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CLASSIFICATION OF LOW-DIMENSIONAL ORBIT CLOSURES IN VARIETIES OF QUIVER REPRESENTATIONS

BY

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Abstract. We classify the affine varieties of dimension at most 4 which occur as orbit closures with an invariant point in varieties of representations of quivers. Moreover, we show that they are normal and Cohen–Macaulay.

1. Introduction. Throughout the paper, k denotes an algebraically closed field and $\mathbb{M}_{c\times d}(k)$ stands for the vector space of $c \times d$-matrices with coefficients in k for any $c, d \in \mathbb{N}$. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver, i.e. Q_0 is a finite set of vertices and Q_1 is a finite set of arrows $\alpha : s(\alpha) \to t(\alpha)$, where $s(\alpha)$ and $t(\alpha)$ denote the starting and ending vertices of α, respectively. Given a dimension vector $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$, we define the vector space

$$\mathrm{rep}_Q(\mathbf{d}) = \prod_{\alpha\in Q_1} \mathbb{M}_{d_{t(\alpha)}\times d_{s(\alpha)}}(k)$$

together with the regular action of the group

$$\mathrm{GL}(\mathbf{d}) = \prod_{i\in Q_0} \mathrm{GL}_{d_i}(k)$$

via

$$(g_i)_{i\in Q_0} \star (V_\alpha)_{\alpha\in Q_1} = (g_{t(\alpha)} \cdot V_\alpha \cdot g_{s(\alpha)}^{-1})_{\alpha\in Q_1}.$$

Let $\mathcal{O}_V = \mathrm{GL}(\mathbf{d}) \star V$ for any $V = (V_\alpha)_{\alpha\in Q_1} \in \mathrm{rep}_Q(\mathbf{d})$. Our main object of interest is the Zariski closure $\overline{\mathcal{O}}_V$ of the orbit $\mathcal{O}_V$ in $\mathrm{rep}_Q(\mathbf{d})$. The family of such orbit closures contains the classical determinantal varieties of matrices of bounded rank as well as the closures of conjugacy classes of square matrices. An interesting problem is to study singularities of $\overline{\mathcal{O}}_V$ (see [2]–[4], [6], [8], [12]–[14], [17], [19]–[21] for results in this direction). The orbit closure $\overline{\mathcal{O}}_V$ contains a unique closed orbit (see Section 2), say $\mathcal{O}_U$. Therefore $\overline{\mathcal{O}}_V$ has a $\mathrm{GL}(\mathbf{d})$-invariant point if and only if $\mathcal{O}_U$ consists of only one point. As we explain in Section 2, $\overline{\mathcal{O}}_V$ is isomorphic to an associated fibre bundle $\mathrm{GL}(\mathbf{d}) \times^{\mathrm{GL}(\mathbf{e})} \overline{\mathcal{O}}_W$ of an orbit closure $\overline{\mathcal{O}}_W$ having a $\mathrm{GL}(\mathbf{e})$-invariant point.

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Since the associated fibre bundles preserve singularities up to smooth morphisms, we can restrict our attention to the orbit closures with an invariant point. Our first main result is a classification, up to isomorphism, of the orbit closures having an invariant point, which as varieties have dimension at most 4 (see Theorem 3.2). As a consequence of the classification, we will prove that the $\mathrm{GL}(\mathbf{d})$-orbit closures in $\mathrm{rep}_Q(\mathbf{d})$ are normal and Cohen–Macaulay varieties provided their dimension is at most 4 (see Theorem 3.3). An open problem is if 4 can be replaced by a greater integer. There are examples of a 12-dimensional orbit closure which is not a normal variety (see [17, Section 6]) and a 14-dimensional orbit closure which is not a Cohen–Macaulay variety (see [18]).

Section 2 contains preliminaries on representations of quivers and related algebras, and the reduction of orbit closures to those having invariant points (and then to the orbit closures of admissible representations). In Section 3 we consider affine varieties appearing in the classification of orbit closures (Theorem 3.2). Section 4 contains reduction techniques which simplify the proof of the main result. In Section 5 we classify the orbit closures of admissible representations with special dimension vectors. The proof of the main result is finished in Section 6.

We refer to [1] for basic background on the representation theory of algebras and quivers. An introduction to toric varieties can be found in [9].

2. Representations of quivers and algebras. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver. A *representation* V of Q over k is a collection $(V_i;\ i \in Q_0)$ of finite-dimensional k-vector spaces together with a collection $(V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)};\ \alpha \in Q_1)$ of k-linear maps. Thus we may identify a point of $\mathrm{rep}_Q(\mathbf{d})$ with a representation V having $V_i = k^{d_i}$ for all $i \in Q_0$, where $\mathbf{d} = (d_i)_{i\in Q_0}$ is a dimension vector in $\mathbb{N}^{Q_0}$. A homomorphism $f : V \to W$ between two representations is a collection $(f_i : V_i \to W_i;\ i \in Q_0)$ of k-linear maps such that

$$f_{t(\alpha)} \circ V_\alpha = W_\alpha \circ f_{s(\alpha)} \quad \text{for all } \alpha \in Q_1.$$

It follows that two representations V and W in $\mathrm{rep}_Q(\mathbf{d})$ are isomorphic if and only if they belong to the same $\mathrm{GL}(\mathbf{d})$-orbit.

One calls a sequence

$$\omega = \alpha_n\alpha_{n-1}\cdots\alpha_2\alpha_1, \quad s(\alpha_{i+1}) = t(\alpha_i),\ i = 1, \ldots, n-1,$$

a *path* of length n in Q starting at $s(\omega) = s(\alpha_1)$ and ending at $t(\omega) = t(\alpha_n)$. We also consider a path ε_i of length 0 with $s(\varepsilon_i) = t(\varepsilon_i) = i$ for each vertex $i \in Q_0$. All paths of Q form a linear basis of the so-called *path algebra* kQ of Q. The elements ε_i, $i \in Q_0$, are idempotents of kQ and

$$1_{kQ} = \sum_{i\in Q_0} \varepsilon_i.$$

Any element $\varrho \in \varepsilon_j \cdot kQ \cdot \varepsilon_i$, $i,j \in Q_0$, is a linear combination of paths starting at i and ending at j; the coefficients of this combination form a matrix $M_\varrho \in \mathbb{M}_{d_j\times d_i}(k)$. We associate with a representation M in $\operatorname{rep}_Q(\mathbf{d})$ the algebra homomorphism

$$\widehat{M} : kQ \to \operatorname{End}_k\Big(\bigoplus_{i\in Q_0} k^{d_i}\Big)$$

such that the matrix $\widehat{M}(\varrho)$ is built of the blocs $M_{\varepsilon_j\cdot\varrho\cdot\varepsilon_i}$ for any $\varrho \in kQ$. Then the kernel of $\widehat{M}$ is called the *annihilator* of M and denoted by $\operatorname{Ann}(M)$. Moreover, we associate with M the algebra $A_M = kQ/\operatorname{Ann}(M)$, which is finite-dimensional ($\dim_k A_M \le (\sum d_i)^2$).

By a result due to Artin and Voigt (see [10, Corollary 1.3] and its proof), a representation V in $\operatorname{rep}_Q(\mathbf{d})$ is semisimple if and only if the orbit $\mathcal{O}_V$ is closed. Moreover, any orbit closure $\overline{\mathcal{O}}_M$ in $\operatorname{rep}_Q(\mathbf{d})$ contains exactly one closed orbit, say $\mathcal{O}_N$, where N is the semisimple representation of Q associated with M, i.e. N is the direct sum of the simple factors of a composition series of M. A special role is played by the orbit closures $\overline{\mathcal{O}}_M$ having a (necessarily unique) $\operatorname{GL}(\mathbf{d})$-invariant point, or equivalently, when $\mathcal{O}_N = \{N\}$. One says that an endomorphism of a finite-dimensional vector space is *equipotent* if all its eigenvalues are equal.

Lemma 2.1. *Let M be a point in* $\operatorname{rep}_Q(\mathbf{d})$. *Then the following conditions are equivalent*:

(1) $\overline{\mathcal{O}}_M$ *contains a* $\operatorname{GL}(\mathbf{d})$*-invariant point*;
(2) *the endomorphism M_ϱ is equipotent for any $\varrho \in \varepsilon_i \cdot kQ \cdot \varepsilon_i$, $i \in Q_0$, and M_ϱ is nilpotent for any $\varrho \in \varepsilon_i \cdot kQ \cdot \varepsilon_j \cdot kQ \cdot \varepsilon_i$, $i,j \in Q_0$, $i \ne j$*;
(3) *the algebra A_M is basic and the idempotents $\varepsilon_i + \operatorname{Ann}(M) \in A_M$, $i \in Q_0$, are primitive.*

Proof. Let $\mathcal{O}_N$ be the unique closed orbit in $\overline{\mathcal{O}}_M$. It is easy to see that $\mathcal{O}_N$ consists of one point ($\mathcal{O}_N = \{N\}$) if and only if

(1′) for each arrow $\alpha \in Q_1$, we have $N_\alpha = c_\alpha \cdot 1_{d_{s(\alpha)}}$, $c_\alpha \in k$, provided $s(\alpha) = t(\alpha)$, and $N_\alpha = 0$ otherwise.

Since taking the characteristic polynomial of X_ϱ is a $\operatorname{GL}(\mathbf{d})$-invariant function on $\operatorname{rep}_Q(\mathbf{d})$, condition (2) can be replaced by

(2′) the endomorphism N_ϱ is equipotent for any $\varrho \in \varepsilon_i \cdot kQ \cdot \varepsilon_i$, $i \in Q_0$, and N_ϱ is nilpotent for any $\varrho \in \varepsilon_i \cdot kQ \cdot \varepsilon_j \cdot kQ \cdot \varepsilon_i$, $i,j \in Q_0$, $i \ne j$.

Since N is the semisimple representation associated to M, we have $A_N = A_M/\operatorname{rad}(A_M)$ and therefore (3) can be replaced by

(3′) the (semisimple) algebra A_N is basic and the idempotents $\varepsilon_i + \mathrm{Ann}(N)$, $i \in Q_0$, are primitive.

The proofs of the implications (1′)⇒(2′)⇒(3′)⇒(1′) are straightforward. ∎

Let (Q, I) be a bound quiver, i.e. Q is a finite quiver and I is a two-sided ideal I of kQ. We write $\mathrm{rep}_{Q,I}(\mathbf{d})$ for the subset of $\mathrm{rep}_Q(\mathbf{d})$ consisting of the representations V such that $I \subseteq \mathrm{Ann}(V)$. In fact, the subset $\mathrm{rep}_{Q,I}(\mathbf{d})$ is closed and $\mathrm{GL}(\mathbf{d})$-invariant. Hence, the orbit closure $\overline{\mathcal{O}}_M$ is contained in the variety $\mathrm{rep}_{Q,\mathrm{Ann}(M)}(\mathbf{d})$.

Assume we have two bound quivers (Q, I) and (P, J) such that $Q_0 = P_0$. Given an algebra homomorphism

$$\Phi : kP/J \to kQ/I$$

such that $\Phi(\varepsilon_i + J) = \varepsilon_i + I$ for any $i \in Q_0$, we have an induced regular $\mathrm{GL}(\mathbf{d})$-equivariant morphism

$$\Phi(\mathbf{d}) : \mathrm{rep}_{Q,I}(\mathbf{d}) \to \mathrm{rep}_{P,J}(\mathbf{d}), \quad \Phi(\mathbf{d})(V)_\beta = V_\varrho,$$

for each dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, where $V = (V_\alpha) \in \mathrm{rep}_{Q,I}(\mathbf{d})$, $\beta \in P_1$ and $\varrho + I = \Phi(\beta + J)$. Obviously $\Phi(\mathbf{d})$ is an isomorphism provided Φ is an algebra isomorphism.

Let A be a finitely generated k-algebra and $e = (e_l)_{l\in L}$ be a finite collection of pairwise orthogonal idempotents of A whose sum equals 1_A. Observe that there is a bound quiver (Q, I) satisfying $Q_0 = L$ together with an isomorphism $A \simeq kQ/I$ sending e_l to $\varepsilon_l + I$ for each $l \in L$. Then we may write $\mathrm{rep}_{A,e}(\mathbf{d})$ for the $\mathrm{GL}(\mathbf{d})$-variety $\mathrm{rep}_{Q,I}(\mathbf{d})$, for any dimension vector $\mathbf{d} \in \mathbb{N}^L$.

Let us return to our point $M \in \mathrm{rep}_Q(\mathbf{d})$. The orbit closure $\overline{\mathcal{O}}_M$ can be considered as a point of the variety $\mathrm{rep}_{A_M,\varepsilon}(\mathbf{d})$, where $\varepsilon = (\varepsilon_i + \mathrm{Ann}(M))_{i\in Q_0}$. We choose a maximal semisimple subalgebra C of the finite-dimensional algebra A_M containing the collection ε. The embedding $\Phi : C \to A_M$ leads to the regular $\mathrm{GL}(\mathbf{d})$-equivariant morphism

$$\Phi(\mathbf{d}) : \mathrm{rep}_{A_M,\varepsilon}(\mathbf{d}) \to \mathrm{rep}_{C,\varepsilon}(\mathbf{d}).$$

Since the algebra C is semisimple, the $\mathrm{GL}(\mathbf{d})$-orbits in $\mathrm{rep}_{C,\varepsilon}(\mathbf{d})$ are closed. Consequently, $\Phi(\mathbf{d})(\overline{\mathcal{O}}_M)$ is a closed orbit in $\mathrm{rep}_{C,\varepsilon}(\mathbf{d})$, say $\mathcal{O}_D$. Let $\Psi : \overline{\mathcal{O}}_M \to \mathcal{O}_D$ denote the restriction of Φ. Then $\overline{\mathcal{O}}_M$ is isomorphic to an associated fibre bundle $\mathrm{GL}(\mathbf{d}) \times^H \Psi^{-1}(D)$, where H is the isotropy group of D (see [16, Lemma 3.7.4]). Let η be a maximal set of pairwise orthogonal, primitive and nonconjugate idempotents of A and let e stand for their sum. Then eAe is the basic algebra of A and eM is the eAe-module corresponding to M. Let $\mathbf{d}'$ be the dimension vector of eM. Then $H \simeq \mathrm{GL}(\mathbf{d}')$ and the H-variety $\Psi^{-1}(D)$ is isomorphic to the orbit closure $\overline{\mathcal{O}}_{eM}$ in $\mathrm{rep}_{eAe,\eta}(\mathbf{d}')$ (cf. [5, Section 2]).

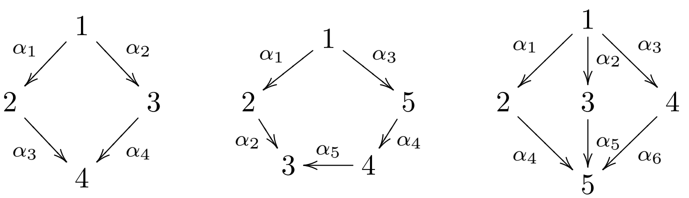
Given a finite quiver Q, we denote by $\mathcal{R}_Q$ the two-sided ideal in kQ generated by the paths of length one, i.e. the paths in kQ of positive length form a linear basis of $\mathcal{R}_Q$. It is well known that any basic algebra B can be represented in the form kQ/I, where the ideal I is *admissible*, i.e. $(\mathcal{R}_Q)^h \subseteq I \subseteq (\mathcal{R}_Q)^2$ for some $h \geq 2$. We call a representation $M \in \mathrm{rep}_Q(\mathbf{d})$ *admissible* if $\mathrm{Ann}(M)$ is an admissible ideal in kQ. Hence any orbit closure in $\mathrm{rep}_Q(\mathbf{d})$ having a $\mathrm{GL}(\mathbf{d})$-invariant point is isomorphic to the orbit closure of an admissible representation. Summarizing, any orbit closure in $\mathrm{rep}_Q(\mathbf{d})$ is isomorphic to an associated fibre bundle of the orbit closure of some admissible representation. Observe that an admissible representation M is nilpotent, i.e. the following equivalent conditions hold:

(1) $0 \in \overline{\mathcal{O}}_M$;
(2) the endomorphism M_ϱ is nilpotent for any $\varrho \in \varepsilon_i \cdot \mathcal{R}_Q \cdot \varepsilon_i$, $i \in Q_0$;
(3) $(\mathcal{R}_Q)^h \subseteq \mathrm{Ann}(M)$ for some positive integer h.

