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AN EXAMPLE OF A SIMPLE DERIVATION IN TWO VARIABLES

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Abstract. Let k be a field of characteristic zero. We prove that the derivation $D = \partial/\partial x + (y^s + px)(\partial/\partial y)$, where $s \geq 2$, $0 \neq p \in k$, of the polynomial ring $k[x, y]$ is simple.

1. Introduction. Throughout the paper k is a field of characteristic zero. Assume that d is a derivation of a commutative k-algebra R. We say that d is *simple* if R has no d-invariant ideals other than 0 and R.

Simple derivations are useful for constructions of simple noncommutative rings which are not fields. It is well known ([2]) that if $R[t, d]$ is the Ore extension of R with respect to d ([11], [5]), then $R[t, d]$ is a simple ring (that is, $R[t, d]$ has no two-sided ideals other than 0 and $R[t, d]$) if and only if the derivation d is simple.

We can use simple derivations to construct simple Lie rings. Recall that a Lie ring L is said to be *simple* if it has no Lie ideals other than 0 and L. Denote by R_0 the Lie ring whose elements are the elements of R, with the product $[a, b] = ad(b) - d(a)b$ for all $a, b \in R_0$. It is known ([4], [9]) that R_0 is simple if and only if d is simple.

A. Seidenberg [13] showed that if R is a finitely generated domain and d is simple, then R is regular. R. Hart [3] showed that if R is a finitely generated local domain, then R is regular if and only if there exists a simple derivation of R.

Examples, applications and various properties of simple derivations can be found in many other papers (see, for example, [12], [7], [6], [10], [8], [1]).

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring over k in n variables and let $d(x_1) = f_1, \dots, d(x_n) = f_n$. It would be of considerable interest to find necessary and sufficient conditions on $f_1, \dots, f_n$ for d to be simple. The answer is obvious only for $n = 1$.

If $n = 2$, then only some sporadic examples of simple derivations of $R = k[x, y]$ are known.

The problem seems to be difficult even if we assume that $d(x) = 1$. In [10] and [1], there is a description of all simple derivations d of $k[x, y]$ such that

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$d(x) = 1$ and $d(y) = a(x)y + b(x)$, where $a(x), b(x) \in k[x]$. A. Maciejewski, J. Moulin-Ollagnier and the author [8] gave an algebraic characterization of simple derivations d of $k[x, y]$ such that $d(x) = 1$ and $d(y) = y^2 + a(x)y + b(x)$, where $a(x), b(x) \in k[x]$. Analytic proofs of our results with more precise characterizations of simple derivations of such forms were given by H. Żołądek in [14].

Recently, P. Brumatti, Y. Lequain and D. Levcovitz [1] constructed examples of simple derivations d of the local ring $k[x, y]_{(x,y)}$ such that $d(x) = 1$ and $\deg_y d(y) = s$, where s is an arbitrary positive integer. Most of the published examples of simple derivations d of $k[x, y]$ with $d(x) = 1$ are of the type $d = \partial/\partial x + F(x, y)(\partial/\partial y)$, where $F(x, y) \in k[x, y]$ and $\deg_y F(x, y) \leq 2$. In particular, there does not seem to be any example with $\deg_y F(x, y)$ an arbitrary positive integer. The aim of this paper is to provide such an example. We prove, in an elementary way, that if $s \geq 2$ and $0 \neq p \in k$, then the derivation $\partial/\partial x + (y^s + px)(\partial/\partial y)$ is simple.

2. Preliminaries and notations. Let d be a derivation of $k[x, y]$. We say (as in [8]) that a polynomial $F \in k[x, y]$ is a *Darboux polynomial* of d if $F \notin k$ and $d(F) = \Lambda F$ for some $\Lambda \in k[x, y]$, or equivalently (F) is a proper d-invariant ideal of $k[x, y]$. Note the following easy observation.

