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CO-ANALYTIC, RIGHT-INVERTIBLE OPERATORS ARE SUPERCYCLIC

BY

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Abstract. Let $\mathcal{H}$ denote a complex, infinite-dimensional, separable Hilbert space, and for any such Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. We show that for any co-analytic, right-invertible T in $\mathcal{B}(\mathcal{H})$, αT is hypercyclic for every complex α with $|\alpha| > \beta^{-1}$, where $\beta \equiv \inf_{\|x\|=1} \|T^*x\| > 0$. In particular, every co-analytic, right-invertible T in $\mathcal{B}(\mathcal{H})$ is supercyclic.

Complete set of eigenvectors and hypercyclicity. We denote by $\bigvee\{w : w \in W\}$ the smallest closed linear subspace of $\mathcal{H}$ generated by the subset W of $\mathcal{H}$. We say that T in $\mathcal{B}(\mathcal{H})$

(1) is *co-analytic* if T^* is analytic, that is, $\bigcap_{k\ge 0} T^{*k}\mathcal{H} = \{0\}$;
(2) admits a *complete set of eigenvectors* if

$$\mathcal{H} = \bigvee_{\mu\in\mathbb{D}_r} \mathrm{null}(T-\mu) \quad \text{for every positive real } r,$$

where $\mathrm{null}(S)$ denotes the null-space of S and $\mathbb{D}_r$ stands for the open disc $\{z \in \mathbb{C} : |z| < r\}$;
(3) is *supercyclic* if there exists $f \in \mathcal{H}$ such that $\{\lambda T^n f : n \in \mathbb{Z}_+,\ \lambda \in \mathbb{C}\}$ is dense in $\mathcal{H}$, where $\mathbb{Z}_+$ denotes the set of non-negative integers;
(4) is *hypercyclic* if there exists $f \in \mathcal{H}$ such that $\{T^n f : n \in \mathbb{Z}_+\}$ is dense in $\mathcal{H}$.

There is a voluminous amount of literature on hypercyclic and supercyclic operators (see, for instance, [5], and the references cited therein).

LEMMA. *Let T denote a right-invertible operator in $\mathcal{B}(\mathcal{H})$. Then the following statements are true:*

(A) *If T is co-analytic then T admits a complete set of eigenvectors.*
(B) *If T admits a complete set of eigenvectors then αT is hypercyclic for every complex α with $|\alpha| > \beta^{-1}$, where $\beta \equiv \inf_{\|x\|=1} \|T^*x\| > 0$.*

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Proof. (A) Suppose T is co-analytic and let $S \equiv T^*(TT^*)^{-1}$. Since $(\bigcap_{n\geq 0} T^{*n}\mathcal{H})^{\perp} = \bigvee_{n\geq 0} S^n(\mathrm{null}(T))$ ([10, Proposition 2.7]), it follows that

$$\mathcal{H} = \bigvee_{n\geq 0} S^n(\mathrm{null}(T)). \tag{1}$$

Note that the series

$$e_{\mu,h} \equiv \sum_{n=0}^{\infty} \mu^n S^n h \qquad (h \in \mathrm{null}(T))$$

is absolutely convergent in $\mathcal{H}$ for every $\mu \in \mathbb{D}_{r_0}$, where $r_0 \equiv \|S\|^{-1}$. Since $TS = I$, it follows that for any $r \in (0, r_0]$,

$$\bigvee\{e_{\mu,h} : \mu \in \mathbb{D}_r,\ h \in \mathrm{null}(T)\} \subseteq \bigvee\{h \in \mathrm{null}(T-\mu) : \mu \in \mathbb{D}_r\}. \tag{2}$$

For $x \in \mathcal{H}$ and $h \in \mathrm{null}(T)$, define $f_{x,h} : \mathbb{D}_{r_0} \to \mathbb{C}$ by

$$f_{x,h}(\mu) = \sum_{n=0}^{\infty} \overline{\langle x, S^n h\rangle}\, \mu^n \qquad (\mu \in \mathbb{D}_{r_0}),$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Since $e_{\mu,h}$ is absolutely convergent, $f_{x,h}$ is a well-defined function analytic in $\mathbb{D}_{r_0}$. Fix $r \in (0, r_0]$ and let $x \in (\bigvee\{e_{\mu,h} : \mu \in \mathbb{D}_r,\ h \in \mathrm{null}(T)\})^{\perp}$. Thus $\langle x, e_{\mu,h}\rangle = 0$ for every $\mu \in \mathbb{D}_r$ and every $h \in \mathrm{null}(T)$. It follows that

$$\sum_{n=0}^{\infty} \langle x,\, S^n h\rangle \overline{\mu}^n = 0 \qquad \text{for every } \mu \in \mathbb{D}_r.$$

