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ABSOLUTELY CONVERGENT FOURIER SERIES AND GENERALIZED LIPSCHITZ CLASSES OF FUNCTIONS

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Abstract. We investigate the order of magnitude of the modulus of continuity of a function f with absolutely convergent Fourier series. We give sufficient conditions in terms of the Fourier coefficients in order that f belong to one of the generalized Lipschitz classes $\mathrm{Lip}(\alpha, L)$ and $\mathrm{Lip}(\alpha, 1/L)$, where $0 \le \alpha \le 1$ and $L = L(x)$ is a positive, nondecreasing, slowly varying function such that $L(x) \to \infty$ as $x \to \infty$. For example, a 2π-periodic function f is said to belong to the class $\mathrm{Lip}(\alpha, L)$ if

$$|f(x+h) - f(x)| \le Ch^\alpha L(1/h) \quad \text{for all } x \in \mathbb{T},\ h > 0,$$

where the constant C does not depend on x and h. The above sufficient conditions are also necessary in the case of a certain subclass of Fourier coefficients. As a corollary, we deduce that if a function f with Fourier coefficients in this subclass belongs to one of these generalized Lipschitz classes, then the conjugate function $\tilde{f}$ also belongs to the same generalized Lipschitz class.

1. Introduction. Let $\{c_k : k \in \mathbb{Z}\}$ be a sequence of complex numbers (in symbols, $\{c_k\} \subset \mathbb{C}$) such that

$$\sum_{k\in\mathbb{Z}} |c_k| < \infty. \tag{1.1}$$

Then the trigonometric series

$$\sum_{k\in\mathbb{Z}} c_k e^{ikx} =: f(x) \tag{1.2}$$

converges uniformly in x and it is the Fourier series of its sum f.

We recall (see, e.g., [1, p. 6]) that a positive measurable function L defined on some neighborhood $[a, \infty)$ of infinity is said to be *slowly varying*

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(in Karamata's sense) if

$$\frac{L(\lambda x)}{L(x)} \to 1 \quad \text{as } x \to \infty \qquad \text{for every } \lambda > 0. \tag{1.3}$$

The neighborhood $[a, \infty)$ is of little importance. One may suppose that L is defined on $(0, \infty)$, for instance, by setting $L(x) := L(a)$ on $(0, a)$. A typical slowly varying function is

$$L(x) := \begin{cases} 1 & \text{for } 0 < x < 2, \\ \log x & \text{for } x \geq 2, \end{cases}$$

where the logarithm is to base 2.

In this paper, we consider positive, nondecreasing, slowly varying functions. In this case, it is enough to require (1.3) only for the single value $\lambda := 2$. To be more specific, condition $(*)$ below will be required in our theorems and lemmas.

Condition $(*)$. L is a positive nondecreasing function defined on $(0, \infty)$ and satisfying the limit relations

$$L(x) \to \infty \quad \text{and} \quad \frac{L(2x)}{L(x)} \to 1 \qquad \text{as } x \to \infty. \tag{1.4}$$

Given $\alpha > 0$ and a function L satisfying condition $(*)$, a periodic function f is said to belong to the *generalized Lipschitz class* $\operatorname{Lip}(\alpha, L)$ if its modulus of continuity satisfies

$$\omega(f; h) := \sup_{x \in \mathbb{T}} |f(x+h) - f(x)| \leq C h^\alpha L(1/h) \qquad \text{for all } h > 0, \tag{1.5}$$

where the constant $C = C(f)$ does not depend on h. Given $\alpha \geq 0$ and L with condition $(*)$, f is said to belong to the *generalized Lipschitz class* $\operatorname{Lip}(\alpha, 1/L)$ if

$$\omega(f; h) \leq C \, \frac{h^\alpha}{L(1/h)} \qquad \text{for all } h > 0. \tag{1.6}$$

Remark 1. Clearly, a function f satisfying (1.5) for some $\alpha > 0$, or (1.6) for some $\alpha \geq 0$, is continuous. Furthermore, if $f \in \operatorname{Lip}(\alpha, L)$ for some $\alpha > 1$, or if $f \in \operatorname{Lip}(\alpha, 1/L)$ for some $\alpha \geq 1$, then $f \equiv$ constant (cf. [7, p. 42]).

Remark 2. Various kinds of "generalized" Lipschitz classes of periodic functions were introduced in [2, 3, 4], where necessary and sufficient conditions were proved in order that the sum of an absolutely convergent sine or cosine series with nonnegative coefficients belong to a generalized Lipschitz class of order α for some $0 < \alpha < 1$.