PROPOSITION 1. *If $d : k[x, y] \to k[x, y]$ is a derivation such that $d(x) = 1$, then d is simple if and only if d has no Darboux polynomials.*

Proof. This is well known (see, for example, Proposition 2.1 in [8]) if the field k is algebraically closed. In the general case we use standard arguments (see [10]). ∎

Throughout the paper, D denotes the derivation of $k[x, y]$ defined by

$$D = \partial/\partial x + (y^s + px)(\partial/\partial y),$$

where $s \geq 0$ and $p \in k \setminus \{0\}$. If $s = 0$, then this derivation is not simple, because $D(y - x - \frac{1}{2}px^2) = 0$. If $s = 1$, then D is not simple either, because $D(y + px + p) = y + px + p$. We will assume that $s \geq 2$. Note that if $s = 2$, then we know ([8, Theorem 6.2]) that D is simple. We will prove that the same is true for any $s \geq 2$. For the proof we need to show (by Proposition 1) that D has no Darboux polynomials.

Suppose that D has a Darboux polynomial. Let F and Λ be fixed polynomials from $k[x, y]$ such that $F \notin k$ and $D(F) = \Lambda F$. Using these notations we have:

LEMMA 1. *$\Lambda \in k[y] \setminus \{0\}$, $\deg \Lambda = s - 1$ and $\Lambda = ny^{s-1} + \lambda$, where $n = \deg_y F$, $\lambda \in k[y]$ with $\deg \lambda < s - 1$.*

Proof. First suppose that $\Lambda = 0$, that is, $D(F) = 0$. Let $F = Ay^n + G$, where $0 \neq A \in k[x]$, $n \geq 0$ and $G \in k[x, y]$ with $\deg_y G < n$. If $n = 0$, then $F = A \in k[x]$ and $0 = D(F) = A'$, where A' is the derivative of A with respect to x. So, if $n = 0$, then $F \in k$, and we have a contradiction. If $n > 0$, then $0 = D(F) = nAy^{(n-1)+s} + H$ for some $H \in k[x, y]$ with $\deg_y H < n + s - 1$, and again we have a contradiction. Therefore, $\Lambda \neq 0$.

Let $F = a(y)x^m + G$ and $\Lambda = b(y)x^r + H$, where $a(y), b(y) \in k[y] \setminus \{0\}$, $m, r \geq 0$, $G, H \in k[x, y]$, $\deg_x G < m$ and $\deg_x H < r$. Then $D(F) = pa(y)'x^{m+1} + U$ and $\Lambda F = a(y)b(y)x^{m+r} + V$ for some $U, V \in k[x, y]$ with $\deg_x U < m+1$ and $\deg_x V < m+r$, where $a(y)'$ is the derivative of $a(y)$ with respect to y. But $D(F) = \Lambda F$. So, if $r > 1$ then we have the contradiction $0 = a(y)b(y) \neq 0$, and if $r = 1$ then we have the equality $pa(y)' = a(y)b(y)$, which is also an evident contradiction. Hence, $r = 0$ (and $a(y)' = 0$), which means that $\Lambda = b(y) \in k[y]$.

Now, comparing in $D(F) = \Lambda F$ the leading terms with respect to powers of y, we see that $\deg_y \Lambda = s - 1$ and that the leading coefficient of Λ is equal to $\deg_y F$. ■

By the above lemma we may fix the following notations. Assume that $n = \deg_y F$,

$$F = A_0 y^n + A_1 y^{n-1} + \cdots + A_n,$$

where $A_0, \ldots, A_n \in k[x]$ with $A_0 \neq 0$, and

$$\Lambda = ny^{s-1} - a_1 y^{s-2} - a_2 y^{s-3} + \cdots + a_{s-2} y + a_{s-1},$$

where $a_1, \ldots, a_{s-1} \in k$. It is obvious that $n \geq 1$. Since every polynomial of the form cF, where $0 \neq c \in k$, is also a Darboux polynomial of D, we may assume that A_0 is monic. Assume also that $A_i = 0$ if $i > n$ or $i < 0$.

If u is a polynomial from $k[x]$, then we denote by u' the derivative du/dx, by $|u|$ the degree of u, and by u^* the leading monomial of u. Moreover, if u and v are polynomials from $k[x]$, then we write $u \sim v$ if there exists a positive rational number q such that $u = qv$. Let r be the degree of A_0. Thus, $|A_0| = r \geq 0$ and $A_0^* = x^r$.

