Abstract. We prove that the unit sphere of every infinite-dimensional Banach space $X$ contains an $\alpha$-separated sequence, for every $0 < \alpha < 1 + \delta_X(1)$, where $\delta_X$ denotes the modulus of asymptotic uniform convexity of $X$.

1. Introduction. Elton and Odell [2] proved that the unit sphere $S_X$ of every infinite-dimensional normed linear space $X$ contains a $(1 + \varepsilon)$-separated sequence, for some $\varepsilon > 0$ depending on the space $X$. Recall that a sequence $(x_n)$ in $(X, \|\cdot\|)$ is said to be $\alpha$-separated, for some $\alpha > 0$, if $\|x_n - x_m\| \geq \alpha$ for every $n \neq m$. Diestel [1, p. 254] asked whether this $\varepsilon$ can be quantified. Kryczka and Prus [4] answered this question for the class of non-reflexive Banach spaces, proving that the unit sphere of such a space contains a $4^{1/5}$-separated sequence. Van Neerven [6] studied the class of uniformly convex Banach spaces and connected together $\varepsilon$ and the modulus of convexity (see comments below).

In this note, we are interested in the class of asymptotically uniformly convex Banach spaces. We connect $\varepsilon$ and the modulus of asymptotic uniform convexity. This modulus has been introduced by Milman [5] and investigated by Johnson, Lindenstrauss, Preiss and Schechtman [3]. The modulus of asymptotic uniform convexity of an infinite-dimensional Banach space $X$ is given for $t > 0$ by

$$\overline{\delta}_X(t) = \inf_{\|x\| = 1} \sup_{H \subset X, \text{codim } H < \infty} \inf_{h \in H} \|h\| \geq t, \|x + h\| - 1.$$ 

The Banach space $X$ is said to be asymptotically uniformly convex if $\overline{\delta}_X(t) > 0$ for all $0 < t < 1$. If $X$ is a subspace of $\ell_p$, $1 \leq p < \infty$, then $\overline{\delta}_X(t) = (1+t^p)^{1/p} - 1$, and $X$ is asymptotically uniformly convex. If $X$ is a subspace
of $c_0$ then $\delta_X(t) = 0$ for every $t \in (0; 1]$, so that $X$ is not asymptotically uniformly convex.

2. Statement and proof of the main result

**Theorem 1.** The unit sphere of every infinite-dimensional Banach space $X$ contains an $\alpha$-separated sequence for every $0 < \alpha < 1 + \delta_X(1)$.

**Proof.** Fix $0 < \alpha < 1 + \delta_X(1)$ and $x_1 \in S_X$. Write $X = H_0$. There exists a finite-codimensional subspace $H_1 \subset H_0$ such that for every $h \in S_{H_1}$, $\|x_1 - h\| = \|x_1 + (-h)\| > \alpha$. Take $x_2 \in S_{H_1}$. It is easy to see that $\delta_{H_1}(1) \geq \delta_X(1)$. As before, there exists a finite-codimensional subspace $H_2 \subset H_1$ such that for every $h \in S_{H_2}$, $\|x_2 - h\| > \alpha$. Take $x_3 \in S_{H_2}$. As $S_{H_2} \subset S_{H_1}$, we have $\|x_1 - x_3\| > \alpha$ too. Inductively, we construct an $\alpha$-separated sequence $(x_n)$ in $S_X$ along with a corresponding non-increasing sequence of finite-codimensional subspaces $(H_n)$. These sequences are chosen so that $x_{n+1} \in S_{H_n}$ with $H_n \subset H_{n-1}$ such that $\|x_n - h\| > \alpha$ for every $h \in S_{H_n}$.

3. Comments. If $X$ is asymptotically uniformly convex, then $\delta_X(1) > 0$ and the $\varepsilon$ obtained by Elton and Odell [2] is quantified. In $X = \ell_p$, $1 \leq p < \infty$, the sequence of unit vectors is $2^{1/p}$-separated and $1 + \delta_X(1) = 2^{1/p}$. This answers the question raised in [6] whether Theorem 1.2 therein can be improved. This improvement occurs in two ways. Indeed, according to Proposition 2.3(3) in [3], for every infinite-dimensional Banach space $X$ and every $0 < \varepsilon < 1$, we have $\delta_X(\varepsilon) \leq \delta_X(1)$ (where $\delta_X$ denotes the modulus of convexity of $X$). First, every infinite-dimensional uniformly convex Banach space is asymptotically uniformly convex, but the converse is false (consider $\ell_1$). Secondly, quantitatively we have $1 + \delta_X(1) \geq 1 + \frac{1}{2}\delta_X(2/3)$ where the right-hand side is the separation constant obtained in [6].

**REFERENCES**


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