3. Special varieties and the main result. Let Q be a quiver without oriented cycles and $\mathbf{d} \in \mathbb{N}^{Q_0}$ be the dimension vector with all $d_i = 1$. Let M be an admissible representation in $\mathrm{rep}_Q(\mathbf{d})$. It is proved in [4] that $\overline{\mathcal{O}}_M$ is a normal toric variety with coordinate ring

$$k[\overline{\mathcal{O}}_M] = k[x_\alpha]_{\alpha \in Q_1}/I_Q,$$

where the ideal I_Q is generated by binomials coming from the primitive nonoriented cycles in Q (see [4, Section 2] for details). For instance, the quivers



give the orbit closures (denoted by $\mathcal{C}(2,2)$, $\mathcal{C}(2,3)$ and $\mathcal{C}(2,2,2)$, respectively) with coordinate algebras

$$k[x_1,x_2,x_3,x_4]/(x_3x_1 - x_4x_2), \quad k[x_1,x_2,x_3,x_4,x_5]/(x_2x_1 - x_5x_4x_3)$$

and

$$k[x_1,x_2,x_3,x_4,x_5,x_6]/(x_4x_1 - x_5x_2, x_5x_2 - x_6x_3),$$

respectively. Let $Q = 1 \overset{\alpha}{\longleftarrow} 2$, $\mathbf{d} = (p,q)$ and M be a representation in $\mathrm{rep}_Q(\mathbf{d})$ given by a matrix of rank one. We denote by $\mathcal{D}(p,q)$ the orbit closure $\overline{\mathcal{O}}_M$. It is well known (see for instance [7]) that $\mathcal{D}(p,q)$ is a normal and Cohen–Macaulay variety with the coordinate algebra

$$k[x_{i,j}]_{i \leq p,\, j \leq q}/(x_{i',j'}x_{i'',j''} - x_{i',j''}x_{i'',j'})_{i'<i'',\, j'<j''}.$$

In particular, $\mathcal{D}(2,2) \simeq \mathcal{C}(2,2)$.

Let $\mathcal{D}(2,2,2)$ denote the image of the multilinear map

$$k^2 \times k^2 \times k^2 \to k^2 \otimes k^2 \otimes k^2, \qquad (v_1, v_2, v_3) \mapsto v_1 \otimes v_2 \otimes v_3.$$

The torus $T = (k^*)^6$ acts on $\mathcal{D}(2,2,2)$ via

$$(t_1, t_2, t_3, t_4, t_5, t_6) \star \left(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} x_3 \\ x_4 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} x_5 \\ x_6 \end{smallmatrix}\right]\right) = \left(\left[\begin{smallmatrix} t_1 x_1 \\ t_2 x_2 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} t_3 x_3 \\ t_4 x_4 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} t_5 x_5 \\ t_6 x_6 \end{smallmatrix}\right]\right).$$

One can check that $\mathcal{D}(2,2,2)$ is a normal toric variety of dimension 4. Let $Q = 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$, $\mathbf{d} = (2,2)$ and $M = (\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]) \in \operatorname{rep}_Q(\mathbf{d})$. Direct calculations shows that the map

$$\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} x_3 \\ x_4 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} x_5 \\ x_6 \end{smallmatrix}\right] \mapsto \left(\left[\begin{smallmatrix} x_1x_3x_5 & x_1x_3x_6 \\ x_1x_4x_5 & x_1x_4x_6 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_2x_4x_6 & -x_2x_3x_6 \\ -x_2x_4x_5 & x_2x_3x_5 \end{smallmatrix}\right]\right)$$

gives an isomorphism between $\mathcal{D}(2,2,2)$ and $\overline{\mathcal{O}}_M$.

Let $p, q \geq 2$. Let H be a linear hyperplane in $\mathbb{M}_{p\times q}(k)$ given by

$$\sum_{i=1}^{p} \sum_{j=1}^{q} b_{i,j} \cdot x_{i,j} = 0,$$

where the matrix $B = [b_{i,j}] \in \mathbb{M}_{p\times q}(k)$ is nonzero. Since $\mathcal{D}(p,q)$ is invariant under the action of $G = \mathrm{GL}_p(k) \times \mathrm{GL}_q(k)$, the intersection $H \cap \mathcal{D}(p,q)$, up to isomorphism, depends only on the rank of B. Thus we obtain the varieties

$$\mathcal{HD}^{[r]}(p,q) = \{M = [m_{i,j}] \in \mathcal{D}(p,q);\ m_{1,1} + m_{2,2} + \cdots + m_{r,r} = 0\}$$

for $r = 1, \ldots, \min\{p,q\}$. It is easy to see that $\mathcal{HD}^{[1]}(p,q)$ has two irreducible components isomorphic to $\mathcal{D}(p-1,q)$ and $\mathcal{D}(p,q-1)$.

LEMMA 3.1. *Assume that $r \geq 2$. Then $\mathcal{HD}^{[r]}(p,q)$ is an irreducible, normal and Cohen–Macaulay variety with*

$$k[\mathcal{HD}^{[r]}(p,q)] = k[\mathcal{D}(p,q)]/(x_{1,1} + x_{2,2} + \cdots + x_{r,r}).$$

Proof. Since $k[\mathcal{D}(p,q)]$ is a Cohen–Macaulay domain, the quotient ring $R = k[\mathcal{D}(p,q)]/(x_{1,1} + \cdots + x_{r,r})$ is Cohen–Macaulay as well. Moreover,

$$\dim_N \mathcal{HD}^{[r]}(p,q) = \dim \mathcal{D}(p,q) - 1 = p + q - 2$$

for any $N \in \mathcal{HD}^{[r]}(p,q)$. To prove that R is reduced and normal, it suffices to show that the singular locus of $\mathrm{Spec}(R)$ has codimension greater than 1 (see [15]). A straightforward calculation shows that this singular locus consists of the matrices $N = [n_{i,j}] \in \mathcal{HD}^{[r]}(p,q)$ such that $n_{i,j} = 0$ if $i \leq r$ or $j \leq r$. Hence it is isomorphic to $\mathcal{D}(p-r, q-r)$ and its codimension equals

$$(p + q - 2) - (p - r + q - r - 1) = 2r - 1 \geq 3 > 1.$$

Since $\mathcal{HD}^{[r]}(p,q)$ is a cone, it is connected. Thus it must be irreducible as it is a normal variety. ■

Now we are ready to formulate the main result of the paper.

THEOREM 3.2. *Let M be a representation whose orbit closure $\overline{\mathcal{O}}_M$ is of dimension at most 4 and has an invariant point. Then it is isomorphic to a product of the following varieties:*

dim	
1	k
2	$\mathcal{HD}^{[2]}(2,2)$
3	$\mathcal{D}(2,2)$, $\mathcal{HD}^{[2]}(2,3)$
4	$\mathcal{D}(2,3)$, $\mathcal{HD}^{[2]}(2,4)$, $\mathcal{HD}^{[2]}(3,3)$, $\mathcal{HD}^{[3]}(3,3)$, $\mathcal{D}(2,2,2)$,$\mathcal{C}(2,3)$, $\mathcal{C}(2,2,2)$

By the well known Hochster theorem (see [11]) all normal toric varieties are Cohen–Macaulay. Combining this with Lemma 3.1 we see that all varieties appearing in Theorem 3.2 are normal and Cohen–Macaulay. Since the associated fibre bundles preserve the above local geometric properties, Theorem 3.2 implies the following result:

THEOREM 3.3. *Let M be a representation with* $\dim \mathcal{O}_M \leq 4$. *Then the variety $\overline{\mathcal{O}}_M$ is normal and Cohen–Macaulay.*

The rest of the paper is devoted to the proof of Theorem 3.2.

Let $M \in \operatorname{rep}_Q(\mathbf{d})$. Since the isotropy group of M can be identified with the automorphism group $\operatorname{Aut}_Q(M)$ and the latter is an open subset of the vector space $\operatorname{End}_Q(M)$, we get the well known formula

$$\begin{aligned}\dim \overline{\mathcal{O}}_M &= \dim \mathrm{GL}(\mathbf{d}) - \dim \operatorname{Aut}_Q(M) \\ &= \sum_{i\in Q_0} d_i^2 - \dim_k \operatorname{End}_Q(M).\end{aligned} \tag{3.1}$$

In order to find the defining equations for $\overline{\mathcal{O}}_M$ we shall often use the following method. We take two sequences $(i_1,\ldots,i_q)$ and $(j_1,\ldots,j_p)$ of vertices from Q_0 together with elements $\omega_{u,v} \in \varepsilon_{j_u} \cdot kQ \cdot \varepsilon_{i_v}$ for $u \leq p$ and $v \leq q$. Let $d' = \sum_{u=1}^{p} d_{j_u}$ and $d'' = \sum_{v=1}^{q} d_{i_v}$. We consider the regular morphism

$$\mathcal{F} : \operatorname{rep}_Q(\mathbf{d}) \to \mathbb{M}_{d'\times d''}(k), \qquad M \mapsto \begin{pmatrix} M_{\omega_{1,1}} & \cdots & M_{\omega_{1,q}} \\ \vdots & \ddots & \vdots \\ M_{\omega_{p,1}} & \cdots & M_{\omega_{p,q}} \end{pmatrix}.$$

Observe that $\operatorname{rk}(\mathcal{F}(g\star M)) = \operatorname{rk}(\mathcal{F}(M))$ for any $g \in \mathrm{GL}(\mathbf{d})$. We denote by X a set of variables corresponding bijectively to the entries of the matrices $\mathbb{M}_{d_{t(\alpha)}\times d_{s(\alpha)}}(k)$, $\alpha \in Q_1$, so $k[X]$ can be identified with the polynomial ring of $\operatorname{rep}_Q(\mathbf{d})$. Then the above implies the following fact:

LEMMA 3.4. *The variety $\overline{\mathcal{O}}_M$ satisfies the equations given by vanishing of all minors of size* $\operatorname{rk}(\mathcal{F}(M)) + 1$ *in the matrix* $\mathcal{F}(X)$.

4. Reduction techniques. Let Q' be a subquiver of a quiver Q. We may regard kQ' as a subalgebra of kQ. Let $\mathbf{d}|_{Q'} \in \mathbb{N}^{Q'_0}$, $g|_{Q'} \in \mathrm{GL}(\mathbf{d}|_{Q'})$ and $M|_{Q'} \in \mathrm{rep}_{Q'}(\mathbf{d}|_{Q'})$ denote the restrictions of $\mathbf{d} \in \mathbb{N}^{Q_0}$, $g \in \mathrm{GL}(\mathbf{d})$ and $M \in \mathrm{rep}_Q(\mathbf{d})$, respectively. The inclusion $kQ' \subseteq kQ$ induces the linear and surjective map

$$\varrho : \mathrm{rep}_Q(\mathbf{d}) \to \mathrm{rep}_{Q'}(\mathbf{d}|_{Q'}), \quad \varrho(M) = M|_{Q'}.$$

Since $\varrho(g \star M) = g|_{Q'} \star M|_{Q'}$, $\varrho(\mathcal{O}_M) = \mathcal{O}_{M'}$ and we get the following fact.

Corollary 4.1. *Let M be a representation of Q, and Q' be a subquiver of Q. Then*

$$\dim \mathcal{O}_M \geq \dim \mathcal{O}_{M|_{Q'}}.$$

Note that $\mathrm{Ann}(M|_{Q'}) = \mathrm{Ann}(M) \cap kQ'$. Since $(\mathcal{R}_Q)^n \cap kQ' = (\mathcal{R}_{Q'})^n$ for all $n \geq 1$, we get the following corollary.

Corollary 4.2. *A restriction of an admissible (resp. nilpotent) representation is an admissible (resp. nilpotent) representation.*

The *opposite quiver* Q^{op} is $Q^{\mathrm{op}} = (Q_0, Q_1, s', t')$ where, for $\alpha \in Q_1$, $s'(\alpha) = t(\alpha)$ and $t'(\alpha) = s(\alpha)$.

Remark 4.3. To classify the varieties which occur as orbit closures of nilpotent representations, it is enough to consider quivers up to duality. Indeed, consider the two maps

$$\begin{aligned} &\Phi : \mathrm{rep}_Q(\mathbf{d}) \to \mathrm{rep}_{Q^{\mathrm{op}}}(\mathbf{d}), \quad &&\Phi((M_\alpha)_{\alpha \in Q_0}) = ((M_\alpha)^t)_{\alpha \in Q_0},\\ &\Psi : \mathrm{GL}(\mathbf{d}) \to \mathrm{GL}(\mathbf{d}), &&\Psi((g_i)_{i \in Q_0}) = ((g_i^{-1})^t)_{i \in Q_0}. \end{aligned}$$

The former is a linear isomorphism while the latter is a group automorphism. Moreover, a representation M is admissible or nilpotent if and only if $\Phi(M)$ has the same property. The formula $\Phi(g \star M) = \Psi(g) \star \Phi(M)$ leads to the isomorphism of orbits $\mathcal{O}_M \simeq \mathcal{O}_{\Phi(M)}$ and their closures $\overline{\mathcal{O}}_M \simeq \overline{\mathcal{O}}_{\Phi(M)}$.

Let $\alpha_1, \ldots, \alpha_{l_{i,j}}$ be the arrows starting at i and ending at j, where $i, j \in Q_0$. The group $\mathrm{GL}_{l_{i,j}}(k)$ acts on the $l_{i,j}$-dimensional linear space

$$\langle \alpha_1, \ldots, \alpha_{l_{i,j}} \rangle \subseteq kQ$$

in a natural way. This extends easily to an action of the group

$$\mathrm{GL}((l_{i,j})_{i,j \in Q_0}) = \prod_{i,j \in Q_0} \mathrm{GL}_{l_{i,j}}(k)$$

on kQ. We have the induced action of $\mathrm{GL}((l_{i,j})_{i,j \in Q_0})$ on $\mathrm{rep}_Q(\mathbf{d})$ given by

$$(h \circ M)_\alpha = M_{h^{-1} \cdot \alpha}$$

for $h \in \mathrm{GL}((l_{i,j})_{i,j \in Q_0})$, $M \in \mathrm{rep}_Q(\mathbf{d})$ and $\alpha \in Q_1$. Note that

$$h \circ (g \star M) = g \star (h \circ M)$$

for any $g \in \mathrm{GL}(\mathbf{d})$, $h \in \mathrm{GL}((l_{i,j})_{i,j\in Q_0})$ and $M \in \mathrm{rep}_Q(\mathbf{d})$. Moreover, $\mathrm{Ann}(h \circ M) = h \circ \mathrm{Ann}(M)$ and $h \circ (\mathcal{R}_Q)^m = (\mathcal{R}_Q)^m$. Thus we get the following corollary.

Corollary 4.4. *With the above notation*

$$\mathcal{O}_M \simeq \mathcal{O}_{h\circ M} \quad \text{and} \quad \overline{\mathcal{O}}_M \simeq \overline{\mathcal{O}}_{h\circ M},$$

for any $h \in \mathrm{GL}((l_{i,j})_{i,j\in Q_0})$. *Moreover,* M *is an admissible representation if and only if* $h \circ M$ *is.*

Remark 4.5. To classify the varieties which occur as orbit closures of nilpotent representations, it is enough to consider connected quivers. Indeed, if a quiver Q is a disjoint union of subquivers Q' and Q'', then for fixed $\mathbf{d} \in \mathbb{N}^{Q_0}$, the map

$$\mathrm{rep}_Q(\mathbf{d}) \to \mathrm{rep}_{Q'}(\mathbf{d}|_{Q'}) \times \mathrm{rep}_{Q''}(\mathbf{d}|_{Q''}),$$
$$((M_\alpha)_{\alpha\in Q_1}) \mapsto ((M_\alpha)_{\alpha\in(Q')_1}, (M_\alpha)_{\alpha\in(Q'')_1}),$$

is a linear $\mathrm{GL}(\mathbf{d})$-isomorphism. Moreover, if $M \in \mathrm{rep}_Q(\mathbf{d})$, then

$$\overline{\mathcal{O}}_M \simeq \overline{\mathcal{O}}_{M|_{Q'}} \times \overline{\mathcal{O}}_{M|_{Q''}}.$$

There is another special case where an orbit closure is isomorphic to a product of two other orbit closures.