3. The proof of the main result. Comparing in $D(F) = \Lambda F$ the coefficients (belonging to $k[x]$) of y^j for $j = n + s - 1, \ldots, 2, 1, 0$, we obtain

$$\begin{aligned} \sigma A_\sigma = {} & a_1 A_{\sigma-1} + a_2 A_{\sigma-2} + \cdots + a_{s-1} A_{\sigma-(s-1)} \\ & + A'_{\sigma-(s-1)} + (n + s - \sigma) A_{\sigma-s} px \end{aligned} \tag{1}$$

for all $\sigma = 1, \ldots, n + s - 1$. Putting $\sigma = \tau + s$ we obtain

$$\begin{aligned} (\tau + s) A_{\tau+s} = {} & a_1 A_{\tau+s-1} + a_2 A_{\tau+s-2} + \cdots + a_{s-1} A_{\tau+1} \\ & + A'_{\tau+1} + (n - \tau) A_\tau px \end{aligned} \tag{2}$$

for all $\tau = -(s - 1), -(s - 2), \ldots, -1, 0, 1, \ldots, n - 1$.

The above equalities will play an important role in our proof. Observe that we have the following sequence of equalities:

$$(3)\quad \begin{cases} A_1 = a_1A_0, \\ 2A_2 = a_1A_1 + a_2A_0, \\ 3A_3 = a_1A_2 + a_2A_1 + a_3A_0, \\ \vdots \\ (s-2)A_{s-2} = a_1A_{s-3} + a_2A_{s-4} + \cdots + a_{s-2}A_0, \\ (s-1)A_{s-1} = a_1A_{s-2} + a_2A_{s-3} + \cdots + a_{s-1}A_0 + A_0', \\ sA_s = a_1A_{s-1} + a_2A_{s-2} + \cdots + a_{s-1}A_1 + A_1' + nA_0px, \\ (s+1)A_{s+1} = a_1A_s + a_2A_{s-1} + \cdots + a_{s-1}A_2 + A_2' + (n-1)A_1px, \\ \vdots \\ nA_n = a_1A_{n-1} + a_2A_{n-2} + \cdots + a_{s-1}A_{n+1-s} + A_{n+1-s}' + sA_{n-s}px, \\ 0 = a_1A_n + a_2A_{n-1} + \cdots + a_{s-1}A_{n+2-s} + A_{n+2-s}' + (s-1)A_{n+1-s}px, \\ 0 = a_2A_n + a_3A_{n-1} + \cdots + a_{s-1}A_{n+3-s} + A_{n+3-s}' + (s-2)A_{n+2-s}px, \\ \vdots \\ 0 = a_{s-2}A_n + a_{s-1}A_{n-1} + A_{n-1}' + 2A_{n-2}px, \\ 0 = a_{s-1}A_n + A_n' + A_{n-1}px. \end{cases}$$

for $\sigma = 1, \ldots, s-1$.

LEMMA 2. (a) *If i is an integer such that $0 \le is \le n$, then*

$$A_{is} \neq 0, \quad |A_{is}| = r + i \quad \textit{and} \quad A_{is}^* \sim p^i x^{r+i}.$$

(b) *If i, j are integers such that $0 \le is + j \le n$ and $0 < j < s$, then $|A_{is+j}| \le r + i$.*

Proof. Since $A_0 \neq 0$ and $A_0^* = x^r$, statement (a) is true for $i = 0$. Since $a_1, \ldots, a_{s-1} \in k$, the initial equalities of (3) imply that for $i = 0$ statement (b) is also true.

Assume now that both (a) and (b) hold for some $i \ge 0$. Assume also that $(i+1)s \le n$. Then, by (2), $A_{(i+1)s} = A_{is+s} \sim B$, where

$$B = a_1A_{is+s-1} + a_2A_{is+s-2} + \cdots + a_{s-1}A_{is+1} + A_{is+1}' + (n - is)A_{is}px.$$

So, by induction, $A_{(i+1)s}^* \sim (n - is)A_{is}^* px \sim p^i x^{r+i} pi = p^{i+1}x^{r+(i+1)}$. This means that (a) holds for $i + 1$.

