ and $t < 0$.

Proof of Theorem 1.4. (i) Suppose u is a solution of (1.3) which blows up at finite time $T^* > 0$ and $t_n \uparrow T^*$ as $n \to \infty$. Let

$$u_n(t,x) = u(t_n + t, x).$$

Then $\{u_n\}_{n=1}^{\infty}$ is a family of solutions on $I_n = [-t_n, T^* - t_n)$. Moreover,

$$\lim_{n\to\infty} \|u_n\|_{L^3_{t,x}\in[0,T^*-t_n)} = \lim_{n\to\infty} \|u_n\|_{L^3_{t,x}\in[-t_n,0]} = \infty.$$

Since $\|u_n\|_{L^2}$ is bounded due to L^2 conservation, we can apply Lemma 4.1 and then Theorem 4.1 on $I_n = [0, T^* - t_n)$ to deduce that there exists some j_0 such that the nonlinear profile $\{U^{j_0}, \rho_n^{j_0}, s_n^{j_0}, \xi_n^{j_0}, x_n^{j_0}\}$ satisfies

$$\lim_{n\to\infty} \|U^{j_0}\|_{L^3_{t,x}[I_n^{j_0}]} = \infty, \tag{4.16}$$

where

$$I_n^{j_0} := [s_n^{j_0}, (\rho_n^{j_0})^2(T^* - t_n) + s_n^{j_0}).$$

In fact, let $s^{j_0} = \lim_{n\to\infty} s_n^{j_0}$. Then $s^{j_0} \neq \infty$, since otherwise $I_n^{j_0} \to \emptyset$ and (4.16) is impossible. This implies either $s^{j_0} = -\infty$ or $s^{j_0} = 0$ (up to translation). If $s^{j_0} = 0$, let U^{j_0} be the solution of (1.4) with initial data V^{j_0}. Then (4.16) implies that U^{j_0} blows up at time $T^*_{j_0} \in (0, \infty)$ and

$$\lim_{n\to\infty} (\rho_n^{j_0})^2(T^* - t_n) \geq T^*_{j_0}. \tag{4.17}$$

If we also assume that $\|u_0\|_{L^2} < \sqrt{2}\,\delta_0$, then there is at most one linear profile with L^2 norm greater than δ_0 thanks to (4.3). That means that the profile U^{j_0} found above is the only blow-up nonlinear profile (since all the other profiles have L^2 norm less than δ_0 and so they are global). By repeating the same argument in $I_n = [-t_n, 0]$, we get

$$\lim_{n\to\infty} \|U^{j_0}\|_{L^3_{t,x}[I_n^{j_0}]} = \infty, \qquad I_n^{j_0} = [-(\rho_n^{j_0})^2 t_n + s_n^{j_0}, s_n^{j_0}].$$

This implies that $s^{j_0} \neq -\infty$. Hence $s^{j_0} = 0$ and the solution U^{j_0} of (1.3) with initial data $V^{j_0}(0, \cdot)$ blows up also for $t < 0$. Thus the nonlinear profile U^{j_0} is the solution of (1.3) which blows up for both $t < 0$ and $t > 0$.

(ii) The linear decomposition yields

$$(\mathbf{\Gamma}_n^{j_0})^{-1}(e^{it\Delta}(u(t_n, \cdot))) = V^{j_0} + \sum_{1\leq j\leq l,\, j\neq j_0} (\mathbf{\Gamma}_n^{j_0})^{-1}\mathbf{\Gamma}_n^{j} V^j + (\mathbf{\Gamma}_n^{j_0})^{-1} w_n^l.$$

The family $\{\mathbf{\Gamma}_n^j\}_{j=1}^{\infty}$ is pairwise orthogonal, so for every $j \neq j_0$,

$$(\mathbf{\Gamma}_n^{j_0})^{-1}\mathbf{\Gamma}_n^{j} V^j \xrightarrow{n\to\infty} 0 \quad \text{weakly in } L^2.$$

Then

$$(\mathbf{\Gamma}_n^{j_0})^{-1}(e^{it\Delta}(u(t_n, \cdot))) \xrightarrow{n\to\infty} V^{j_0} + \tilde{w}^l \quad \text{weakly},$$

where $\tilde{w}^l$ denotes the weak limit of $(\mathbf{\Gamma}_n^{j_0})^{-1} w_n^l$. However,

$$\|\tilde{w}^l\|_{L^3_{t,x}} \leq \lim_{n\to\infty} \|w_n^l\|_{L^3_{t,x}} \xrightarrow{l\to\infty} 0.$$

By uniqueness of the weak limit, we get $\tilde{w}^l = 0$ for every $l \geq j_0$. Hence,

$$(\mathbf{\Gamma}_n^{j_0})^{-1}(e^{it\Delta}(u(t_n, \cdot))) \xrightarrow{n\to\infty} V^{j_0}.$$

We need the following lemma:

LEMMA 4.2 ([23]). *Let $\{\varphi_n\}_{n\geq 1}$ and φ be in $L^2(\mathbb{R}^4)$. The following statements are equivalent:*

(1) $\varphi_n \rightharpoonup \varphi$ *weakly in* $L^2(\mathbb{R}^4)$.
(2) $e^{it\Delta}\varphi_n \rightharpoonup e^{it\Delta}\varphi$ *in* $L^3_{t,x}(\mathbb{R}^{4+1})$.