Lemma 4.6. *Let* Q' *and* Q'' *be subquivers of a quiver* Q *such that*

$$Q_0 = Q'_0 \cup Q''_0, \quad Q_1 = Q'_1 \cup Q''_1, \quad Q'_0 \cap Q''_0 = \{a\}, \quad Q'_1 \cap Q''_1 = \emptyset.$$

Let $\mathbf{d} \in \mathbb{N}^{Q_0}$ *be a dimension vector with* $d_a = 1$, M *be a representation in* $\mathrm{rep}_Q(\mathbf{d})$ *and* $M' = M|_{Q'}$, $M'' = M|_{Q''}$. *Then*

$$\overline{\mathcal{O}}_M \simeq \overline{\mathcal{O}}_{M'} \times \overline{\mathcal{O}}_{M''}.$$

Proof. Since $Q = Q' \cup Q''$, the linear map

$$\mathrm{End}_Q(M) \to \mathrm{End}_{Q'}(M') \times \mathrm{End}_{Q''}(M''), \quad (h_i)_{i\in Q_0} \mapsto ((h_i)_{i\in Q'_0}, (h_i)_{i\in Q''_0}),$$

is injective with image

$$\{((h'_i)_{i\in Q'_0}, (h''_i)_{i\in Q''_0}) \in \mathrm{End}_{Q'}(M') \times \mathrm{End}_{Q''}(M'');$$
$$h'_i = h''_i \text{ for all } i \in Q'_0 \cap Q''_0\}.$$

Since $Q'_0 \cap Q''_0 = \{a\}$ and $d_a = 1$, this image is described by one equation $h'_a = h''_a$. This equation does not hold at $((1_{d_i})_{i\in Q'_0}, (0)_{i\in Q''_0})$ and hence

$$\dim_k \mathrm{End}_Q(M) = \dim_k(\mathrm{End}_{Q'}(M') \times \mathrm{End}_{Q''}(M'')) - 1.$$

The above partition of Q leads to the linear isomorphism

$$\mathcal{F} : \mathrm{rep}_Q(\mathbf{d}) \to \mathrm{rep}_{Q'}(\mathbf{d}|_{Q'}) \times \mathrm{rep}_{Q''}(\mathbf{d}|_{Q''}),$$
$$\mathcal{F}((M_\alpha)_{\alpha\in Q_1}) = ((M_\alpha)_{\alpha\in Q'_1}, (M_\alpha)_{\alpha\in Q''_1}).$$

Of course $\mathcal{F}(M) = (M', M'')$. Note that $\mathcal{F}(\mathcal{O}_M) \subseteq \mathcal{O}_{M'} \times \mathcal{O}_{M''}$ and

$$\mathcal{F}(\overline{\mathcal{O}}_M) \subseteq \overline{\mathcal{O}_{M'} \times \mathcal{O}_{M''}} = \overline{\mathcal{O}}_{M'} \times \overline{\mathcal{O}}_{M''},$$

because $\mathcal{F}$ is continuous. On the other hand,

$$\begin{aligned}
\dim \mathcal{F}(\overline{\mathcal{O}}_M) &= \dim \mathcal{O}_M = \sum_{i \in Q_0} d_i^2 - \dim_k \operatorname{End}_Q(M) \\
&= \Big(\sum_{i \in Q_0'} d_i^2 + \sum_{i \in Q_0''} d_i^2 - 1 \Big) - (\dim_k \operatorname{End}_{Q'}(M') + \dim_k \operatorname{End}_{Q''}(M'') - 1) \\
&= \Big(\sum_{i \in Q_0'} d_i^2 - \dim_k \operatorname{End}_{Q'}(M') \Big) + \Big(\sum_{i \in Q_0''} d_i^2 - \dim_k \operatorname{End}_{Q''}(M'') \Big) \\
&= \dim \mathcal{O}_{M'} + \dim \mathcal{O}_{M''} = \dim(\overline{\mathcal{O}}_{M'} \times \overline{\mathcal{O}}_{M''}).
\end{aligned}$$

Since $\mathcal{F}(\overline{\mathcal{O}}_M)$ is a closed subset of the irreducible variety $\overline{\mathcal{O}}_{M'} \times \overline{\mathcal{O}}_{M''}$, we conclude that $\mathcal{F}(\overline{\mathcal{O}}_M) = \overline{\mathcal{O}}_{M'} \times \overline{\mathcal{O}}_{M''}$. ∎

Remark 4.7. Fix a finite set Q_0 and $\mathbf{d} \in \mathbb{N}^{Q_0}$. There are only finitely many quivers (up to isomorphism) such that $\operatorname{rep}_Q(\mathbf{d})$ contains an admissible representation. Indeed, if M is an admissible representation and i, j are vertices, then the matrices of the form M_α, where $s(\alpha) = i$ and $t(\alpha) = j$, are linearly independent in $\mathbb{M}_{d_j \times d_i}(k)$. Consequently, there are at most $d_i \cdot d_j$ arrows satisfying $s(\alpha) = i$ and $t(\alpha) = j$.

Lemma 4.8. *Let* $Q = 1 \stackrel{\alpha}{\leftarrow} 2$, $\mathbf{d} = (p, q)$, $M \in \operatorname{rep}_Q(\mathbf{d})$ *and* $r = \operatorname{rk} M_\alpha$. *Then*

$$\dim \mathcal{O}_M = r(p + q - r).$$

Moreover, if $r = 1$, *then* $\overline{\mathcal{O}}_M \simeq \mathcal{D}(p, q)$.

Proof. The second part was mentioned in Section 3, and the first part is well known (see for instance [7, Section 1C]). In fact, we can easily calculate the dimension of $\mathcal{O}_M$ using (3.1) and replacing M by an isomorphic representation $M' = \left(\left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right] \right)$. ∎

Corollary 4.9. *Let* $Q = 1 \stackrel{\alpha}{\leftarrow} 2$, $\mathbf{d} = (p, q)$, $M, M' \in \operatorname{rep}_Q(\mathbf{d})$ *and* $r = \operatorname{rk}(M_\alpha)$, $r' = \operatorname{rk}(M'_\alpha)$. *If* $r > r'$ *then*

$$\dim \mathcal{O}_M > \dim \mathcal{O}_{M'}.$$

Corollary 4.10. *Let* $Q = 1 \stackrel{\alpha}{\leftarrow} 2$. *If* $M \in \operatorname{rep}_Q(\mathbf{d})$ *is nonzero, then*

$$\dim \overline{\mathcal{O}}_M \geq |\mathbf{d}| - 1.$$

We shall prove in this section that this inequality holds for any quiver Q and any admissible representation M in $\operatorname{rep}_Q(\mathbf{d})$.

Lemma 4.11. *Let* $Q = a \circlearrowleft \alpha$, $\mathbf{d} = (d)$. *If* $M \in \mathrm{rep}_Q(\mathbf{d})$ *is nonzero and nilpotent, then* $d \geq 2$ *and*

$$\dim \mathcal{O}_M \geq 2d - 2.$$

Moreover, equality holds if and only if the matrix M_α *is conjugate to* $E_{1,2}$, *where* $E_{i,j}$ *denotes the matrix whose only nonzero element is* 1 *in the ith row and jth column.*

Proof. This is well known (see for instance [13, Section 2.3]). ∎

Lemma 4.12. *Let* Q *be a subquiver of a quiver* $\widetilde{Q}$ *such that* $\widetilde{Q}_0 \setminus Q_0 = \{a\}$, $\widetilde{\mathbf{d}} \in \mathbb{N}^{\widetilde{Q}_0}$, $\widetilde{M} \in \mathrm{rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}})$ *and* $M = \widetilde{M}|_Q$. *Moreover, let* $U = \bigcap_{\alpha \in A} \mathrm{Ker}(\widetilde{M}_\alpha)$, *where* $A = \{\alpha \in \widetilde{Q}_1;\ s(\alpha) = a \neq t(\alpha)\}$, *and* V *be the subspace generated by the images* $\mathrm{Im}(\widetilde{M}_\beta)$, *where* $\beta \in B = \{\beta \in \widetilde{Q}_1;\ t(\beta) = a \neq s(\beta)\}$. *Then*

$$\dim \mathcal{O}_{\widetilde{M}} \geq \dim \mathcal{O}_M + d_a^2 - \dim_k U \cdot (d_a - \dim_k V).$$

Proof. Consider the linear map

$$\pi : \mathrm{End}_{\widetilde{Q}}(\widetilde{M}) \to \mathrm{End}_Q(M), \qquad \pi((f_i)_{i \in \widetilde{Q}_0}) = (f_i)_{i \in Q_0}.$$

Then

$$\dim_k \mathrm{End}_{\widetilde{Q}}(\widetilde{M}) \leq \dim_k \mathrm{Ker}(\pi) + \dim_k \mathrm{End}_Q(M).$$

Applying (3.1) we get

$$\begin{aligned} \dim \mathcal{O}_{\widetilde{M}} &= \sum_{i \in \widetilde{Q}_0} d_i^2 - \dim_k \mathrm{End}_{\widetilde{Q}}(\widetilde{M}) \\ &\geq \sum_{i \in Q_0} d_i^2 - \dim_k \mathrm{End}_Q(M) + d_a^2 - \dim_k \mathrm{Ker}(\pi) \\ &= \dim \mathcal{O}_M + d_a^2 - \dim_k \mathrm{Ker}(\pi). \end{aligned}$$

Hence, it suffices to show that $\dim_k \mathrm{Ker}(\pi) \leq \dim_k U \cdot (d_a - \dim_k V)$. But this is evident, since

$$\begin{aligned} \mathrm{Ker}(\pi) \subseteq \{ &f_a \in \mathrm{End}_k(k^{d_a});\ \widetilde{M}_\alpha \circ f_a = 0 = f_a \circ \widetilde{M}_\beta \\ &\qquad \text{for all } \alpha \in A \text{ and } \beta \in B\} \\ = \{ &f_a \in \mathrm{End}_k(k^{d_a});\ \mathrm{Im}(f) \subseteq \mathrm{Ker}(\widetilde{M}_\alpha) \text{ and } \mathrm{Im}(\widetilde{M}_\beta) \subseteq \mathrm{Ker}(f) \\ &\qquad \text{for all } \alpha \in A \text{ and } \beta \in B\}, \\ \simeq \ &\mathrm{Hom}_k(k^{d_a}/V, U). \ ∎ \end{aligned}$$

Corollary 4.13. *Let* Q *be a subquiver of a quiver* $\widetilde{Q}$ *such that* $\widetilde{Q}_0 \setminus Q_0 = \{a\}$, $\widetilde{\mathbf{d}} \in \mathbb{N}^{\widetilde{Q}_0}$, *and let* $\widetilde{M} \in \mathrm{rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}})$ *satisfy* $\widetilde{M}_\alpha \neq 0$ *for all* $\alpha \in \widetilde{Q}_1$. *Let* $M = \widetilde{M}|_Q$. *If there exists an arrow* α *in* $\widetilde{Q}_1$ *such that* $s(\alpha) = a \neq t(\alpha)$ *or* $t(\alpha) = a \neq s(\alpha)$, *then*

$$\dim \mathcal{O}_{\widetilde{M}} \geq \dim \mathcal{O}_M + d_a.$$

Furthermore, if there exist arrows $\alpha, \beta \in \widetilde{Q}_1$ such that $t(\alpha) \neq s(\alpha) = a = t(\beta) \neq s(\beta)$, then

$$\dim \mathcal{O}_{\widetilde{M}} \geq \dim \mathcal{O}_M + 2d_a - 1.$$

In particular, the last corollary holds for any admissible representation.

COROLLARY 4.14. *If $M \simeq k \underset{M_{\alpha_r}}{\overset{M_{\alpha_1}}{\vdots}} k^n$ is admissible, then $r \leq n$ and $\overline{\mathcal{O}}_M \simeq k^{nr}$.*

Proof. The claim follows easily from Remark 4.7 and Lemma 4.12. ∎

LEMMA 4.15. *Let Q' and Q'' be subquivers of Q such that $Q'_0 \cap Q''_0 = \emptyset$ and there exists an arrow α satisfying $s(\alpha) \in Q'_0$ and $t(\alpha) \in Q''_0$. Let $M \in \operatorname{rep}_Q(\mathbf{d})$ be such that $M_\alpha \neq 0$ and let $M' = M|_{Q'}$ and $M'' = M|_{Q''}$. Then*

$$\dim \overline{\mathcal{O}}_M > \dim \overline{\mathcal{O}}_{M'} + \dim \overline{\mathcal{O}}_{M''}.$$

Proof. Consider the linear injective map

$$h : \operatorname{End}_Q(M) \to \operatorname{End}_{Q'}(M') \times \operatorname{End}_{Q''}(M''),$$
$$(h_i)_{i \in Q_0} \mapsto ((h_i)_{i \in Q'_0}, (h_i)_{i \in Q''_0}).$$

The element $((1_{d_i})_{i \in Q'_0}, (0)_{i \in Q''_0}) \in \operatorname{End}_{Q'}(M') \times \operatorname{End}_{Q''}(M'')$ does not belong to the image of h, and thus

$$\dim_k \operatorname{End}_Q(M) < \dim_k \operatorname{End}_{Q'}(M') + \dim_k \operatorname{End}_{Q''}(M'').$$

The result now follows easily from (3.1). ∎

REMARK 4.16. If $M_\beta \neq 0$ for each $\beta \in Q_1$, then we can replace the assumption on α in the above lemma by the assumption that Q is connected. Indeed, by Corollary 4.1, we can assume that $Q'_0 \cup Q''_0 = Q_0$.

Let Q be a connected quiver. Recall that for any subquiver $Q' \subseteq Q$, if $M \in \operatorname{rep}_Q(\mathbf{d})$ is admissible and $M' = M|_{Q'}$, then $\dim \overline{\mathcal{O}}_{M'} \leq \dim \overline{\mathcal{O}}_M$ (Corollary 4.1). We also know that if $Q'_0 \neq Q_0$, then $\dim \overline{\mathcal{O}}_{M'} < \dim \overline{\mathcal{O}}_M$ (Corollary 4.13). However, if $Q'_0 = Q_0$ and $Q'_1 \neq Q_1$, these dimensions may be equal, as one can see in the following example.

EXAMPLE 4.17. Consider

$$Q = \alpha \circlearrowright a \underset{\delta}{\overset{\beta}{\rightleftarrows}} b \circlearrowleft \gamma \qquad M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \circlearrowright k^2 \underset{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}{\overset{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}{\rightleftarrows}} k^2 \circlearrowleft \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the subquiver Q' obtained from Q by removing β. Then

$$\dim \mathcal{O}_M = \dim \mathcal{O}_{M|_{Q'}} = 5.$$

LEMMA 4.18. *Let Q be a connected quiver such that $Q_0 = \{a, b\}$ and $Q_1 = \{\alpha, \beta\}$. Let $M \in \operatorname{rep}_Q(\mathbf{d})$ be admissible and M' be its restriction to the subquiver Q' obtained from Q by removing β. Then*

$$\dim \overline{\mathcal{O}}_M > \dim \overline{\mathcal{O}}_{M'}.$$

Proof. If $s(\alpha) = t(\alpha)$, then the statement follows immediately from Corollary 4.13 applied to the vertex different from $s(\alpha)$. Thus, we have to consider three cases (up to duality, see Remark 4.3):

$$Q = a \underset{\beta}{\overset{\alpha}{\rightleftarrows}} b \qquad Q = a \underset{\beta}{\overset{\alpha}{\rightrightarrows}} b \qquad Q = a \xrightarrow{\alpha} b \circlearrowleft \beta$$

We may assume that $M_\alpha = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Suppose that $\dim \overline{\mathcal{O}}_M = \dim \overline{\mathcal{O}}_{M'}$. This means that

$$\begin{aligned} &\operatorname{End}_Q(M) \simeq \operatorname{End}_{Q'}(M') \\ &\quad \simeq \left\{ \left(\begin{bmatrix} G_1 & 0 \\ G_3' & G_4' \end{bmatrix}, \begin{bmatrix} G_1 & G_2'' \\ 0 & G_4'' \end{bmatrix} \right) \in \mathbb{M}_{d_a \times d_a}(k) \times \mathbb{M}_{d_b \times d_b}(k);\, G_1 \in \mathbb{M}_{r \times r}(k) \right\}. \end{aligned}$$

Let $M_\beta = \begin{bmatrix} M_{\beta 1} & M_{\beta 2} \\ M_{\beta 3} & M_{\beta 4} \end{bmatrix}$, where $M_{\beta 1} \in \mathbb{M}_{r \times r}(k)$. Then in the first case

$$\begin{bmatrix} G_1 & 0 \\ G_3' & G_4' \end{bmatrix} \cdot \begin{bmatrix} M_{\beta 1} & M_{\beta 2} \\ M_{\beta 3} & M_{\beta 4} \end{bmatrix} = \begin{bmatrix} M_{\beta 1} & M_{\beta 2} \\ M_{\beta 3} & M_{\beta 4} \end{bmatrix} \cdot \begin{bmatrix} G_1 & G_2'' \\ 0 & G_4'' \end{bmatrix}$$

for all G_1, G_3', G_4', G_2'' and G_4''. Thus $M_\beta = 0$, and we get a contradiction, because M is admissible. In the second case

$$\begin{bmatrix} M_{\beta 1} & M_{\beta 2} \\ M_{\beta 3} & M_{\beta 4} \end{bmatrix} \cdot \begin{bmatrix} G_1 & 0 \\ G_3' & G_4' \end{bmatrix} = \begin{bmatrix} G_1 & G_2'' \\ 0 & G_4'' \end{bmatrix} \cdot \begin{bmatrix} M_{\beta 1} & M_{\beta 2} \\ M_{\beta 3} & M_{\beta 4} \end{bmatrix}$$

for all G_1, G_3', G_4', G_2'' and G_4''. Thus $M_{\beta 4} = 0$, $M_{\beta 3} = 0$, $M_{\beta 2} = 0$ and $M_{\beta 1} G_1 = G_1 M_{\beta 1}$ for any G_1. Hence $M_{\beta 1} = t \cdot I_r$ for some $t \in k$, and $t \cdot \alpha - \beta \in \operatorname{Ann}(M)$, a contradiction. Finally, in the third case

$$\begin{bmatrix} G_1 & G_2'' \\ 0 & G_4'' \end{bmatrix} \cdot \begin{bmatrix} M_{\beta 1} & M_{\beta 2} \\ M_{\beta 3} & M_{\beta 4} \end{bmatrix} = \begin{bmatrix} M_{\beta 1} & M_{\beta 2} \\ M_{\beta 3} & M_{\beta 4} \end{bmatrix} \cdot \begin{bmatrix} G_1 & G_2'' \\ 0 & G_4'' \end{bmatrix}$$

for all G_1, G_2'' and G_4''. Thus $M_{\beta 3} = 0$, $M_{\beta 1} = t \cdot I_r$ for some $t \in k$, and $M_{\beta 4} = s \cdot I_{d_b - r}$ for some $s \in k$. Since M_β is nilpotent, $t = s = 0$. This yields $M_{\beta 2} = 0$. Hence $M_\beta = 0$, a contradiction once again. ∎

LEMMA 4.19. *Let $Q = a \underset{\beta}{\overset{\alpha}{\rightrightarrows}} b$, $\mathbf{d} = (p, q)$. If $M \in \operatorname{rep}_Q(\mathbf{d})$ is admissible, then*

$$\dim \overline{\mathcal{O}}_M \geq p + q + \min\{p, q\} - 1.$$

Proof. If p or q is equal to 1, then, by Corollary 4.14,

$$\dim \overline{\mathcal{O}}_M = 2 \max\{p, q\} > \max\{p, q\} + 1 = p + q + \min\{p, q\} - 1.$$

The inequality is strict, because p and q cannot be equal to 1 simultaneously (Remark 4.7).