Thus the analytic function $f_{x,h}$ is identically zero in $\mathbb{D}_r$. Hence $\langle x, S^n h\rangle = 0$ for every integer $n \geq 0$ and every $h \in \mathrm{null}(T)$. It follows from (1) that $x = 0$. Hence, by (2), we must have $\bigvee\{\mathrm{null}(T-\mu) : \mu \in \mathbb{D}_r\} = \mathcal{H}$ for any positive real r.

(B) Assume that $\mathcal{H} = \bigvee\{\mathrm{null}(T-\mu I) : \mu \in \mathbb{D}_r\}$ for every positive real r. Hence the linear manifold $\mathcal{M}_r \equiv \mathrm{linspan}\{h \in \mathrm{null}(T-\mu I) : \mu \in \mathbb{D}_r\}$ invariant for T is dense in $\mathcal{H}$. In view of the Hypercyclicity Criterion ([5, Section 2]), it now suffices to check the following: For every $\alpha \in \mathbb{C}$ such that $|\alpha| > \beta^{-1}$ with $\beta \equiv \inf_{\|x\|=1} \|T^* x\| > 0$, there exists a positive real r such that

$$\|\alpha^n T^n h\| \to 0, \quad \|\alpha^{-n} S^n h\| \to 0 \quad \text{ as } n \to \infty\ (h \in \mathcal{M}_r), \tag{3}$$

where $S = T^*(TT^*)^{-1}$. Let $\alpha \in \mathbb{C}$ be such that $|\alpha| > \beta^{-1}$ and let r be a positive real such that $|\alpha| r < 1$. Since $T^n h = \mu^n h$ $(h \in \mathrm{null}(T-\mu))$ and $\|S\| \leq \beta^{-1}$, (3) follows at once. ■

The following, inspired mainly by Rolewicz's Theorem ([7, Theorem 1.1]), provides a new class of supercyclic operators, which includes in particular completely non-unitary operators close to isometries, adjoints of multiplication operators on reproducing kernel Hilbert spaces, and many backward weighted shift operators.

THEOREM 1. *For a co-analytic right-invertible T in $\mathcal{B}(\mathcal{H})$, αT is hypercyclic for every complex α with $|\alpha| > \beta^{-1}$, where $\beta \equiv \inf_{\|x\|=1} \|T^*x\| > 0$. In particular, every co-analytic right-invertible T in $\mathcal{B}(\mathcal{H})$ is supercyclic.*

Applications. To see the first application of Theorem 1, we recall some notions needed to define the operators close to isometries. It has been pointed out by the anonymous referee that the assumption of left-invertibility in the definition of the so-called restriction classes is unnecessary. Accordingly, the definitions to follow are more general than those that appeared in [2].

A class $\mathcal{U}$ of bounded linear operators on Hilbert spaces is said to be a *restriction class* if for every non-zero invariant subspace $\mathcal{M}$ of a bounded linear operator S in $\mathcal{B}(\mathcal{H})$, the restriction operator $S|_{\mathcal{M}}$ belongs to $\mathcal{U}$ whenever S belongs to $\mathcal{U}$.

We are interested in those restriction classes $\mathcal{U}$ for which the following hold:

PROPERTY I. If S belongs to $\mathcal{U}$, and S is invertible, then S is unitary.

PROPERTY II. If S belongs to $\mathcal{U}$, and $\mathcal{M}$ is a non-zero invariant subspace of S such that $S|_{\mathcal{M}}$ is unitary, then S reduces $\mathcal{M}$.