Applying this lemma to $(\mathbf{\Gamma}_n^{j_0})^{-1}(e^{it\Delta}(u(t_n,\cdot)))$, we get

$$e^{-is_n\Delta}(\rho_n^2 e^{ix\cdot\xi_n}e^{i\theta_n}u(t_n,\rho_n x+x_n)) \rightharpoonup V^{j_0}(0,\cdot)$$

with

$$s_n = s_n^{j_0}, \qquad \rho_n = \frac{1}{\rho_n^{j_0}}, \qquad \theta_n = \frac{x_n^{j_0}\xi_n^{j_0}}{\rho_n^{j_0}}, \qquad x_n = \frac{-x_n^{j_0}}{\rho_n^{j_0}}, \qquad \xi_n = -\frac{\xi_n^{j_0}}{\rho_n^{j_0}}.$$

Up to a subsequence, we can assume that $e^{i\theta_n} \to e^{i\theta}$. Since $s_n \to 0$, we get

$$\rho_n^2 e^{ix\cdot\xi_n}u(t_n,\rho_n x+x_n) \rightharpoonup e^{-i\theta}V^{j_0}(0,\cdot). \tag{4.18}$$

The associated solution is $e^{-i\theta}U^{j_0}$. (4.17) gives

$$\lim_{n\to\infty}\frac{\rho_n}{\sqrt{T^*-t_n}} \leq \frac{1}{\sqrt{T^*_{j_0}}}.$$

This completes the proof of Theorem 1.4(ii).

(iii) Let u be a solution of (1.1) with $\|u_0\|_{L^2} < \sqrt{2}\,\delta_0$ which blows up at finite time $T^* > 0$. Let $t_n \uparrow T^*$ as $n\to\infty$. So there exists $V \in L^2(\mathbb{R}^4)$ with $\|V\|_{L^2} \geq \delta_0$ and a sequence $\{\rho_n, \xi_n, x_n\} \subset \mathbb{R}^*_+ \times \mathbb{R}^4 \times \mathbb{R}^4$ such that up to a subsequence,

$$(\rho_n)^2 e^{ix\cdot\xi_n}u(t_n,\rho_n x+x_n) \xrightarrow{n\to\infty} V$$

and

$$\lim_{n\to\infty}\frac{\rho_n}{\sqrt{T^*-t_n}} \leq A$$

for some $A \geq 0$. Thus we have

$$\lim_{n\to\infty}\rho_n^4 \int_{|x|\leq R} |u(t_n,\rho_n x+x_n)|^2\,dx \geq \int_{|x|\leq R} |V|^2\,dx$$

for every $R \geq 0$. This implies that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^4}\int_{|x-y|\leq R\rho_n} |u(t_n,x)|^2\,dx \geq \int_{|x|\leq R}|V|^2\,dx.$$

Since $\sqrt{T^*-t}/\lambda(t) \to 0$ as $t\uparrow T^*$, it follows that $\rho_n/\lambda(t_n) \to 0$ and then

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^4}\int_{|x-y|\leq\lambda(t_n)} |u(t_n,x)|^2\,dx \geq \int|V|^2\,dx \geq \delta_0^2.$$

Since $\{t_n\}_{n=1}^{\infty}$ is an arbitrary sequence, we infer

$$\liminf_{t\to T} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\le\lambda(t)} |u(t,x)|^2\,dx \ge \delta_0^2.$$

However, for every $t \in [0,T)$, the function $y \mapsto \int_{|x-y|\le\lambda(t)} |u(t,x)|^2\,dx$ is continuous and goes to 0 at infinity. As a consequence,

$$\sup_{y\in\mathbb{R}^4} \int_{|x-y|\le\lambda(t)} |u(t,x)|^2\,dx = \int_{|x-x(t)|\le\lambda(t)} |u(t,x)|^2\,dx$$

for some $x(t) \in \mathbb{R}^4$, and this completes the proof of Theorem 1.4.

Proof of Corollary 1.2. In the context of the proof of Theorem 1.4 we also assume that

$$\|u_n\|_{L^2} = \|u_0\|_{L^2} = \delta_0.$$

(4.3) gives $\|V^{j_0}\|_{L^2} \le \delta_0$. It follows that $\|V^{j_0}\|_{L^2} = \delta_0$. This implies that there exists a unique profile V^{j_0} and the weak limit in (4.18) is strong.

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