Now we assume that $p, q \geq 2$ and $\mathrm{rk}(M_\alpha) \geq \mathrm{rk}(M_\beta)$. Let M' be the restriction of M to the subquiver obtained from Q by removing β. If $\mathrm{rk}(M'_\alpha) = r \geq 2$, then by Lemmas 4.18 and 4.8, Corollary 4.9 and the inequality $p+q \geq \min\{p,q\}+2$ we get

$$\begin{aligned}\dim \overline{\mathcal{O}}_M &\geq \dim \overline{\mathcal{O}}_{M'} + 1 = r(p+q-r)+1 \\ &\geq 2(p+q-2)+1 \geq p+q+\min\{p,q\}-1.\end{aligned}$$

To conclude the proof, assume that $\mathrm{rk}(M_\alpha) = \mathrm{rk}(M_\beta) = 1$. We can additionally assume that $\mathrm{rk}(M_\beta + t\cdot M_\alpha) = 1$ for all $t \in k$. Indeed, $\widetilde{M}^{(t)} = (M_\alpha, M_\beta + t\cdot M_\alpha) = \left[\begin{smallmatrix} 1 & 0 \\ -t & 1\end{smallmatrix}\right] \circ (M_\alpha, M_\beta)$, thus the orbit closures of $\widetilde{M}^{(t)}$ and M are isomorphic (Corollary 4.4). Suppose that $\mathrm{Ker}(M_\alpha) \neq \mathrm{Ker}(M_\beta)$ and $\mathrm{Im}(M_\alpha) \neq \mathrm{Im}(M_\beta)$. Take vectors $v_\alpha \in \mathrm{Ker}(M_\beta) \setminus \mathrm{Ker}(M_\alpha)$ and $v_\beta \in \mathrm{Ker}(M_\alpha)\setminus\mathrm{Ker}(M_\beta)$ and note that the images $v'_\alpha = M_\alpha(v_\alpha)$ and $v'_\beta = M_\beta(v_\beta)$ in k^q are linearly independent. Then

$$\widetilde{M}^{(1)}_\beta(v_\alpha) = M_\beta(v_\alpha) + M_\alpha(v_\alpha) = v'_\alpha, \qquad \widetilde{M}^{(1)}_\beta(v_\beta) = M_\beta(v_\beta) + M_\alpha(v_\beta) = v'_\beta,$$

i.e. $\mathrm{rk}(\widetilde{M}^{(1)}_\beta) \geq 2$ and we get a contradiction. Therefore $\mathrm{Ker}(M_\alpha) = \mathrm{Ker}(M_\beta)$ or $\mathrm{Im}(M_\alpha) = \mathrm{Im}(M_\beta)$. In both cases the proof is similar, so consider the latter. Then $\overline{\mathcal{O}}_M$ is isomorphic to the orbit closure of the representation $(E_{1,1}, E_{1,2})$. It is easy to calculate that $\dim_k \mathrm{End}_Q((E_{1,1}, E_{1,2})) = p^2 - 2p + q^2 - q + 1$. Thus by (3.1),

$$\dim \overline{\mathcal{O}}_M = 2p+q-1 \geq p+q+\min\{p,q\}-1. \ \blacksquare$$

LEMMA 4.20. *Let* $Q = a \underset{\beta}{\overset{\alpha}{\rightleftarrows}} b$ *and* $\mathbf{d} = (p,q)$. *If* $M \in \mathrm{rep}_Q(\mathbf{d})$ *is admissible*, *then*

$$\dim \overline{\mathcal{O}}_M \geq 2p+2q-4.$$

Moreover, *equality holds if and only if* $\mathrm{rk}(M_\alpha) = \mathrm{rk}(M_\beta) = 1$, $\mathrm{Im}(M_\alpha) \subseteq \mathrm{Ker}(M_\beta)$ *and* $\mathrm{Im}(M_\beta) \subseteq \mathrm{Ker}(M_\alpha)$.

Proof. Without loss of generality we may assume that $\mathrm{rk}(M_\alpha) \geq \mathrm{rk}(M_\beta)$. If $\mathrm{rk}(M_\alpha) = r \geq 2$, then by Lemmas 4.18 and 4.8, and Corollary 4.9,

$$\dim \overline{\mathcal{O}}_M \geq r(p+q-r)+1 \geq 2(p+q-2)+1.$$

It remains to consider the case when $\mathrm{rk}(M_\alpha) = \mathrm{rk}(M_\beta) = 1$. We may assume that $M_\alpha = E_{1,1}$. Since M is nilpotent, either $\mathrm{Im}(M_\alpha) \subseteq \mathrm{Ker}(M_\beta)$ or $\mathrm{Im}(M_\beta) \subseteq \mathrm{Ker}(M_\alpha)$. If both inclusions hold, then $\overline{\mathcal{O}}_M$ is isomorphic to the orbit closure of $(E_{1,1}, E_{2,2})$ and $\dim \overline{\mathcal{O}}_M = 2p+2q-4$. If $\mathrm{Im}(M_\alpha) \nsubseteq \mathrm{Ker}(M_\beta)$, then $\overline{\mathcal{O}}_M$ is isomorphic to the orbit closure of $(E_{1,1}, E_{2,1})$. If $\mathrm{Im}(M_\beta) \nsubseteq \mathrm{Ker}(M_\alpha)$, then M is isomorphic to $(E_{1,1}, E_{1,2})$. In both cases, computing the dimensions of the endomorphism spaces yields $\dim \overline{\mathcal{O}}_M = 2p+2q-3$. $\blacksquare$

Lemma 4.21. *Let Q be a quiver whose underlying graph G_Q is a cycle of length less than* 4. *If $M \in \operatorname{rep}_Q(\mathbf{d})$ is admissible, then*

$$\dim \mathcal{O}_M \geq |\mathbf{d}|.$$

Proof. If G_Q is a cycle of length 1 (i.e., Q is a loop quiver), then

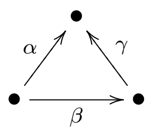
$$\dim \overline{\mathcal{O}}_M \geq 2|\mathbf{d}| - 2 \geq |\mathbf{d}|,$$

by Lemma 4.11. If G_Q is a cycle of length 2, then by Lemma 4.18 and Corollary 4.10,

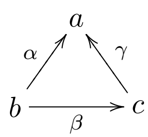
$$\dim \overline{\mathcal{O}}_M \geq \dim \overline{\mathcal{O}}_{M'} + 1 \geq |\mathbf{d}|,$$

where M' is the restriction of M to the quiver obtained from Q by removing an arrow.

Assume that G_Q is a cycle of length 3. Suppose that $\mathbf{d} = (1,1,1)$. Then Q is not a cycle as M is nilpotent. Thus Q is not a cycle and there is a vertex at which two arrows start, say

$Q =$ 

Then there exists $t \in k^*$ satisfying $\alpha - t \cdot \gamma\beta \in \operatorname{Ann}(M)$, and we get a contradiction. Thus $\mathbf{d} = (d_a, d_b, d_c) \neq (1,1,1)$ (cf. Lemma 5.1). Suppose now that $\dim \overline{\mathcal{O}}_M = |\mathbf{d}| - 1$. Without loss of generality, we may assume that $d_a \geq 2$. If there is an arrow starting at a and an arrow ending at a, then $\dim \overline{\mathcal{O}}_M \geq d_b + d_c - 1 + 2d_a - 1 \geq d_b + d_c + d_a = |\mathbf{d}|$, by Corollaries 4.10 and 4.13. Thus we may assume that two arrows end at a (Remark 4.3). Interchanging b and c if necessary, we may assume that

$Q =$ 

The above arguments yield $d_c = 1$. Using Lemma 4.12 and Corollary 4.13, we find that the space generated by $\operatorname{Im}(M_\alpha)$ and $\operatorname{Im}(M_\gamma)$ is one-dimensional and $\dim \operatorname{Ker}(M_\alpha) \cap \operatorname{Ker}(M_\beta) = d_b - 1$. Thus $\operatorname{Im}(M_\alpha) = \operatorname{Im}(M_\gamma)$ and $\operatorname{Ker}(M_\alpha) = \operatorname{Ker}(M_\beta)$. Hence $\alpha - t \cdot \gamma\beta \in \operatorname{Ann}(M)$ for some $t \in k^*$, a contradiction. ∎

Proposition 4.22. *Let Q be a connected quiver with $Q_1 \neq \emptyset$, and $\mathbf{d} \in \mathbb{N}^{Q_0}$. If $M \in \operatorname{rep}_Q(\mathbf{d})$ is admissible, then*

$$\dim \overline{\mathcal{O}}_M \geq |\mathbf{d}| - 1.$$

Moreover, if $\dim \overline{\mathcal{O}}_M = |\mathbf{d}| - 1$, then G_Q does not contain a cycle of length less than 4.

Proof. Let Q' be a subquiver of Q. Then $\dim \overline{\mathcal{O}}_M \geq \dim \overline{\mathcal{O}}_{M'}$, where $M' = M|_{Q'}$ (Remark 4.1). If there are vertices in Q which do not belong to Q'_0, then we can denote them by integers from 1 to j in such a way that, for each $1 \leq i \leq j$, the full subquiver of Q with the set of vertices $Q'_0 \cup \{1, \dots, i\}$ is connected. Applying Corollary 4.13 j times we get

$$\dim \overline{\mathcal{O}}_M \geq \dim \overline{\mathcal{O}}_{M'} + \sum_{i=1}^{j} d_i.$$

Since Q_1 is not empty, it contains an arrow or a loop Q' as a subquiver. By Corollary 4.10 and Lemma 4.11, $\dim \overline{\mathcal{O}}_{M'} \geq |\mathbf{d}'| - 1$, where $\mathbf{d}' = \mathbf{d}|_{Q'}$. Using the above we get the required inequality. If Q contains a subquiver Q' such that $G_{Q'}$ is a cycle of length less than 4, then by Lemma 4.21 we have $\dim \overline{\mathcal{O}}_{M'} \geq |\mathbf{d}'|$, so $\dim \overline{\mathcal{O}}_M \geq |\mathbf{d}|$. ■

Let a be a *sink vertex* in Q, i.e. there is no arrow α satisfying $s(\alpha) = a$. We denote by $\alpha_1, \dots, \alpha_r$ the arrows of Q ending at a. Let $s_a^+ Q$ be the quiver obtained from Q by replacing the arrows $\alpha_1, \dots, \alpha_r$ by the arrows $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_r$ with opposite orientation. For $M \in \operatorname{rep}_Q(\mathbf{d})$ we construct a linear map

$$v = [M_{\alpha_1}, \dots, M_{\alpha_r}] : \bigoplus_{i=1}^{r} M_{a_i} \to M_a.$$

Let M'_{α_i} be the composition of the inclusion and projection

$$\operatorname{Ker}(v) \hookrightarrow \bigoplus_{j=1}^{r} M_{a_j} \to M_{a_i},$$

for each $1 \leq i \leq r$. Let

$$S_a^+ \mathbf{d} = \begin{cases} \dim_k \operatorname{Ker}(v), & b = a, \\ d_b, & b \neq a. \end{cases}$$

Putting

$$S_a^+ M_b = \begin{cases} \operatorname{Ker}(v), & b = a, \\ M_b, & b \neq a, \end{cases} \qquad S_a^+ M_\beta = \begin{cases} M'_{\alpha_i}, & \beta = \widetilde{\alpha}_i, \\ M_\beta, & \beta \notin \{\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_r\}, \end{cases}$$

we define the representation $S_a^+ M \in \operatorname{rep}_{s_a Q}(S_a^+ \mathbf{d})$ called the *Coxeter reflection* of M at a. We define similarly the notion of a source vertex a, the quiver $s_a^- Q$, the vector $S_a^- \mathbf{d}$ and the representation $S_a^- M$ (also called the Coxeter reflection). If a is a sink vertex and M does not contain a direct summand isomorphic to the simple representation $S(a)$ (equivalently v is surjective), then $S_a^- S_a^+ M \simeq M$. We also know (see [1, Corollary 5.7]) that in this case $\operatorname{End}_Q(M) \simeq \operatorname{End}_{s_a^+ Q}(S_a^+ M)$. Thus applying Lemma 3.1 we get

$$\dim \overline{\mathcal{O}}_M - \dim \overline{\mathcal{O}}_{S_a^+ M} = d_a^2 - (\dim_k \operatorname{Ker}(v))^2,$$

and $\dim \overline{\mathcal{O}}_M = \dim \overline{\mathcal{O}}_{S_a^+ M}$ if and only if $\sum_{i=1}^{r} d_{s(\alpha_i)} = 2d_a$, i.e. $S_a^+ \mathbf{d} = \mathbf{d}$.

If a is a source vertex, then by duality $S_a^+ S_a^- M \simeq M$, $\mathrm{End}_Q(M) \simeq \mathrm{End}_{s_a^- Q}(S_a^- M)$,

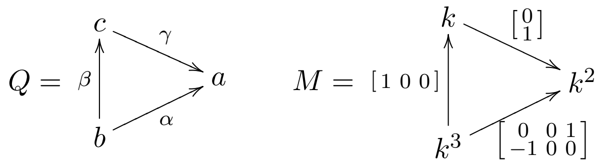
$$\dim \overline{\mathcal{O}}_M - \dim \overline{\mathcal{O}}_{S_a^- M} = d_a^2 - (\dim_k \mathrm{Coker}(v))^2,$$

and $\dim \overline{\mathcal{O}}_M = \dim \overline{\mathcal{O}}_{S_a^- M}$ if and only if $\sum_{i=1}^r d_{t(\alpha_i)} = 2d_a$ (i.e. $S_a^- \mathbf{d} = \mathbf{d}$).

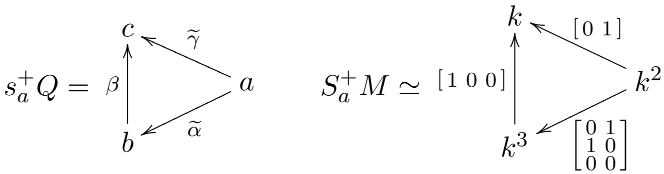
Assume that M is nilpotent. In $s_a^+ Q$ there are no other cycles than those in Q. As $\mathcal{R}_Q^m \subseteq \mathrm{Ann}(M)$ for some $m \geq 0$, we also have $\mathcal{R}_{s_a^+ Q}^m \subseteq \mathrm{Ann}(S_a^+ M)$ and $S_a^+ M$ is nilpotent.

The following example shows that, if M is admissible, then $S_a^+ M$ does not have to be.

EXAMPLE 4.23. Consider



Applying the Coxeter reflection at the vertex a we get



However, since $\beta\widetilde{\alpha} - \widetilde{\gamma} \in \mathrm{Ann}(S_a^+ M)$, $S_a^+ M$ is not admissible.