A restriction class $\mathcal{U}$ is said to be an *almost isometric restriction class* (for short, an *AIR class*) if it satisfies Properties I and II. We will refer to the members of AIR classes as *operators close to isometries.*

Important examples of operators close to isometries include many weighted shifts (e.g. unilateral shift, Bergman shift, and Dirichlet shift) and certain composition operators on discrete measure spaces ([6, Example 4.4]). The results of [2] demonstrate, in particular, that the set-up of AIR classes is specifically useful for guaranteeing the existence of complete sets of eigenvectors and wandering subspaces. In particular, this set-up enables one to obtain the wandering subspace theorems for Bergman shift and Dirichlet shift in one stroke ([2, Remark 4.5]).

We refer the reader to [2] for the basic theory of operators close to isometries. In particular, it follows from [2, proof of Lemma 4.2] that any completely non-unitary left-invertible operator close to isometry is analytic. This simple observation along with Theorem 1 yields a refinement of [2, Corollary 3.6(b)].

COROLLARY 2. *Let T in $\mathcal{B}(\mathcal{H})$ denote a left-invertible operator close to isometry. If T is completely non-unitary then αT^* is hypercyclic for every complex α with $|\alpha| > \beta^{-1}$, where $\beta \equiv \inf_{\|x\|=1} \|Tx\| > 0$. In particular,* *the adjoint of any completely non-unitary left-invertible T in $\mathcal{B}(\mathcal{H})$, which is close to isometry, is supercyclic.*

Example. Let T in $\mathcal{B}(\mathcal{H})$ be a completely non-unitary left-invertible operator. Suppose that one of the following conditions holds true:

(1) $I - 2T^*T + T^{*2}T^2 \leq 0$;
(2) $TT^* \leq T^*T$ and the approximate-point spectrum of T is contained in the unit circle $\partial\mathbb{D}_1$;
(3) $\sum_{0\leq p\leq n}(-1)^p\binom{n}{p}T^{*p}T^p = 0$ for some integer $n \geq 1$ and $T^*T \geq I$.

Then it follows from [2, Example 2.3] and Corollary 2 that T^* is supercyclic. Let us see some concrete operators satisfying conditions (1)–(3). The following example is borrowed from [6, Example 4.4].

Let $X = \{(n,m) : n, m \in \mathbb{Z}$ such that $n \leq m\}$ and let $\{a_k\}_{k=-\infty}^{\infty}$ be a sequence of positive real numbers. Consider the measure μ on the power set of X given by

$$\mu(\{(n,m)\}) = \begin{cases} 1 & \text{if } n = m, \\ a_n & \text{if } n < m. \end{cases}$$

Consider the measurable function $\phi : X \to X$ given by

$$\phi(n,m) = \begin{cases} (n-1, m-1) & \text{if } n = m, \\ (n, m-1) & \text{if } n < m. \end{cases}$$

Define the *composition operator* C_ϕ in $L^2(\mu)$ by

$$C_\phi f = f \circ \phi, \quad f \in \{f \in L^2(\mu) : f \circ \phi \in L^2(\mu)\}.$$

Then C_ϕ is a bounded linear operator on $L^2(\mu)$ if and only if $\{a_k\}_{k=-\infty}^{\infty}$ is a bounded sequence. Moreover, C_ϕ satisfies (1) (resp. (3) with $n = 2$) if and only if $a_{k+1} \leq a_k$ (resp. $a_{k+1} = a_k$) for all integers k. It turns out that the so-called Cauchy dual operator $C_\phi(C_\phi^* C_\phi)^{-1}$ satisfies (2) ([1, Theorem 2.9 and Lemma 2.14]).

An operator T in $\mathcal{B}(\mathcal{H})$ is said to be *expansive* if $\|Tx\| \geq \|x\|$ for every $x \in \mathcal{H}$.

Corollary 3. *Consider the class of bounded Hilbert space operators given by*

$$\mathcal{U} \equiv \{T : T \text{ is expansive with } W(T) \subseteq \overline{\mathbb{D}}_1\}, \tag{4}$$

where $W(T)$ denotes the numerical range of T. Then $\mathcal{U}$ is an AIR class. In particular, the adjoint of any completely non-unitary T in $\mathcal{U}$ is supercyclic.