2. New results

THEOREM 1. *Suppose* $\{c_k\} \subset \mathbb{C}$ *satisfies* (1.1), f *is defined in* (1.2), *and* L *satisfies condition* $(*)$.

(i) *If for some* $0 < \alpha \leq 1$,

$$\sum_{|k| \leq n} |kc_k| = O(n^{1-\alpha} L(n)), \qquad n \in \mathbb{N}, \tag{2.1}$$

then $f \in \operatorname{Lip}(\alpha, L)$.

(ii) *Conversely, if* $\{c_k\}$ *is a sequence of real numbers such that* $kc_k \geq 0$ *for all* k, *and if* $f \in \operatorname{Lip}(\alpha, L)$ *for some* $0 < \alpha \leq 1$, *then* (2.1) *holds.*

REMARK 3. Due to Lemma 3 in Section 3, in case $0 < \alpha < 1$ condition (2.1) is equivalent to

$$\sum_{|k| \geq n} |c_k| = O(n^{-\alpha} L(n)), \qquad n \in \mathbb{N}. \tag{2.2}$$

REMARK 4. In a certain sense, Theorem 1 is a generalization of [6, Theorems 1 and 2] by Németh. Furthermore, in case $L \equiv 1$, Theorem 1 was proved in [5, Theorem 1].

The next theorem is a natural counterpart of Theorem 1.

THEOREM 2. *Suppose* $\{c_k\} \subset \mathbb{C}$ *satisfies* (1.1), f *is defined in* (1.2), *and* L *satisfies condition* $(*)$.

(i) *If for some* $0 \leq \alpha < 1$,

$$\sum_{|k| \geq n} |c_k| = O\left(\frac{n^{-\alpha}}{L(n)}\right), \qquad n \in \mathbb{N}, \tag{2.3}$$

then $f \in \operatorname{Lip}(\alpha, 1/L)$.

(ii) *Conversely, if* $\{c_k\}$ *is a sequence of nonnegative real numbers and if* $f \in \operatorname{Lip}(\alpha, 1/L)$ *for some* $0 \leq \alpha < 1$, *then* (2.3) *holds.*

REMARK 5. Due to Lemma 4 in Section 3, in case $0 < \alpha < 1$ condition (2.3) is equivalent to

$$\sum_{|k| \leq n} |kc_k| = O\left(\frac{n^{1-\alpha}}{L(n)}\right), \qquad n \in \mathbb{N}. \tag{2.4}$$

REMARK 6. In case $\alpha = 0$, Theorem 2 may be considered as a generalization of [6, Theorem 5] by Németh.

3. Auxiliary results. To prove Theorems 1 and 2, we will need six lemmas, which may be useful in other investigations.

Lemma 1. *Suppose L satisfies condition $(*)$. If $\eta < -1$, then*

$$\sum_{k=n}^{\infty} k^\eta L(k) = O(n^{\eta+1} L(n)), \qquad n \in \mathbb{N}. \tag{3.1}$$

Proof. Clearly, it is enough to prove (3.1) in the special case $n := 2^m$, $m \in \mathbb{N}$. We fix a constant C such that

$$1 < C < 2^{-\eta-1}, \tag{3.2}$$

which is possible since $-\eta - 1 > 0$. It follows from (1.4) that there exists $m_0 \in \mathbb{N}$ such that

$$L(2^{m+1}) \le CL(2^m) \quad \text{for } m \ge m_0. \tag{3.3}$$

By forming dyadic sums, an elementary estimation gives

$$\begin{aligned} \sum_{k=2^m}^{\infty} k^\eta L(k) &= \sum_{l=m}^{\infty} \sum_{k=2^l}^{2^{l+1}-1} k^\eta L(k) \le \sum_{l=m}^{\infty} 2^{l(\eta+1)} L(2^{l+1}) \\ &\le L(2^m) \sum_{l=m}^{\infty} 2^{l(\eta+1)} C^{l-m+1} \\ &= C 2^{m(\eta+1)} L(2^m) [1 + 2^{\eta+1} C + 2^{2(\eta+1)} C^2 + \cdots]. \end{aligned} \tag{3.4}$$

Due to (3.2), the geometric series in brackets is convergent. Consequently, (3.4) results in

$$\sum_{k=2^m}^{\infty} k^\eta L(k) = O(2^{m(\eta+1)} L(2^m)),$$

whence (making use of (1.4) again) (3.1) follows. ∎

Lemma 2. *Suppose L satisfies condition $(*)$. If $\eta > -1$, then*

$$\sum_{k=1}^{n} \frac{k^\eta}{L(k)} = O\left(\frac{n^{\eta+1}}{L(n)}\right), \qquad n \in \mathbb{N}. \tag{3.5}$$