We are looking for an admissible representation M whose orbit closure is isomorphic to the orbit closure of a Coxeter reflection of M. In the next lemma we prove that this is the case when one of the following conditions is satisfied:

$$(*) \qquad a \text{ is a sink vertex}, \quad d_a = 1 \quad \text{and} \quad \sum_{i=1}^r d_{s(\alpha_i)} = 2$$

or

$$(**) \qquad a \text{ is a source vertex}, \quad d_a = 1 \quad \text{and} \quad \sum_{i=1}^r d_{t(\alpha_i)} = 2.$$

LEMMA 4.24. *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$. Assume that there exists a vertex $a \in Q_0$ satisfying either $(*)$ or $(**)$ Let $M \in \mathrm{rep}_Q(\mathbf{d})$ be admissible. Then the Coxeter reflection M' at a is admissible and*

$$\overline{\mathcal{O}}_M \simeq \overline{\mathcal{O}}_{M'}.$$

Proof. If a satisfies $(**)$, then a considered as a vertex in $s_a^- Q$ satisfies $(*)$. By our assumptions, since M is admissible, we have $S_a^+ S_a^- M \simeq M$. Thus it suffices to prove the lemma when a satisfies $(*)$. We consider two cases: when exactly one arrow ends at a and when exactly two arrows end at a. In the former case let $\alpha \in Q_1$ satisfy $t(\alpha) = a$. The map

$$\varphi_a^+ : \mathrm{rep}_Q(\mathbf{d}) \to \mathrm{rep}_{s_a^+ Q}(\mathbf{d})$$

such that $\varphi_a^+ N_{\widetilde{\alpha}} = \left[\begin{smallmatrix} -v \\ u \end{smallmatrix}\right]$ provided $N_\alpha = [\,u\ v\,]$ and $\varphi_a^+ N_\gamma = N_\gamma$ provided $\gamma \neq \widetilde{\alpha}$, is a linear isomorphism. If $N \in \mathrm{rep}_Q(\mathbf{d})$ is admissible, then $\varphi_a^+ N \simeq S_a^+ N_{\widetilde{\alpha}}$. Moreover, for each $g \in \mathrm{GL}(\mathbf{d})$,

$$\varphi_a^+(g \star N) = \widetilde{g} \star \varphi_a^+ N,$$

where $\widetilde{g} = (\widetilde{g}_i)_{i \in Q_0}$ is as follows:

$$\widetilde{g}_i = \begin{cases} g_i, & i \neq a, \\ g_a^{-1} \cdot \det(g_{s(\alpha)}), & i = a. \end{cases}$$

Thus $\varphi_a^+(\mathcal{O}_M) = \mathcal{O}_{S_a^+ M}$ and $\varphi_a^+(\overline{\mathcal{O}}_M) = \overline{\mathcal{O}}_{S_a^+ M}$.

Suppose now that $S_a^+ M$ is not admissible. There exists $\omega \in \varepsilon_j k(s_a^+ Q)\varepsilon_i$ such that $\omega \in \mathrm{Ann}(S_a^+ M) \setminus \mathcal{R}^2_{s_a^+ Q}$. Since M is admissible, $i = a$ and $j = s(\alpha)$. Multiplying by a nonzero scalar if necessary, we may assume that $\omega = \omega' \widetilde{\alpha} - \widetilde{\alpha}$, where $\omega' \in \varepsilon_{s(\alpha)} \cdot \mathcal{R}_{s_a^+ Q} \cdot \varepsilon_{s(\alpha)}$. It follows from

$$0 = S_a^+ M_\omega = S_a^+ M_{\omega'} \left[\begin{smallmatrix} -v \\ u \end{smallmatrix}\right] - \left[\begin{smallmatrix} -v \\ u \end{smallmatrix}\right]$$

that $\left[\begin{smallmatrix} -v \\ u \end{smallmatrix}\right] \neq 0$ is an eigenvector with eigenvalue 1 of the nilpotent endomorphism $S_a^+ M_{\omega'}$, a contradiction.

If there exist arrows α, β satisfying $t(\alpha) = t(\beta) = a$, then $d_{s(\alpha)} = d_{s(\beta)} = 1$. Since M is admissible, $s(\alpha) \neq s(\beta)$ (Remark 4.7). The map

$$\psi_a^+ : \mathrm{rep}_Q(\mathbf{d}) \to \mathrm{rep}_{s_a^+ Q}(\mathbf{d}), \qquad \psi_a^+ N_\gamma = \begin{cases} N_\gamma, & \gamma \notin \{\widetilde{\alpha}, \widetilde{\beta}\}, \\ N_\beta, & \gamma = \widetilde{\alpha}, \\ -N_\alpha, & \gamma = \widetilde{\beta}, \end{cases}$$

is a linear isomorphism. If N is admissible, then $\psi_a^+ N \simeq S_a^+ N$. For all $g \in \mathrm{GL}(\mathbf{d})$ we have

$$\psi_a^+(g \star N) = \widetilde{g} \star \psi_a^+ N,$$

where $\widetilde{g} = (\widetilde{g}_i)_{i \in Q_0}$ is given by

$$\widetilde{g}_i = \begin{cases} g_i, & i \neq a, \\ g_{s(\beta)} g_{s(\alpha)} g_a^{-1}, & i = a. \end{cases}$$

Thus $\psi_a^+(\mathcal{O}_M) = \mathcal{O}_{S_a^+ M}$ and $\psi_a^+(\overline{\mathcal{O}}_M) = \overline{\mathcal{O}}_{S_a^+ M}$.

Suppose that $S_a^+ M$ is not admissible. There exists $\omega \in \varepsilon_j k(s_a^+ Q)\varepsilon_i$ such that $\omega \in \mathrm{Ann}(S_a^+ M) \setminus \mathcal{R}^2_{s_a^+ Q}$. Since M is admissible, we have $i = a$. Thus $j = s(\alpha)$ or $j = s(\beta)$. If $\omega' \in \varepsilon_j \cdot \mathcal{R}_{S_a^+ Q} \cdot \varepsilon_j$ then $S_a^+ M_{\omega'} = [0]$. Therefore there is a path ω' satisfying either $s(\omega') = s(\alpha)$ and $t(\omega') = s(\beta)$, or $s(\omega') = s(\beta)$

and $t(\omega') = s(\alpha)$, such that $S_a^+ M_{\omega'} = M_{\omega'} \neq 0$. Then there exists $t \in k^*$ satisfying $\alpha\omega' - t\cdot\beta \in \operatorname{Ann}(M)$ or $\beta\omega' - t\cdot\alpha \in \operatorname{Ann}(M)$, which is impossible. ∎

5. Orbit closures of representations with low dimension vectors. Our primary aim is the classification of varieties isomorphic to orbit closures with an invariant point. Of course we can restrict our investigations to admissible representations. We have also noticed already that it suffices to consider connected quivers (see Remark 4.5). Let b be a fixed positive integer. Note that there exist only finitely many pairs $(Q, \mathbf{d})$ consisting of a connected quiver Q and a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$ such that there exists an admissible representation $M \in \operatorname{rep}_Q(\mathbf{d})$ satisfying $\dim \mathcal{O}_M = b$. Indeed, the sum of coordinates of $\mathbf{d}$, by Proposition 4.22, cannot exceed $b + 1$, and there are bounds on the set Q_1 (see Remark 4.7).

First we focus on the classification when the dimension vector belongs to the set

$$\{(1), (1,1), (1,1,1), (1,1,1,1), (1,1,1,1,1), (2), (1,2)\}$$

(and partially for the vector $(1,1,2)$). This enables us to get through the classification in Theorem 3.2 more efficiently.

LEMMA 5.1. *Let Q be a connected quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$ satisfies $d_i = 1$ for all $i \in Q_0$, and $M \in \operatorname{rep}_Q(\mathbf{d})$ be admissible. Then*:

(1) $\dim \overline{\mathcal{O}}_M = \#Q_0 - 1$.
(2) *Q does not contain any cycle.*
(3) *Each arrow $\alpha \in Q_1$ is the only path from $s(\alpha)$ to $t(\alpha)$.*
(4) *G_Q does not contain any cycle of length less than* 4.
(5) *If G_Q is a connected tree, then $\overline{\mathcal{O}}_M \simeq k^{\#Q_0 - 1}$.*

Proof. (1) Since $\operatorname{End}_Q(M) \neq \{0\}$ and $\#Q_0 = |\mathbf{d}|$ we get

$$\#Q_0 - 1 \le \dim \overline{\mathcal{O}}_M = \sum_{i \in Q_0} d_i^2 - \dim_k \operatorname{End}_Q(M) \le \sum_{i \in Q_0} d_i^2 - 1 = \#Q_0 - 1$$

by Lemma 4.22 and (3.1).

(2) Note that $M_\omega \in k^*$ for any path ω in Q of positive length. Since M is nilpotent, Q does not contain any cycles.

(3) Suppose that $\omega \neq \alpha$ is a path in Q from $s(\alpha)$ to $t(\alpha)$. Since $M_\omega \in k^*$, there exists $t \in k^*$ such that $\omega - t \cdot \alpha \in \operatorname{Ann}(M)$, and we get a contradiction.

(4) This is a consequence of (1) and Proposition 4.22.

(5) Observe that $\#Q_1 = \#Q_0 - 1$, and consequently $\operatorname{rep}_Q(\mathbf{d}) \simeq k^{\#Q_0 - 1}$. Hence the claim follows from the fact that $\overline{\mathcal{O}}_M$ is a closed subset of $\operatorname{rep}_Q(\mathbf{d})$ of dimension $\#Q_0 - 1$. ∎

The classification for the dimension vectors (1), $(1,1)$ and $(1,1,1)$ follows easily from Lemma 5.1 and its proof.

Corollary 5.2. *Let Q be connected and $M \in \mathrm{rep}_Q(\mathbf{d})$ be admissible.*

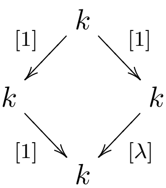
(1) *If $\mathbf{d} = (1)$ then $Q_1 = \emptyset$ and $\mathcal{O}_M = \overline{\mathcal{O}}_M = \{0\}$.*

(2) *If $\mathbf{d} = (1,1)$ then $M \simeq k \xrightarrow{[1]} k$ and $\overline{\mathcal{O}}_M \simeq k$.*

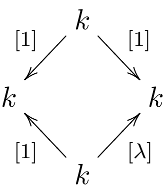
(3) *If $\mathbf{d} = (1,1,1)$ then M is isomorphic to $k \xrightarrow{[1]} k \xrightarrow{[1]} k$ or $k \xrightarrow{[1]} k \xleftarrow{[1]} k$ or $k \xleftarrow{[1]} k \xrightarrow{[1]} k$. Moreover, $\overline{\mathcal{O}}_M \simeq k^2$.*

Lemma 5.3. *Let Q be a connected quiver with four vertices and $M \in \mathrm{rep}_Q(\mathbf{d})$ be admissible, where $\mathbf{d} = (1,1,1,1)$.*

(1) *If G_Q is a tree then $\overline{\mathcal{O}}_M \simeq k^3$.*
(2) *If G_Q is not a tree then M is isomorphic to*

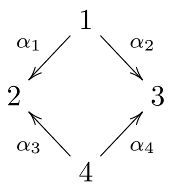


or



for some $\lambda \in k^*$. Moreover $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2)$.

Proof. The first part is a special case of Lemma 5.1(5). If G_Q is not a tree, then by Lemma 5.1(4), it is a cycle of length 4. Since Q is not a cycle (see Lemma 5.1(2)), there exists at least one vertex at which two arrows start. Lemma 5.1(3) implies that there are two essentially different cases



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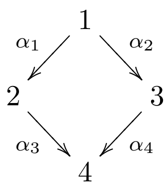
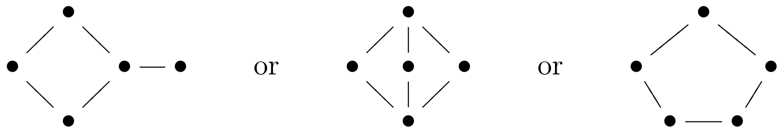


which lead to admissible representations given in the lemma. The orbit closures of all these representations are isomorphic, by Lemma 4.24. Thus $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2)$ (see Section 3). ∎

Lemma 5.4. *Let Q be a connected quiver with five vertices and $M \in \mathrm{rep}_Q(\mathbf{d})$ be admissible, where $\mathbf{d} = (1,1,1,1,1)$. Then $\overline{\mathcal{O}}_M$ is isomorphic to one of the following five varieties*:

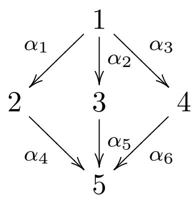
$$k^4; \quad \mathcal{D}(2,2) \times k; \quad \mathcal{D}(2,3); \quad \mathcal{C}(2,2,2); \quad \mathcal{C}(2,3).$$

Proof. By Lemma 5.1(1), $\dim \overline{\mathcal{O}}_M = 4$. If G_Q is a tree, then $\overline{\mathcal{O}}_M \simeq k^4$, by Lemma 5.1(5). Otherwise, using Lemma 5.1(4), we deduce that G_Q is of the form

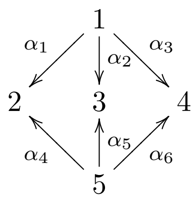


In the first case, $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2) \times k$, by Lemmas 5.3 and 4.6.

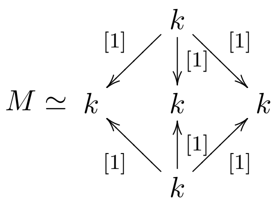
Consider the second case. If there exists a vertex at which one arrow ends and one arrow starts, then by Lemma 5.1(3), Q is of the form



and consequently $\overline{\mathcal{O}}_M \simeq \mathcal{C}(2,2,2)$ (see Section 3). If such a vertex does not exist, then Lemma 4.24 implies that it suffices to check Q of the form



We may assume that

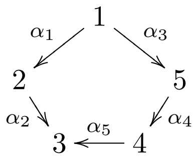


If $M' = ([\,m'_{\alpha_1}\,], [\,m'_{\alpha_2}\,], [\,m'_{\alpha_3}\,], [\,m'_{\alpha_4}\,], [\,m'_{\alpha_5}\,], [\,m'_{\alpha_6}\,]) \in \overline{\mathcal{O}}_M$ then

$$\operatorname{rk}\left(\begin{bmatrix} m'_{\alpha_1} & m'_{\alpha_4} \\ m'_{\alpha_2} & m'_{\alpha_5} \\ m'_{\alpha_3} & m'_{\alpha_6} \end{bmatrix}\right) \leq \operatorname{rk}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 1,$$

by Lemma 3.4. Hence $\overline{\mathcal{O}}_M \subseteq \mathcal{D}(2,3)$. In fact, the inclusion is an equality as the latter variety is irreducible of dimension 4.

In the third case, by Lemmas 5.1(3) and 4.24, it suffices to consider Q of the form



Then $\overline{\mathcal{O}}_M \simeq \mathcal{C}(2,3)$ (see Section 3), which completes the proof. ■

LEMMA 5.5. *If $M \in \operatorname{rep}_Q((2))$ is admissible, then*

$$M \simeq k^2 \circlearrowright \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \quad \text{and} \quad \overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[2]}(2,2).$$

Proof. Consider the natural homomorphism of algebras

$$\varrho : kQ \to \mathbb{M}_2(k), \qquad \alpha \mapsto M_\alpha, \qquad \text{for all } \alpha \in Q_1.$$

Since M is nilpotent and Q has one vertex, the algebra $B = \operatorname{Im} \varrho$ has two idempotents: 0 and 1. Since B is finite-dimensional, it is local ([1, I.4.6]). This shows that $B \simeq \operatorname{rad}(B) \oplus k \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ as a linear space and the ideal $\operatorname{rad} B$ is nilpotent. Fix nonzero $N \in \operatorname{rad} B$. Since N is nilpotent, we may assume that $N = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. If $N' = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \operatorname{rad} B$, then $\left[\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right] \in \operatorname{rad} B$ and $a = d = 0$. Since $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} c & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \operatorname{rad} B$, $c = 0$. Thus, if $\alpha \in Q_1$, then $M_\alpha = \left[\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right]$ for some $b \in k^*$. This implies that Q has only one arrow, say α (see Remark 4.7). Since M is nilpotent, we have

$$M \simeq k^2 \circlearrowright \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$$

The space $\operatorname{End}_Q(M)$ is isomorphic to $\{\left[\begin{smallmatrix} t & s \\ 0 & t \end{smallmatrix}\right];\ t, s \in k\}$ and by (3.1) we get $\dim \overline{\mathcal{O}}_M = 2$. For every $g \in \operatorname{GL}((2))$,

$$\operatorname{tr}((g \star M)_\alpha) = \operatorname{tr}(M_\alpha) = 0 \quad \text{and} \quad \det((g \star M)_\alpha) = \det(M_\alpha) = 0,$$

thus $\overline{\mathcal{O}}_M \subseteq \mathcal{HD}^{[2]}(2,2)$. As $\mathcal{HD}^{[2]}(2,2)$ is irreducible of dimension 2 (see Lemma 3.1), the conclusion follows. ■

PROPOSITION 5.6. *Let Q be a connected quiver with two vertices. If $M \in \operatorname{rep}_Q((1,2))$ is admissible, then it is isomorphic to one of the following nine representations*:

(1) $k \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} k^2$, $k \xleftarrow{[0\ 1]} k^2$, *and then* $\overline{\mathcal{O}}_M \simeq k^2$;

(2) $k \underset{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}{\rightrightarrows}} k^2$, $k \underset{[1\ 0]}{\overset{[0\ 1]}{\leftleftarrows}} k^2$, *and then* $\overline{\mathcal{O}}_M \simeq k^4$;

(3) $k \underset{[0\ 1]}{\overset{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}{\rightleftarrows}} k^2$, *and then* $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2)$;

(4) $k \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} k^2 \circlearrowright \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $k \xleftarrow{[1\ 0]} k^2 \circlearrowright \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, *and then* $\overline{\mathcal{O}}_M \simeq k^2 \times \mathcal{HD}^{[2]}(2,2)$;

(5) $k \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} k^2 \circlearrowright \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$, $k \xleftarrow{[0\ 1]} k^2 \circlearrowright \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$, *and then* $\overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[2]}(2,3)$.