Proof. Clearly, $\mathcal{U}$ is a restriction class. If T belongs to $\mathcal{U}$ and T is invertible then it follows from a result of Donoghue [3] that T is unitary. Thus Property I in the definition of AIR classes is satisfied. The fact that $\mathcal{U}$ satisfies Property II is immediate from [2, Lemma 2.5]. The remaining part is a special case of Corollary 2. ■

Unlike the method employed in [4, Section 3], ours is confined to the Hilbert space set-up. However, in contrast with [4, Theorem 3.6], Theorem 1 above is true for any co-analytic right-invertible T without any condition on the null-space of T. Indeed, the adjoint C_ϕ^* of the composition operator C_ϕ discussed in the example prior to Corollary 3 may have infinite-dimensional null-space. On the other hand, because of its generality, Theorem 1 does not readily provide the best results in some concrete situations. In particular, we are able to recover important cases of [4, Corollary 4.6] and [8, Theorem 2.8].

EXAMPLE. The assertion of Theorem 1 is also applicable to the adjoints of multiplication operators on "analytic" RKHS. To see that, let Ω denote an open connected subset of $\mathbb{C}^n$, and let $\mathcal{H}_r$ denote a non-zero Hilbert space of functions analytic in Ω. Let $\phi : \Omega \to \mathbb{C}$ be a non-constant bounded analytic function such that $\phi f \in \mathcal{H}_r$ whenever $f \in \mathcal{H}_r$, where ϕf is given by $(\phi f)(\mu) = \phi(\mu) f(\mu)$ $(\mu \in \Omega)$. Further, assume the following:

(1) for any $\mu \in \Omega$, there exists $0 \neq \kappa_\mu \in \mathcal{H}_r$ such that $\langle f, \kappa_\mu \rangle = f(\mu)$ $(f \in \mathcal{H}_r)$;
(2) there exists some $\lambda \in \Omega$ such that for any $f \in \mathcal{H}_r$ with $f(\lambda) = 0$, one can find some $g \in \mathcal{H}_r$ such that $f = (\phi - \phi(\lambda))g$.

Define $M_\phi : \mathcal{H}_r \to \mathcal{H}_r$ by $M_\phi f = \phi f$ $(f \in \mathcal{H}_r)$. It then follows from (1) and the Closed Graph Theorem that M_ϕ is a bounded linear operator on $\mathcal{H}_r$. Also, if $f \in \bigcap_{n\geq 0}(M_\phi - \phi(\lambda))^n \mathcal{H}_r$ then it is easy to see that all mixed partial derivatives of f are zero at λ. Hence, by the Identity Theorem ([9, p. 10]), $M_\phi - \phi(\lambda)$ is analytic. Furthermore, because of (2), $M_\phi - \phi(\lambda)$ is left-invertible with a bounded left inverse $L_\phi : \mathcal{H}_r \to \mathcal{H}_r$ given by

$$L_\phi f = \frac{f - f(\lambda)\|\kappa_\lambda\|^{-2}\kappa_\lambda}{\phi - \phi(\lambda)} \quad (f \in \mathcal{H}_r).$$

Hence, by Theorem 1, $M_\phi^* - \overline{\phi(\lambda)}$ is supercyclic.

EXAMPLE. Let $\mathcal{H}$ be a complex separable Hilbert space and let $\{e_n\}_{n\geq 0}$ be an orthonormal basis for $\mathcal{H}$. Define the *backward weighted shift operator* B in $\mathcal{B}(\mathcal{H})$ with the weight sequence $\{\alpha_n\}_{n\geq 0}$ through the relations $Be_n = \alpha_n e_{n-1}$ for every $n \geq 1$ and $Be_0 = 0$. Assume that the infimum $\inf_{n\geq 0} \alpha_n$ of the weight sequence $\{\alpha_n\}_{n\geq 0}$ is positive. Notice that B is a co-analytic right-invertible operator. Hence, by Theorem 1, αB is hypercyclic for every complex α with $|\alpha| > \beta^{-1}$, where $\beta \equiv \inf_{n\geq 0} \alpha_n$. In particular, B is hypercyclic if $\inf_{n\geq 0} \alpha_n > 1$.

Finally, we would like to invite the interested reader to specialize Theorem 1 to other subclasses of co-analytic right-invertible bounded Hilbert space operators.

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