Proof. Clearly, it is enough to prove (3.5) in the special case $n := 2^m$. This time we fix another constant C for which

$$1 < C < 2^{\eta+1}, \tag{3.6}$$

which is possible since $\eta + 1 > 0$. By (1.4), there exists another $m_0 \in \mathbb{N}$ such that (3.3) holds. Let $m > m_0$; then we may write

$$\sum_{k=1}^{2^m} \frac{k^\eta}{L(k)} = \left\{ \sum_{k=1}^{2^{m_0}} + \sum_{k=2^{m_0}+1}^{2^m} \right\} \frac{k^\eta}{L(k)} =: A_{m_0} + B_m, \tag{3.7}$$

say. We form dyadic sums again, and making use of (3.3) gives

$$B_m = \sum_{l=m_0+1}^{m} \sum_{k=2^{l-1}+1}^{2^l} \frac{k^\eta}{L(k)} \le \sum_{l=m_0+1}^{m} \frac{2^{l(\eta+1)-1}}{L(2^{l-1})} \le \frac{2^{m(\eta+1)-1}}{L(2^m)} C\left[1 + \frac{C}{2^{\eta+1}} + \frac{C^2}{2^{2(\eta+1)}} + \cdots\right], \tag{3.8}$$

provided that $\eta \ge 0$. Due to (3.6), the geometric series in brackets is convergent. Consequently, (3.8) results in

$$B_m = O\left(\frac{2^{m(\eta+1)}}{L(2^m)}\right), \qquad m \in \mathbb{N}, \tag{3.9}$$

provided that $\eta \ge 0$.

In the remaining case when $-1 < \eta < 0$, we make use of the inequality (analogous to one in (3.8))

$$B_m \le \sum_{l=m_0+1}^{m} \frac{2^{(l-1)(\eta+1)}}{L(2^{l-1})},$$

where B_m is defined in (3.7). Then an estimation similar to the one which led to (3.8) gives (3.9) in the case $-1 < \eta < 0$ as well.

Taking into account that the ratio in parentheses on the right-hand side of (3.9) tends to ∞ as $m \to \infty$ (since $\eta + 1 > 0$), by (3.7) and (3.9) we conclude that

$$\sum_{k=1}^{2^m} \frac{k^\eta}{L(k)} = O\left(\frac{2^{m(\eta+1)}}{L(2^m)}\right), \qquad m \in \mathbb{N}.$$

Making use of (1.4) again, we deduce (3.5). ∎

LEMMA 3. *Suppose $\{a_k : k \in \mathbb{N}\}$ is a sequence of nonnegative real numbers (in symbols, $\{a_k\} \subset \mathbb{R}_+$) with $\sum a_k < \infty$, and L satisfies condition* $(*)$.

(i) *If for some $\delta > \gamma \ge 0$,*

$$\sum_{k=1}^{n} k^\delta a_k = O(n^\gamma L(n)), \tag{3.10}$$

then

$$\sum_{k=n}^{\infty} a_k = O(n^{\gamma-\delta} L(n)), \qquad n \in \mathbb{N}. \tag{3.11}$$

(ii) *Conversely, if* (3.11) *holds for some $\delta \ge \gamma > 0$, then* (3.10) *also holds.*

Remark 7. Clearly, in case $\delta > \gamma > 0$ conditions (3.10) and (3.11) are equivalent, while in case $\delta = \gamma \geq 0$ both are trivially satisfied, due to the assumption $\sum a_k < \infty$.

Proof of Lemma 3. (i) Suppose (3.10) is satisfied for some $\delta > \gamma \geq 0$. Then there exists a constant C such that

$$s_n := \sum_{k=1}^{n} k^\delta a_k \leq C n^\gamma L(n), \qquad n \in \mathbb{N}.$$

A summation by parts gives

$$\begin{aligned}
(3.12) \qquad r_n := \sum_{k=n}^{\infty} a_k &= \sum_{k=n}^{\infty} \frac{s_k - s_{k-1}}{k^\delta} \\
&= -\frac{s_{n-1}}{n^\delta} + \sum_{k=n}^{\infty} \left(\frac{1}{k^\delta} - \frac{1}{(k+1)^\delta} \right) s_k \\
&\leq \sum_{k=n}^{\infty} \frac{\delta}{k^{\delta+1}} C k^\gamma L(k) = \delta C \sum_{k=n}^{\infty} k^{\gamma-\delta-1} L(k), \qquad n \in \mathbb{N},\ s_0 := 0.
\end{aligned}$$

Applying Lemma 1 (with $\eta := \gamma - \delta - 1$) yields (3.11).