Proof. It is sufficient to consider the quivers up to duality (Remark 4.3). Denote the vertices of Q by a and b in such a way that $(d_a, d_b) = (1,2)$. First we show that if M is admissible, then Q has at most two arrows.

Suppose that Q contains a subquiver Q' having three arrows. Recall that $M' = M|_{Q'}$ is admissible (see Corollary 4.2). We know that in Q there exist at most two arrows from a to b and at most two arrows from b to a (Remark 4.7). By Lemmas 5.1(2) and 5.5, Q does not have a loop at a and has at most one loop at b. Thus, up to duality, it suffices to consider the following three cases:

(1) $Q' = a \underset{\gamma}{\overset{\alpha,\ \beta}{\rightleftarrows}} b$. Then $M'_{\gamma\alpha} \in k^*$ or $M'_{\gamma\beta} \in k^*$, because M'_α and M'_β are linearly independent (Remark 4.7) and $\dim_k \operatorname{Ker}(M'_\gamma) = 1$. We get a contradiction since M' is nilpotent.

(2) $Q' = a \underset{\beta}{\overset{\alpha}{\rightrightarrows}} b \circlearrowright \gamma$. Then

$$M' \simeq k \underset{\left[\begin{smallmatrix}z\\t\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]}{\rightrightarrows}} k^2 \circlearrowright \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

If $y = 0$, then $x \neq 0$, $t \neq 0$ and $\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}z\\t\end{smallmatrix}\right] = \left[\begin{smallmatrix}t\\0\end{smallmatrix}\right]$, thus $\gamma\beta - tx^{-1}\cdot\alpha \in \operatorname{Ann}(M')$, a contradiction. Thus $y \neq 0$ and it is easily seen that

$$M' \simeq k \underset{\left[\begin{smallmatrix}z'\\t'\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\rightrightarrows}} k^2 \circlearrowright \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

for some $z', t' \in k$. Then $z' \cdot \gamma\alpha + t' \cdot \alpha - \beta \in \operatorname{Ann}(M')$, and we get a contradiction once again.

(3) $Q' = a \underset{\beta}{\overset{\alpha}{\rightleftarrows}} b \circlearrowright \gamma$. Then

$$M' \simeq k \underset{[z\ t]}{\overset{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]}{\rightleftarrows}} k^2 \circlearrowright \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

If $y \neq 0$, then as above

$$M' \simeq k \underset{[z'\ t']}{\overset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\rightleftarrows}} k^2 \circlearrowright \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

Since M' is nilpotent, then $[z'\ t']\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] = [0]$ and $[z'\ t']\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] = [0]$, i.e.

$[\,z'\ t'\,] = [\,0\ 0\,]$, a contradiction. Thus $y = 0$ and

$$M' \simeq k \underset{[\,z'\ t'\,]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightleftarrows}} k^2 \circlearrowright \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

for some $z', t' \in k$. Since M' is nilpotent $[\,z'\ t'\,]\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = [\,0\,]$, we have $z' = 0$. Then $t' \neq 0$ and $\alpha\beta - t' \cdot \gamma \in \operatorname{Ann}(M')$, and we get a contradiction once again.

Therefore Q has at most two arrows and it is sufficient to consider the following cases:

(1) $Q = a \longrightarrow b$ or $Q = a \rightrightarrows b$. Then M is isomorphic to either

$$k \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} k^2 \qquad \text{or} \qquad k \underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightrightarrows}} k^2$$

respectively. By Corollary 4.14, we obtain $\overline{\mathcal{O}}_M \simeq k^2$ or $\overline{\mathcal{O}}_M \simeq k^4$.

(2) $Q = a \underset{\beta}{\overset{\alpha}{\rightleftarrows}} b$. Then $M_\beta M_\alpha = M_{\beta\alpha} = [\,0\,]$, because M is nilpotent. If $M' = \left(\left[\begin{smallmatrix} m'_{\alpha 1}\\ m'_{\alpha 2}\end{smallmatrix}\right], [\,m'_{\beta 1}\ m'_{\beta 2}\,]\right) \in \overline{\mathcal{O}}_M$, then by Lemma 3.4 we get

$$\operatorname{rk}\left([\,m'_{\beta 1}\ m'_{\beta 2}\,] \cdot \left[\begin{smallmatrix} m'_{\alpha 1}\\ m'_{\alpha 2}\end{smallmatrix}\right]\right) \leq \operatorname{rk}(M_{\beta\alpha}) = 0.$$

Thus

$$\overline{\mathcal{O}}_M \subseteq \{(a_1, a_2, b_1, b_2) \in k^4;\ b_1 a_1 + b_2 a_2 = 0\} \simeq \mathcal{D}(2,2).$$

Since the last variety is irreducible of dimension 3 and $\dim \overline{\mathcal{O}}_M \geq 3$ (by Proposition 4.22), we obtain $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2)$.

(3) $Q = a \xrightarrow{\alpha} b \circlearrowright \beta$. Then

$$M \simeq k \xrightarrow{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]} k^2 \circlearrowright \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

If $y \neq 0$, then

$$M \simeq k \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} k^2 \circlearrowright \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

Consequently, $\dim \overline{\mathcal{O}}_M = 4$ as

$$\begin{aligned}\operatorname{End}_Q(M) &\simeq \{([\,c\,], \left[\begin{smallmatrix}d&e\\0&d\end{smallmatrix}\right]) \in \mathbb{M}_{1\times 1}(k) \times \mathbb{M}_{2\times 2}(k);\ \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right][\,c\,] = \left[\begin{smallmatrix}d&e\\0&d\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}\\ &\simeq \{([\,c\,], \left[\begin{smallmatrix}d&e\\0&d\end{smallmatrix}\right]) \in \mathbb{M}_{1\times 1}(k) \times \mathbb{M}_{2\times 2}(k);\ \left[\begin{smallmatrix}0\\c\end{smallmatrix}\right] = \left[\begin{smallmatrix}e\\d\end{smallmatrix}\right]\} \simeq k.\end{aligned}$$

Note that $\overline{\mathcal{O}}_M$ is contained in $k^2 \times \mathcal{HD}^{[2]}(2,2)$, which is also an irreducible variety of dimension 4, thus $\overline{\mathcal{O}}_M \simeq k^2 \times \mathcal{HD}^{[2]}(2,2)$.

If $y = 0$, then $x \neq 0$ and

$$M \simeq k \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} k^2 \circlearrowleft \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

If $M' = \left(\begin{bmatrix} m'_{\alpha 1} \\ m'_{\alpha 2}\end{bmatrix}, \begin{bmatrix} m'_{\beta 1} & m'_{\beta 2} \\ m'_{\beta 3} & m'_{\beta 4}\end{bmatrix}\right) \in \overline{\mathcal{O}}_M$, then by Lemma 3.4 we get

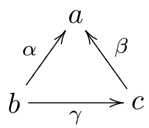
$$\operatorname{rk}\left(\begin{bmatrix} m'_{\beta 1} & m'_{\beta 2} & m'_{\alpha 1} \\ m'_{\beta 3} & m'_{\beta 4} & m'_{\alpha 2}\end{bmatrix}\right) \leq \operatorname{rk}(\left[\begin{smallmatrix}0&1&1\\0&0&0\end{smallmatrix}\right]) = 1.$$

Moreover, $\operatorname{tr}\left(\begin{bmatrix} m'_{\beta 1} & m'_{\beta 2} \\ m'_{\beta 3} & m'_{\beta 4}\end{bmatrix}\right) = 0$. Thus $\overline{\mathcal{O}}_M \subseteq \mathcal{HD}^{[2]}(2,3)$. As the latter variety is irreducible of dimension 3 and by Proposition 4.22 we have $\dim \overline{\mathcal{O}}_M \geq 3$, it follows that $\overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[2]}(2,3)$. ∎

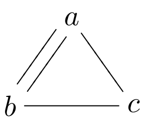
PROPOSITION 5.7. *Let Q be a connected quiver with three vertices. If $M \in \operatorname{rep}_Q((1,1,2))$ is admissible and $\dim \overline{\mathcal{O}}_M \leq 4$, then $\overline{\mathcal{O}}_M$ is isomorphic to one of the following nine varieties*:

$$k^3, \quad \mathcal{D}(2,2), \quad k^4, \quad \mathcal{HD}^{[2]}(2,3) \times k, \quad \mathcal{D}(2,2) \times k,$$
$$\mathcal{HD}^{[2]}(2,4), \quad \mathcal{D}(2,3), \quad \mathcal{HD}^{[2]}(3,3), \quad \mathcal{D}(2,2,2).$$

Proof. It is sufficient to consider quivers up to duality (Remark 4.3). We know that $\dim \overline{\mathcal{O}}_M \geq 3$ (Proposition 4.22). Suppose that Q contains



and $d_a = 2$. Then, by Lemma 4.12 applied to a, the dimension of the subspace generated by $\operatorname{Im}(M_\alpha)$ and $\operatorname{Im}(M_\beta)$ is 1. Thus $\operatorname{Im}(M_\alpha) = \operatorname{Im}(M_\beta)$ and $\alpha - t \cdot \beta\gamma \in \operatorname{Ann}(M)$ for some $t \in k^*$, a contradiction. Thus $d_a = 1$, and $d_b = 1$ by duality. Suppose now that G_Q contains



By Lemma 5.1(4), $d_a > 1$ or $d_b > 1$. The restriction M' of M to the full subquiver Q' with vertices a and b is admissible, belongs to $\operatorname{rep}_{Q'}((1,2))$ and satisfies $\dim \overline{\mathcal{O}}_{M'} \leq 3$ (Corollary 4.13). By Proposition 5.6, Q' is a cycle. This implies that Q contains a subquiver which we have just excluded, a contradiction.

Now we can show that Q has at most four arrows. Indeed, by Lemmas 5.5 and 5.1(4), G_Q may contain at most one loop. If it contains a loop, then by Proposition 5.6 and Lemma 5.1(4), it does not contain a cycle of length 2, and this implies that Q has at most four arrows. If G_Q contains a cycle of length 2 then, as we have seen above, it does not contain a cycle of length 3 and does not contain a loop (by Proposition 5.6), thus Q has at most four arrows. If G_Q does not contain a cycle of length less than 3, then Q has at most three arrows.

Let a be the vertex of Q satisfying $d_a = 2$. We consider three cases according to the number of arrows of Q.

Case $\#Q_1 = 2$. Then G_Q is of the form

$$a \text{———} b \text{———} c \quad \text{or} \quad b \text{———} a \text{———} c$$

In the former case, $\overline{\mathcal{O}}_M \simeq k^3$, by Lemma 4.6 and Corollary 4.14. In the latter case, by Lemma 4.24, it suffices to compute the orbit closure when

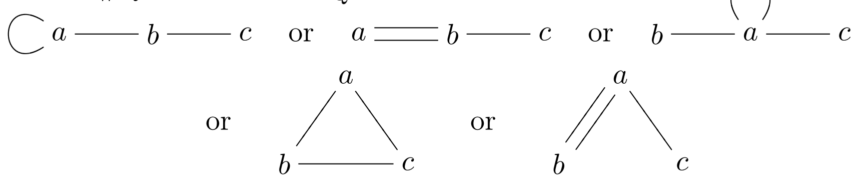
$$Q = b \xrightarrow{\alpha} a \xleftarrow{\beta} c$$

If $M' = \left(\begin{bmatrix} m'_{\alpha 1} \\ m'_{\alpha 2} \end{bmatrix}, \begin{bmatrix} m'_{\beta 1} \\ m'_{\beta 2} \end{bmatrix} \right) \in \overline{\mathcal{O}}_M$, then

$$\operatorname{rk}\left(\begin{bmatrix} m'_{\alpha 1} & m'_{\beta 1} \\ m'_{\alpha 2} & m'_{\beta 2} \end{bmatrix} \right) \leq \operatorname{rk}\left(\begin{bmatrix} m_{\alpha 1} & m_{\beta 1} \\ m_{\alpha 2} & m_{\beta 2} \end{bmatrix} \right),$$

by Lemma 3.4. If $\operatorname{rk}(\begin{bmatrix} m_{\alpha 1} & m_{\beta 1} \\ m_{\alpha 2} & m_{\beta 2} \end{bmatrix}) = 1$, then $\overline{\mathcal{O}}_M \subseteq \mathcal{D}(2,2)$. Since $\mathcal{D}(2,2)$ is irreducible of dimension 3, $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2)$. If $\operatorname{rk}(\begin{bmatrix} m_{\alpha 1} & m_{\beta 1} \\ m_{\alpha 2} & m_{\beta 2} \end{bmatrix}) = 2$, then $\overline{\mathcal{O}}_M \subseteq k^4$. By Lemma 4.12 applied to a we get $\dim \overline{\mathcal{O}}_M \geq 4$. Thus $\overline{\mathcal{O}}_M \simeq k^4$.

Case $\#Q_1 = 3$. Then G_Q is



Observe that $\dim \mathcal{O}_M = 4$ (Proposition 4.22). By Corollary 4.13, the dimension of the orbit of the restriction M' of M to any full subquiver Q' with two vertices is at most 3. In the first two cases we apply Lemma 4.6 for the vertex b, Proposition 5.6 and Lemma 5.1(5). Then

$$\overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[2]}(2,3) \times k \quad \text{or} \quad \overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2) \times k,$$

respectively. In the third case, by Lemma 4.24, it is sufficient to compute the orbit closure when

$$Q = b \xrightarrow{\beta} \overset{\overset{\alpha}{\circlearrowright}}{a} \xleftarrow{\gamma} c$$

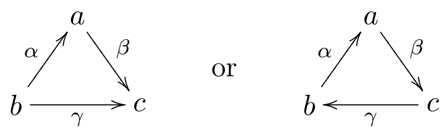
Applying Proposition 5.6 it is easily seen that

$$M \simeq k \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} \overset{\overset{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]}{\circlearrowright}}{k^2} \xleftarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} k$$

If $M' = \left(\begin{bmatrix} m'_{\beta1} \\ m'_{\beta2} \end{bmatrix}, \begin{bmatrix} m'_{\alpha1} & m'_{\alpha2} \\ m'_{\alpha3} & m'_{\alpha4} \end{bmatrix}, \begin{bmatrix} m'_{\gamma1} \\ m'_{\gamma2} \end{bmatrix}\right) \in \overline{\mathcal{O}}_M$, then by Lemma 3.4 we get

$$\operatorname{rk}\left(\begin{bmatrix} m'_{\alpha1} & m'_{\alpha2} & m'_{\beta1} & m'_{\gamma1} \\ m'_{\alpha3} & m'_{\alpha4} & m'_{\beta2} & m'_{\gamma2} \end{bmatrix}\right) \le \operatorname{rk}\left(\left[\begin{smallmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]\right) = 1.$$

Moreover, $\operatorname{tr}\left(\begin{bmatrix} m'_{\alpha1} & m'_{\alpha2} \\ m'_{\alpha3} & m'_{\alpha4} \end{bmatrix}\right) = 0$, thus $\overline{\mathcal{O}}_M \subseteq \mathcal{HD}^{[2]}(2,4)$. The last variety is irreducible of dimension 4, so $\overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[2]}(2,4)$. In the fourth case, as noticed at the beginning of the proof, Q is of the form