Let j be an integer such that $0 < j < s$ and $(i+1)s + j \le n$. If $j = 1$ then, by (2), $A_{(i+1)s+1} = A_{(is+1)+s} \sim B$, where

$$B = a_1A_{(i+1)s} + a_2A_{is+(s-1)} + \cdots + a_{s-1}A_{is+2} + A_{is+2}' + (n - (is+1))A_{is+1}px.$$

We already know that $|A_{(i+1)s}| = r + (i+1)$, so $|a_1 A_{(i+1)s}| \leq r + (i+1)$. We also know that the degrees $|a_2 A_{is+(s-1)}|, \dots, |a_{s-1} A_{is+2}|$ are smaller than $r + (i+1)$. Moreover, $|(n-(is+1))A_{is+1}px| = |A_{is+1}| + 1 \leq (r+i) + 1 = r + (i+1)$. Hence, $|A_{(i+1)s+1}| \leq r + (i+1)$. Repeating the same argument successively for $j = 2, \dots, s-1$ (using a new induction) we deduce that $|A_{(i+1)s+j}| \leq r + (i+1)$. This completes the proof. ■

LEMMA 3. *The number s divides n.*

Proof. Suppose that $n = is + j$, where $i \geq 0$ and $0 < j < s$, and consider the equality (4) for $\sigma = j$. We have

$$0 = a_{s-j} A_{is+j} + a_{(s-j)+1} A_{is+(j-1)} + \dots + a_{s-1} A_{is+1} + A'_{is+1} + j A_{si} px.$$

By Lemma 2, $j A_{si} px$ is a nonzero polynomial of degree $r + (i+1)$. Moreover, also by that lemma, the remaining terms of the right side have degrees smaller than $r + (i+1)$. So, we have a contradiction. ■

It follows from the above lemma that

$$n = ts, \tag{5}$$

where t is a positive integer.

LEMMA 4. *The coefficient a_1 is equal to zero.*

Proof. Suppose that $a_1 \neq 0$. Then, by (2), $A_1^* = a_1 x^r$ (because, as we assumed, $A_0^* = x^r$). We will show, by induction, that if $is + 1 \leq n$, then

$$A_{is+1}^* \sim a_1 p^i x^{r+i}. \tag{6}$$

For $i = 0$, this is clear. Let $(i+1)s + 1 \leq n$. Then, by (2), $A_{(i+1)s+1} = A_{(is+1)+s} \sim B$, where

$$B = a_1 A_{(i+1)s} + a_2 A_{is+(s-1)} + \dots + a_{s-1} A_{is+2} + A'_{is+2} + (n-(is+1)) A_{is+1} px.$$

Observe that, by Lemma 2, $(a_1 A_{(i+1)s})^* \sim a_1 p^{i+1} x^{r+(i+1)}$ and, by induction,

$$((n-(is+1)) A_{is+1} px)^* \sim a_1 p^i x^{r+i} px = a_1 p^{i+1} x^{r+(i+1)}.$$

The degrees of the remaining components of B are, by Lemma 2, smaller than $r + (i+1)$. So, $A_{(i+1)s+1}^* \sim B^* \sim a_1 p^{i+1} x^{r+(i+1)} + a_1 p^{i+1} x^{r+(i+1)} \sim a_1 p^{i+1} x^{r+(i+1)}$. Thus, (6) is proven.