If M satisfies $M_{\beta\alpha} \neq 0$ then, in the former case $\beta\alpha - t \cdot \gamma \in \operatorname{Ann}(M)$ for some $t \in k^*$, and M is not admissible, while in the latter case the matrix $M_{\gamma\beta\alpha}$ is not nilpotent. Thus $M_{\beta\alpha} = 0$. Let M' be the restriction of M to the subquiver Q' obtained from Q by removing γ. Since $\#Q'_1 = 2$, we have $\overline{\mathcal{O}}_{M'} \simeq \mathcal{D}(2,2)$. Therefore $\overline{\mathcal{O}}_M \subseteq \mathcal{D}(2,2) \times k$ and as the latter variety is irreducible of dimension 4, $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2) \times k$. In the fifth case, by Proposition 5.6 and Lemma 4.24, it is sufficient to compute the orbit closure for

$$Q = b \underset{\beta}{\overset{\alpha}{\rightleftarrows}} a \xleftarrow{\ \gamma\ } c \quad \text{and} \quad M \simeq k \underset{[\,0\ 1\,]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightleftarrows}} k^2 \xleftarrow{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]} k$$

Since $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][\,0\ 1\,] = \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$, we have

$$\begin{aligned}
\operatorname{End}_Q(M) \simeq \Big\{ & \left([\,d\,], \begin{bmatrix} f & g \\ 0 & f \end{bmatrix}, [\,e\,]\right) \in \mathbb{M}_{1\times1}(k) \times \mathbb{M}_{2\times2}(k) \times \mathbb{M}_{1\times1}(k); \\
& [\,d\,][\,0\ 1\,] = [\,0\ 1\,]\begin{bmatrix} f & g \\ 0 & f \end{bmatrix},\ \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][\,d\,] = \begin{bmatrix} f & g \\ 0 & f \end{bmatrix}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\ \begin{bmatrix} f & g \\ 0 & f \end{bmatrix}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}x\\y\end{smallmatrix}\right][\,e\,]\Big\} \\
\simeq \Big\{ & \left([\,d\,], \begin{bmatrix} f & g \\ 0 & f \end{bmatrix}, [\,e\,]\right);\ [\,0\ d\,] = [\,0\ f\,],\ \left[\begin{smallmatrix}d\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}f\\0\end{smallmatrix}\right],\ \begin{bmatrix} fx+gy \\ fy \end{bmatrix} = \left[\begin{smallmatrix}xe\\ye\end{smallmatrix}\right]\Big\} \\
\simeq \Big\{ & \left([\,f\,], \begin{bmatrix} f & g \\ 0 & f \end{bmatrix}, [\,e\,]\right);\ \begin{bmatrix} fx+gy \\ fy \end{bmatrix} = \left[\begin{smallmatrix}xe\\ye\end{smallmatrix}\right]\Big\}.
\end{aligned}$$

If $y \neq 0$, then $f = e$, $g = 0$ and $\dim \overline{\mathcal{O}}_M = 5$. Thus $y = 0$, $e = f$, $\dim \overline{\mathcal{O}}_M = 4$ and

$$M \simeq k \underset{[\,0\ 1\,]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightleftarrows}} k^2 \xleftarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} k$$

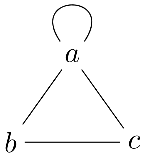
If $M' = \left(\begin{bmatrix} m'_{\alpha1} \\ m'_{\alpha2} \end{bmatrix}, [\,m'_{\beta1}\ m'_{\beta2}\,], \begin{bmatrix} m'_{\gamma1} \\ m'_{\gamma2} \end{bmatrix}\right) \in \overline{\mathcal{O}}_M$, then by Lemma 3.4,

$$\operatorname{rk}\left(\begin{bmatrix} m'_{\alpha1} & m'_{\gamma1} \\ m'_{\alpha2} & m'_{\gamma2} \end{bmatrix}\right) \le \operatorname{rk}\left(\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]\right) = 1, \quad \operatorname{rk}\left([\,m'_{\beta1}\ m'_{\beta2}\,] \cdot \begin{bmatrix} m'_{\alpha1} \\ m'_{\alpha2} \end{bmatrix}\right) \le \operatorname{rk}(M_{\beta\alpha}) = 0,$$

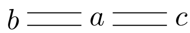
$$\operatorname{rk}\left([\,m'_{\beta1}\ m'_{\beta2}\,] \cdot \begin{bmatrix} m'_{\gamma1} \\ m'_{\gamma2} \end{bmatrix}\right) \le \operatorname{rk}(M_{\beta\gamma}) = 0.$$

Thus
$$\overline{\mathcal{O}}_M \subseteq \left\{(m'_{\alpha 1}, m'_{\alpha 2}, m'_{\beta 1}, m'_{\beta 2}, m'_{\gamma 1}, m'_{\gamma 2}) \in k^6;\ \operatorname{rk}\left(\begin{bmatrix} m_{\alpha_1} & m_{\gamma_1} & -m_{\beta_2} \\ m_{\alpha_2} & m_{\gamma_2} & m_{\beta_1} \end{bmatrix}\right) \leq 1\right\}.$$
Since the latter variety is isomorphic to $\mathcal{D}(2,3)$, we get $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,3)$.

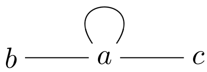
CASE $\#Q_1 = 4$. Then G_Q is of the form



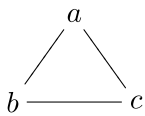
or



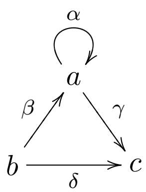
Thus $\dim \overline{\mathcal{O}}_M \geq 4$ (Proposition 4.22). Consider the first case. We have already treated the case when Q has three arrows and G_Q is of the form

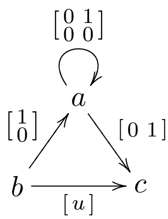


or

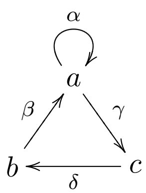


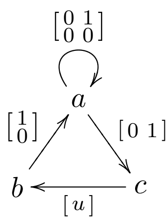
These considerations together with Lemma 4.24 show that we need to check the two possibilities:

$Q =$


$M \simeq$


and

$Q =$


$M \simeq$


By Corollary 4.4, we may assume that $u = 1$. In the latter case $\alpha - \beta\delta\gamma$ belongs to $\operatorname{Ann}(M)$, and we get a contradiction. In the former case, if $M' = \left(\begin{bmatrix} m'_{\alpha_1} & m'_{\alpha_2} \\ m'_{\alpha_3} & m'_{\alpha_4} \end{bmatrix}, \begin{bmatrix} m'_{\beta_1} \\ m'_{\beta_2} \end{bmatrix}, [\, m'_{\gamma_1}\ m'_{\gamma_2} \,], [\, m'_{\delta} \,]\right) \in \overline{\mathcal{O}}_M$, then
$$\operatorname{rk}\left(\begin{bmatrix} m'_{\alpha_1} & m'_{\alpha_2} & m'_{\beta_1} \\ m'_{\alpha_3} & m'_{\alpha_4} & m'_{\beta_2} \\ m'_{\gamma_1} & m'_{\gamma_2} & m'_{\delta} \end{bmatrix}\right) \leq \operatorname{rk}\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = 1,$$
by Lemma 3.4. Moreover, $\operatorname{tr}(M'_\alpha) = 0$, thus $\overline{\mathcal{O}}_M \subseteq \mathcal{HD}^{[2]}(3,3)$. The last

variety is irreducible of dimension 4, thus $\overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[2]}(3,3)$. Consider the second case. By Proposition 5.6,

$$Q = b \underset{\beta}{\overset{\alpha}{\rightleftarrows}} a \underset{\delta}{\overset{\gamma}{\leftrightarrows}} c$$

We have already studied the case when Q has three arrows and G_Q is of the form

$$a = b - c$$

These considerations imply that we may assume

$$M \simeq k \underset{[0\ 1]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightleftarrows}} k^2 \underset{[0\ 1]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\leftrightarrows}} k$$

Let $Q' = x \rightleftarrows y$. Consider the linear isomorphism

$$\varphi : \mathrm{rep}_Q((1,2,1)) \to \mathrm{rep}_{Q'}((2,2)),$$

$$k \underset{[m_{\beta 1}\ m_{\beta 2}]}{\overset{\left[\begin{smallmatrix}m_{\alpha 1}\\m_{\alpha 2}\end{smallmatrix}\right]}{\rightleftarrows}} k^2 \underset{[m_{\delta 1}\ m_{\delta 2}]}{\overset{\left[\begin{smallmatrix}m_{\gamma 1}\\m_{\gamma 2}\end{smallmatrix}\right]}{\leftrightarrows}} k \ \mapsto\ k^2 \underset{\left[\begin{smallmatrix}-m_{\beta 1} & -m_{\beta 2}\\ m_{\delta 1} & m_{\delta 2}\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}m_{\alpha 1} & m_{\gamma 1}\\ m_{\alpha 2} & m_{\gamma 2}\end{smallmatrix}\right]}{\rightleftarrows}} k^2$$

and the injective group homomorphism

$$\psi : \mathrm{GL}((1,2,1)) \to \mathrm{GL}(2,2),$$

$$([\,h\,], \left[\begin{smallmatrix}g_1 & g_2\\ g_3 & g_4\end{smallmatrix}\right], [\,f\,]) \mapsto (\left[\begin{smallmatrix}h & 0\\ 0 & f\end{smallmatrix}\right], \left[\begin{smallmatrix}g_1 & g_2\\ g_3 & g_4\end{smallmatrix}\right]).$$

Then, for every $g \in \mathrm{GL}((1,2,1))$ and every $N \in \mathrm{rep}_Q((1,2,1))$, we have

$$\varphi(g \star N) = \psi(g) \star \varphi(N).$$

Thus $\varphi(\mathcal{O}_N) \subseteq \mathcal{O}_{\varphi(N)}$ and $\varphi(\overline{\mathcal{O}}_N) \subseteq \overline{\mathcal{O}}_{\varphi(N)}$. Observe that

$$\varphi(M) = \varphi(k \underset{[0\ 1]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightleftarrows}} k^2 \underset{[0\ 1]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\leftrightarrows}} k) = k^2 \underset{\left[\begin{smallmatrix}0 & -1\\ 0 & 1\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right]}{\rightleftarrows}} k^2 = (\left[\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right], \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]) \star k^2 \underset{\left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]}{\rightleftarrows}} k^2$$

Denote the last representation by W. Thus $\mathcal{O}_{\varphi(M)} \simeq \mathcal{O}_W$, and

$$\varphi(\overline{\mathcal{O}}_M) \subseteq \overline{\mathcal{O}}_{\varphi(M)} \simeq \overline{\mathcal{O}}_W \simeq \mathcal{D}(2,2,2)$$

(see Section 3). Since φ is an isomorphism, $\varphi(\overline{\mathcal{O}}_M)$ is a closed subset with $\dim \varphi(\overline{\mathcal{O}}_M) = \dim \overline{\mathcal{O}}_M \geq 4$ in the irreducible variety of dimension 4. Thus

$$\overline{\mathcal{O}}_M \simeq \varphi(\overline{\mathcal{O}}_M) \simeq \mathcal{D}(2,2,2). \ \blacksquare$$

6. Proof of the main result. Theorem 3.2 follows from the two theorems below. First we classify the orbit closures (with an invariant point) of dimension less than 4, and then those of dimension 4.

Theorem 6.1. *Let M be an admissible representation whose orbit has dimension at most* 3. *Then the orbit closure $\overline{\mathcal{O}}_M$ is isomorphic to one of the following eight varieties*:

$$\{0\},\quad k,\quad k^2,\quad \mathcal{HD}^{[2]}(2,2),\quad k^3,\quad \mathcal{D}(2,2),\quad \mathcal{HD}^{[2]}(2,3),\quad \mathcal{HD}^{[2]}(2,2)\times k.$$

Proof. Proposition 4.22 yields $|\mathbf{d}| \le 4$. By Lemma 4.11, $\mathbf{d} \notin \{(3),(4)\}$. Thus

$$\mathbf{d} \in \{(1),(1,1),(2),(1,2),(1,1,1),(1,1,1,1),(1,1,2),(1,3),(2,2)\}.$$

We may assume that the quiver Q is connected. If $\mathbf{d} \in \{(1),(1,1),(1,1,1)\}$, then, by Corollary 5.2,

$$\overline{\mathcal{O}}_M \simeq \{0\} \quad \text{or} \quad \overline{\mathcal{O}}_M \simeq k \quad \text{or} \quad \overline{\mathcal{O}}_M \simeq k^2.$$

If $\mathbf{d} = (2)$ and $Q_1 \neq \emptyset$, then, by Lemma 5.5,

$$\overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[2]}(2,2).$$

If $\mathbf{d} = (1,2)$, then, by Proposition 5.6,

$$\overline{\mathcal{O}}_M \simeq k^2 \quad \text{or} \quad \overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2) \quad \text{or} \quad \overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[2]}(2,3).$$

In the remaining cases, $\dim \overline{\mathcal{O}}_M \ge 3$, by Proposition 4.22. If $\mathbf{d} = (1,1,1,1)$, then, by Lemma 5.3,

$$\overline{\mathcal{O}}_M \simeq k^3 \quad \text{or} \quad \overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2).$$

If $\mathbf{d} = (1,1,2)$ then, by Proposition 5.7,

$$\overline{\mathcal{O}}_M \simeq k^3 \quad \text{or} \quad \overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2).$$

If $\mathbf{d} = (1,3)$ or $\mathbf{d} = (2,2)$, then by Corollary 4.10 and Lemma 4.18, Q has only one arrow α. By Corollary 4.9, $\mathrm{rk}(M_\alpha) = 1$. Hence, by Lemma 4.8,

$$\overline{\mathcal{O}}_M \simeq k^3 \quad \text{or} \quad \overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2),$$

respectively. ∎

Theorem 6.2. *Let M be admissible and satisfy* $\dim \overline{\mathcal{O}}_M = 4$. *Then $\overline{\mathcal{O}}_M$ is isomorphic to one of the following twelve varieties*:

$$\begin{array}{llll}
k^4; & k\times\mathcal{D}(2,2); & \mathcal{HD}^{[3]}(3,3); & k^2\times\mathcal{HD}^{[2]}(2,2);\\
\mathcal{D}(2,3); & \mathcal{HD}^{[2]}(2,4); & \mathcal{HD}^{[2]}(3,3); & k\times\mathcal{HD}^{[2]}(2,3);\\
\mathcal{C}(2,2,2); & \mathcal{C}(2,3); & \mathcal{D}(2,2,2); & \mathcal{HD}^{[2]}(2,2)\times\mathcal{HD}^{[2]}(2,2).
\end{array}$$

Proof. Let $M \in \mathrm{rep}_Q(\mathbf{d})$ be admissible with $\dim \mathcal{O}_M = 4$. We may assume that Q is connected, by Remark 4.5 and Theorem 6.1. By Proposition 4.22 and Lemmas 5.1, 5.5, 4.11, we conclude that

$$|\mathbf{d}| \le 5 \quad \text{and} \quad \mathbf{d} \notin \{(1),(1,1),(1,1,1),(1,1,1,1),(2),(4),(5)\}.$$

Therefore,

$$\mathbf{d} \in \{(1,2),(3),(1,1,2),(2,2),(1,3),(1,1,1,1,1),\\ (1,1,1,2),(1,2,2),(1,1,3),(1,4),(2,3)\}.$$

If $\mathbf{d} = (1,2)$, then, by Proposition 5.6,

$$\overline{\mathcal{O}}_M \simeq k^4 \quad \text{or} \quad \overline{\mathcal{O}}_M \simeq k^2 \times \mathcal{HD}^{[2]}(2,2).$$

Let $\mathbf{d} = (3)$ and $\alpha \in Q_1$. By Lemma 4.11, we may assume that $M_\alpha = \left[\begin{smallmatrix} 0&1&0\\0&0&0\\0&0&0 \end{smallmatrix}\right]$. We shall show that α is the only arrow in Q. Let Q' be the subquiver of Q satisfying $Q'_1 = \{\alpha\}$. Since $\dim \mathcal{O}_{M|_{Q'}} = \dim \mathcal{O}_M$,

$$\begin{aligned} \operatorname{End}_Q(M) = \operatorname{End}_{Q'}(M|_{Q'}) &\simeq \left\{ G \in \mathbb{M}_{3\times 3}(k);\ G \left[\begin{smallmatrix} 0&1&0\\0&0&0\\0&0&0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0&1&0\\0&0&0\\0&0&0 \end{smallmatrix}\right] G \right\} \\ &\simeq \left\{ \left[\begin{smallmatrix} x&a&b\\0&x&0\\0&c&d \end{smallmatrix}\right];\ x,a,b,c,d \in k \right\}. \end{aligned}$$

This means that

$$M_\beta \left[\begin{smallmatrix} x&a&b\\0&x&0\\0&c&d \end{smallmatrix}\right] = \left[\begin{smallmatrix} x&a&b\\0&x&0\\0&c&d \end{smallmatrix}\right] M_\beta$$