Consider now the equality (4) for $\sigma = s - 1$. We have

$$0 = a_1 A_{ts} + a_2 A_{(t-1)s+(s-1)} + \dots + a_{s-1} A_{(t-1)s+2} + A'_{(t-1)s+2} + (s-1) A_{(t-1)s+1} px.$$

But $(a_1 A_{ts})^* \sim a_1 p^t x^{r+t}$ (Lemma 2) and, by (6), $((s-1) A_{(t-1)s+1} px)^* \sim a_1 p^t x^{r+t}$; moreover, the degrees of all the remaining terms are (by Lemma 2) smaller than $r + t$. So, we have the contradiction $0 = a_1 p^t \neq 0$. ■

LEMMA 5. *All the coefficients $a_1, \dots, a_{s-1}$ are equal to zero.*

Proof. Suppose otherwise, and let $m \in \{1,\dots,s-1\}$ be smallest such that $a_m \neq 0$. Then, by Lemma 4, $m > 1$ and $a_1 = \cdots = a_{m-1} = 0$. Moreover, by (2), $A_m^* \sim a_m x^r$, and repeating the same arguments as in the proof of Lemma 4, we get

$$A_{is+m}^* \sim a_m p^i x^{r+i} \tag{7}$$

for all i with $is + m \leq n$. Consider the equality (4) for $\sigma = s - m$. We have

$$0 = a_m A_{ts} + a_{m+1} A_{ts-1} + \cdots + a_{s-1} A_{(t-1)s+m+1}$$
$$+ A'_{(t-1)s+m+1} + (s-m) A_{(t-1)s+m} px.$$

But $(a_m A_{ts})^* \sim a_m p^t x^{r+t}$ (Lemma 2) and, by (7), $((s-m)A_{(t-1)s+m}px)^* \sim a_m p^t x^{r+t}$; moreover, the degrees of all remaining components are (by Lemma 2) smaller than $r+t$. So, we have the contradiction $0 = a_m p^t \neq 0$. ∎

Now the equalities (3) have simpler forms. We know that $A_1 = \cdots = A_{s-2} = 0$, $A_{s-1} \sim A'_0$ and, by (2),

$$A_{(j+1)s-1} = A_{(js-1)+s} \sim A'_{js} + ((t-j)s+1) A_{js-1} px \tag{8}$$

for all j with $0 \leq (j+1)s - 1 \leq ts$. Moreover, by (4) (for $\sigma = 1$), we have

$$0 = A'_{ts} + A_{ts-1} px. \tag{9}$$

Suppose $t = 1$. Then $0 = A'_s + A_{s-1}px$ and $(A'_s)^* = (r+1)px^r \sim px^r$. If $r = 0$, then $A_{s-1} = 0$ (because $A_{s-1} \sim A'_0$) and so $0 \sim p \neq 0$, a contradiction. If $r > 0$, then $(A_{s-1}px)^* \sim px^r$ and, in this case, $0 \sim px^r \neq 0$, a contradiction again.

Therefore, $t > 1$. Now, using induction and (8), we see that

$$(A_{(j+1)s-1})^* \sim p^j x^{r+j-1}$$

for all j such that $0 \leq (j+1)s - 1 \leq ts$. In particular, $(A_{ts-1}px)^* \sim p^{t-1}x^{r+t-2}px = p^t x^{r+t-1}$. Moreover, by Lemma 2, $(A'_{ts})^* \sim p^t x^{r+t-1}$. So, by (9), we obtain the contradiction $0 \sim p^t x^{r+t-1} \neq 0$.

We have proved the following theorem.

THEOREM 1. *Let k be a field of characteristic zero and let D be a derivation of $k[x,y]$ of the form*

$$D = \frac{\partial}{\partial x} + (y^s + px)\frac{\partial}{\partial y},$$

where $s \geq 2$ and $0 \neq p \in k$. Then D is simple. ∎

Note also the following fact.

THEOREM 2. *Let k be a field of characteristic zero and let d be a derivation of $k[x,y]$ of the form*

$$d = \frac{\partial}{\partial x} + (y^s + px + q)\frac{\partial}{\partial y},$$

where $s \geq 2$, $p, q \in k$, $p \neq 0$. Then d is simple.

Proof. Let $\sigma : k[x, y] \to k[x, y]$ be the automorphism defined by $\sigma(x) = x + p^{-1}q$ and $\sigma(y) = y$. Then $d = \sigma D \sigma^{-1}$, where D is the derivation from Theorem 1. ∎

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