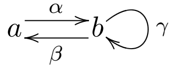
for any $\beta \in Q_1$ and any $x,a,b,c,d \in k$. Thus $M_\beta = u \cdot I + v \cdot E_{1,2}$ for some $u, v \in k$. Since M_β is nilpotent, $u = 0$. Hence $\beta - v\cdot\alpha \in \operatorname{Ann}(M)$ and $\beta = \alpha$, because M is admissible. If $M' = (M'_\alpha) \in \overline{\mathcal{O}}_M$, then $\operatorname{rk}(M'_\alpha) \le \operatorname{rk}(M_\alpha) = 1$ by Lemma 3.4. Thus $\overline{\mathcal{O}}_M \subseteq \mathcal{D}(3,3)$. Moreover, $\operatorname{tr}(M'_\alpha) = \operatorname{tr}(M_\alpha) = 0$, therefore $\overline{\mathcal{O}}_M \subseteq \mathcal{HD}^{[3]}(3,3)$. Since $\mathcal{HD}^{[3]}(3,3)$ is irreducible of dimension 4, it follows that

$$\overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[3]}(3,3).$$

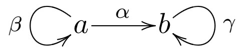
If $\mathbf{d} = (1,1,2)$, then by Proposition 5.7, $\overline{\mathcal{O}}_M$ is isomorphic to one of the following seven varieties:

$$k^4;\quad \mathcal{HD}^{[2]}(2,3)\times k;\quad \mathcal{D}(2,2)\times k;\quad \mathcal{HD}^{[2]}(2,4);\\ \mathcal{D}(2,3);\quad \mathcal{HD}^{[2]}(3,3);\quad \mathcal{D}(2,2,2).$$

If $\mathbf{d} = (2,2)$, then Q has at most two arrows. Indeed, by Lemmas 4.19 and 5.5, Q does not have two different arrows γ, γ' such that $s(\gamma) = s(\gamma')$ and $t(\gamma) = t(\gamma')$. A subquiver Q' of Q having three arrows would be of the form



or



In the first case $\dim \overline{\mathcal{O}}_{M'} \ge 5$ by Lemma 5.5 and Corollary 4.13, applied to the vertex a. In the second case, applying Lemma 4.15 for the subquivers $\beta \circlearrowright a$ and $b \circlearrowleft \gamma$, we get the same inequality. If Q has exactly one arrow $\alpha : a \to b$, then, by Lemma 4.8, $\operatorname{rk}(M_\alpha) = 2$ and $\overline{\mathcal{O}}_M \simeq k^4$. If Q has exactly two arrows, then, by Lemma 4.19, we have to consider two possibilities:

$$Q = a \underset{\beta}{\overset{\alpha}{\rightleftarrows}} b \quad \text{and} \quad Q = a \xrightarrow{\alpha} b \circlearrowleft \beta$$

By Proposition 4.22, $\dim \overline{\mathcal{O}}_M \geq 4$. In the first case, by Lemma 4.20, $M = (M_\alpha, M_\beta)$ is isomorphic to $(\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right])$ and consequently $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2,2)$ (see Section 3). In the second case, by Lemma 4.12 applied to a, we have $\mathrm{rk}(M_\alpha) = 1$. Then it is easily seen that

$$M \simeq k^2 \xrightarrow{\left[\begin{smallmatrix}x&0\\y&0\end{smallmatrix}\right]} k^2 \circlearrowleft \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

If $y \neq 0$, then

$$M \simeq k^2 \xrightarrow{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]} k^2 \circlearrowleft \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

and the space

$$\begin{aligned}\mathrm{End}_Q(M) &\simeq \left\{\left(\left[\begin{smallmatrix}c_1&c_2\\c_3&c_4\end{smallmatrix}\right], \left[\begin{smallmatrix}d&e\\0&d\end{smallmatrix}\right]\right) \in (\mathbb{M}_{2\times 2}(k))^2;\ \left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}c_1&c_2\\c_3&c_4\end{smallmatrix}\right] = \left[\begin{smallmatrix}d&e\\0&d\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\right\} \\ &\simeq \left\{\left(\left[\begin{smallmatrix}c_1&c_2\\c_3&c_4\end{smallmatrix}\right], \left[\begin{smallmatrix}d&e\\0&d\end{smallmatrix}\right]\right);\ \left[\begin{smallmatrix}0&0\\c_1&c_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}e&0\\d&0\end{smallmatrix}\right]\right\} \simeq \left\{\left(\left[\begin{smallmatrix}d&0\\c_3&c_4\end{smallmatrix}\right], \left[\begin{smallmatrix}d&0\\0&d\end{smallmatrix}\right]\right)\right\}\end{aligned}$$

is of dimension 3, i.e. $\dim \overline{\mathcal{O}}_M = 5$. Thus $y = 0$ and

$$M \simeq k^2 \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]} k^2 \circlearrowleft \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$$

If $M' = \left(\begin{bmatrix} m'_{\alpha 11} & m'_{\alpha 12} \\ m'_{\alpha 21} & m'_{\alpha 22}\end{bmatrix}, \begin{bmatrix} m'_{\beta 11} & m'_{\beta 12} \\ m'_{\beta 21} & m'_{\beta 22}\end{bmatrix}\right) \in \overline{\mathcal{O}}_M$, then

$$\mathrm{rk}\left(\begin{bmatrix} m'_{\beta 11} & m'_{\beta 12} & m'_{\alpha 11} & m'_{\alpha 12} \\ m'_{\beta 21} & m'_{\beta 22} & m'_{\alpha 21} & m'_{\alpha 22}\end{bmatrix}\right) \leq \mathrm{rk}(\left[\begin{smallmatrix}0&1&1&0\\0&0&0&0\end{smallmatrix}\right]) = 1,$$

by Lemma 3.4. Moreover, $\mathrm{tr}\left(\begin{bmatrix} m'_{\beta 11} & m'_{\beta 12} \\ m'_{\beta 21} & m'_{\beta 22}\end{bmatrix}\right) = 0$. Thus $\overline{\mathcal{O}}_M \subseteq \mathcal{HD}^{[2]}(2,4)$. Since $\mathcal{HD}^{[2]}(2,4)$ is irreducible of dimension 4, we get $\overline{\mathcal{O}}_M \simeq \mathcal{HD}^{[2]}(2,4)$.

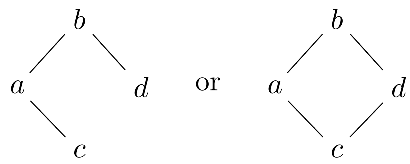
Let $\mathbf{d} = (1,3)$. By Lemma 4.11 and Corollary 4.13, Q does not contain any oriented cycle. Thus $\dim \overline{\mathcal{O}}_M$ is divisible by 3, by Corollary 4.14.

In the remaining cases, $\dim \overline{\mathcal{O}}_M \geq 4$ and G_Q does not contain a cycle of length less than 4, by Proposition 4.22.

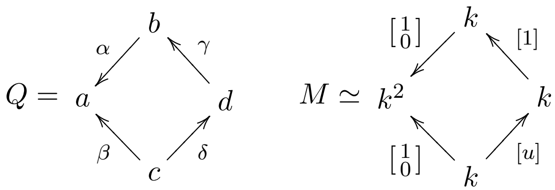
If $\mathbf{d} = (1,1,1,1,1)$, then by Lemma 5.4, $\overline{\mathcal{O}}_M$ is isomorphic to one of the following varieties:

$$k^4; \quad \mathcal{D}(2,2) \times k; \quad \mathcal{D}(2,3); \quad \mathcal{C}(2,2,2); \quad \mathcal{C}(2,3).$$

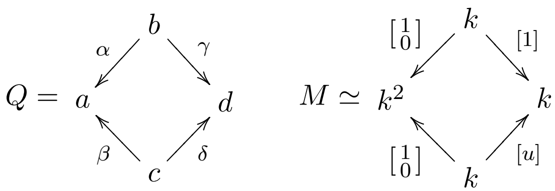
Let $\mathbf{d} = (1,1,1,2)$ and a be the vertex of Q satisfying $d_a = 2$. If a has only one neighbour, then $\overline{\mathcal{O}}_M \simeq k^4$, by Corollary 4.14 and Lemmas 5.1(5) and 4.6. If a has two neighbours, then G_Q is of the form



In the first case, by Lemma 4.6 and 5.1(5), we have $\overline{\mathcal{O}}_M \simeq \overline{\mathcal{O}}_{M'} \times k$, where M' is the restriction of M to the full subquiver of Q with vertices a, b and c. In particular, $\dim \overline{\mathcal{O}}_{M'} = 3$ and $\overline{\mathcal{O}}_{M'} \simeq \mathcal{D}(2,2)$, by Proposition 5.7. Thus $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,2) \times k$. In the second case, the dimension of $\overline{\mathcal{O}}_{M'}$, where M' is the restriction of M to the full subquiver with vertices b, c and d, equals 2 (Lemma 5.1(5)). By Lemma 4.12 applied to a, we know that a is either a sink or a source vertex, and the images (or kernels, respectively) of the corresponding maps coincide. Therefore it suffices to check the following two possibilities (Remark 4.3, Lemma 4.24):



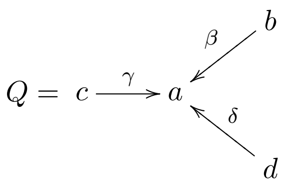
and



We may assume that $u = 1$, by Corollary 4.4. The first representation is not admissible as $\beta - \alpha\gamma\delta \in \mathrm{Ann}(M)$. The second one is admissible. If $M' = \left(\begin{bmatrix} m'_{\alpha 1} \\ m'_{\alpha 2} \end{bmatrix}, \begin{bmatrix} m'_{\beta 1} \\ m'_{\beta 2} \end{bmatrix}, [\, m'_\gamma \,], [\, m'_\delta \,] \right) \in \overline{\mathcal{O}}_M$, then

$$\mathrm{rk}\left(\begin{bmatrix} m'_{\alpha 1} & m'_{\beta 1} \\ m'_{\alpha 2} & m'_{\beta 2} \\ m'_\gamma & m'_\delta \end{bmatrix} \right) \leq \mathrm{rk}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \right) = 1,$$

by Lemma 3.4. Thus $\overline{\mathcal{O}}_M \subseteq \mathcal{D}(2,3)$ and since $\mathcal{D}(2,3)$ is irreducible of dimension 4, we obtain $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,3)$. If a has three neighbours, then, by Lemma 4.24, it suffices to consider the case when three arrows end at a:



Suppose that $\mathrm{Im}(M_\beta) \neq \mathrm{Im}(M_\gamma)$. Let M' be the restriction of M to the full subquiver with vertices a, b and c. By Lemma 4.12 applied to d and next to a, we get $\dim \mathcal{O}_M \geq \dim \mathcal{O}_{M'} + 1 \geq 5$. Thus $\mathrm{Im}(M_\beta) = \mathrm{Im}(M_\gamma)$, and in the same way $\mathrm{Im}(M_\gamma) = \mathrm{Im}(M_\delta)$. Therefore

$$M \simeq \; k \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} k^2 \quad \begin{array}{l} \xleftarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} k \\ \xleftarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} k \end{array}$$

If $M' = \left(\begin{bmatrix} m'_{\beta 1} \\ m'_{\beta 2} \end{bmatrix}, \begin{bmatrix} m'_{\gamma 1} \\ m'_{\gamma 2} \end{bmatrix}, \begin{bmatrix} m'_{\delta 1} \\ m'_{\delta 2} \end{bmatrix}\right) \in \overline{\mathcal{O}}_M$, then

$$\mathrm{rk}\left(\begin{bmatrix} m'_{\beta 1} & m'_{\gamma 1} & m'_{\delta 1} \\ m'_{\beta 2} & m'_{\gamma 2} & m'_{\delta 2} \end{bmatrix}\right) \leq \mathrm{rk}(\left[\begin{smallmatrix}1&1&1\\0&0&0\end{smallmatrix}\right]) = 1,$$

by Lemma 3.4. Therefore $\overline{\mathcal{O}}_M \subseteq \mathcal{D}(2,3)$ and consequently $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,3)$.

If $\mathbf{d} = (1,2,2)$ and the vertex a satisfies $d_a = 1$, then G_Q is of the form

$$b \text{———} a \text{———} c \qquad \text{or} \qquad a \text{———} b \text{———} c$$

In the first case $\overline{\mathcal{O}}_M \simeq k^4$, by Lemmas 4.6 and 4.14. In the second case, by Lemma 4.24 and Remark 4.3, it suffices to compute the orbit closure when

$$Q = a \xleftarrow{\;\alpha\;} b \xrightarrow{\;\beta\;} c$$

We get $\mathrm{rk}(M_\beta) = 1$, by Lemma 4.8 and Corollary 4.13. It is easily seen that

$$M \simeq k \xleftarrow{[\,x\ y\,]} k^2 \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]} k^2$$

Suppose that $y \neq 0$. Then

$$M \simeq k \xleftarrow{[\,0\ 1\,]} k^2 \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]} k^2$$

and

$$\begin{aligned} \mathrm{End}_Q(M) \simeq \Big\{ \Big([\,z\,], \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right], \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix} \Big) \in \mathbb{M}_{1\times 1}(k) \times \mathbb{M}_{2\times 2}(k) \times \mathbb{M}_{2\times 2}(k); \\ [\,z\,][\,0\ 1\,] = [\,0\ 1\,]\left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right],\ \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right] \Big\} \\ \simeq \Big\{ \Big([\,z\,], \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right], \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix} \Big); [\,0\ z\,] = [\,g_3\ g_4\,],\ \left[\begin{smallmatrix} g_1 & g_2 \\ 0 & 0 \end{smallmatrix}\right] = \begin{bmatrix} h_1 & 0 \\ h_3 & 0 \end{bmatrix} \Big\} \\ \simeq \Big\{ \Big([\,z\,], \left[\begin{smallmatrix} g & 0 \\ 0 & z \end{smallmatrix}\right], \begin{bmatrix} g & h_2 \\ 0 & h_4 \end{bmatrix} \Big) \Big\}, \end{aligned}$$

which yields $\dim \overline{\mathcal{O}}_M = 5$. Thus $y = 0$ and

$$M \simeq k \xleftarrow{[\,1\ 0\,]} k^2 \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]} k^2$$

If $M' = \left([\, m'_{\alpha 1}\ m'_{\alpha 2}\,], \begin{bmatrix} m'_{\beta 1} & m'_{\beta 2} \\ m'_{\beta 3} & m'_{\beta 4} \end{bmatrix}\right) \in \overline{\mathcal{O}}_M$, then, by Lemma 3.4,

$$\operatorname{rk}\left(\begin{bmatrix} m'_{\alpha 1} & m'_{\alpha 2} \\ m'_{\beta 1} & m'_{\beta 2} \\ m'_{\beta 3} & m'_{\beta 4} \end{bmatrix}\right) \le \operatorname{rk}\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1.$$

Therefore $\overline{\mathcal{O}}_M \subseteq \mathcal{D}(2,3)$, and consequently, $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,3)$.

If $\mathbf{d} = (1,1,3)$ and $d_a = 3$, then G_Q is of the form

$$a \text{———} b \text{———} c \qquad \text{or} \qquad b \text{———} a \text{———} c$$

In the first case, by Lemmas 4.6 and 4.14, we get $\overline{\mathcal{O}}_M \simeq k^4$. In the second case, by Corollary 4.13, a is either a sink or a source vertex. Thus we can assume (Remark 4.3) that

$$Q = b \xrightarrow{\ \alpha\ } a \xleftarrow{\ \beta\ } c$$

By Lemma 4.12, $\operatorname{Im}(M_\alpha) = \operatorname{Im}(M_\beta)$, thus

$$M \simeq k \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]} k^3 \xleftarrow{\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]} k$$

If $M' = \left(\begin{bmatrix} m'_{\alpha 1} \\ m'_{\alpha 2} \\ m'_{\alpha 3} \end{bmatrix}, \begin{bmatrix} m'_{\beta 1} \\ m'_{\beta 2} \\ m'_{\beta 3} \end{bmatrix}\right) \in \overline{\mathcal{O}}_M$, then

$$\operatorname{rk}\left(\begin{bmatrix} m'_{\alpha 1} & m'_{\beta 1} \\ m'_{\alpha 2} & m'_{\beta 2} \\ m'_{\alpha 3} & m'_{\beta 3} \end{bmatrix}\right) \le \operatorname{rk}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1,$$

by Lemma 3.4. Thus $\overline{\mathcal{O}}_M \subseteq \mathcal{D}(2,3)$ and consequently, $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,3)$.

If $\mathbf{d} = (1,4)$ or $\mathbf{d} = (2,3)$, then, by Lemmas 4.8 and 4.18, Q has one arrow α and $\operatorname{rk}(M_\alpha) = 1$. Moreover, in the former case $\overline{\mathcal{O}}_M \simeq k^4$, while in the latter $\overline{\mathcal{O}}_M \simeq \mathcal{D}(2,3)$. ∎

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