It is worth observing that the assumption $\sum a_k < \infty$ follows from (3.10) holding for some $\delta > \gamma \geq 0$. Indeed, this can be immediately seen if in (3.12) the summation by parts is performed for the finite sum $\sum_{k=n}^{N} a_k$ in place of $\sum_{k=n}^{\infty} a_k$ and then we let $N \to \infty$.

(ii) Conversely, if (3.11) is satisfied for some $\delta \geq \gamma > 0$, then there exists another constant C such that

$$r_n \leq C n^{\gamma-\delta} L(n), \qquad n \in \mathbb{N}.$$

Again, a summation by parts gives

$$\begin{aligned}
(3.13) \qquad s_n := \sum_{k=1}^{n} k^\delta a_k &= \sum_{k=1}^{n} k^\delta (r_k - r_{k+1}) \\
&= \sum_{k=1}^{n} (k^\delta - (k-1)^\delta) r_k - n^\delta r_{n+1} \\
&\leq r_1 + \max\{1, 2^{1-\delta}\} \sum_{k=2}^{n} \delta_k^{\delta-1} r_k \\
&\leq r_1 + \max\{1, 2^{1-\delta}\} \sum_{k=2}^{n} \delta k^{\delta-1} C k^{\gamma-\delta} L(k) \\
&\leq r_1 + \max\{1, 2^{1-\delta}\} \delta C L(n) \sum_{k=2}^{n} k^{\gamma-1} = O(n^\gamma L(n)),
\end{aligned}$$

which is (3.10). ∎

LEMMA 4. *Suppose* $\{a_k\} \subset \mathbb{R}_+$ *with* $\sum a_k < \infty$, *and* L *satisfies condition* $(*)$.

(i) *If for some* $\delta > \gamma > 0$,

$$\sum_{k=1}^{n} k^\delta a_k = O\Big(\frac{n^\gamma}{L(n)}\Big), \tag{3.14}$$

then

$$\sum_{k=n}^{\infty} a_k = O\Big(\frac{n^{\gamma-\delta}}{L(n)}\Big), \qquad n \in \mathbb{N}. \tag{3.15}$$

(ii) *Conversely*, *if* (3.15) *holds for some* $\delta \geq \gamma > 0$, *then* (3.14) *also holds.*

REMARK 8. Clearly, in case $\delta > \gamma > 0$ conditions (3.14) and (3.15) are equivalent.

Proof of Lemma 4. (i) Suppose (3.14) is satisfied for some $\delta > \gamma > 0$. Then there exists a constant C such that

$$s_n := \sum_{k=1}^{n} k^\delta a_k \leq C\, \frac{n^\gamma}{L(n)}, \qquad n \in \mathbb{N}.$$

Similarly to (3.12), we conclude that

$$r_n := \sum_{k=n}^{\infty} a_k \leq \delta C \sum_{k=n}^{\infty} \frac{k^{\gamma-\delta-1}}{L(k)} \leq \frac{\delta C}{L(n)} \sum_{k=n}^{\infty} k^{\gamma-\delta-1} = O\Big(\frac{n^{\gamma-\delta}}{L(n)}\Big),$$

which is (3.15).

It is worth observing again that $\sum a_k < \infty$ follows from (3.14) holding for some $\delta > \gamma > 0$.

(ii) Conversely, if (3.15) is satisfied for some $\delta \geq \gamma > 0$, then there exists another constant C such that

$$r_n \leq C\, \frac{n^{\gamma-\delta}}{L(n)}, \qquad n \in \mathbb{N}.$$

Similarly to (3.13), we find that

$$s_n \leq r_1 + \max\{1, 2^{1-\delta}\}\delta C \sum_{k=2}^{n} \frac{k^{\gamma-1}}{L(k)}.$$

Applying Lemma 2 (with $\eta := \gamma - 1$) yields (3.14). ∎

The last two lemmas may be considered as nondiscrete versions of Lemmas 1 and 2.

LEMMA 5. *If L satisfies condition $(*)$ and $\eta > -1$, then*

$$\int_0^h x^\eta L(1/x)\,dx = O(h^{\eta+1}L(1/h)), \qquad 0 < h < 1. \tag{3.16}$$

Proof. Clearly, it is enough to prove (3.16) in the special case $h := 2^{-m}$, where $m \in \mathbb{N}$. We fix a constant C for which (3.6) is satisfied. By (1.4), there exists $m_0 \in \mathbb{N}$ such that (3.3) holds.

Let $m > m_0$. In case $\eta \geq 0$, we estimate as follows:

$$\begin{aligned}\int_0^{2^{-m}} x^\eta L(1/x)\,dx &= \sum_{k=m}^\infty \int_{2^{-k-1}}^{2^{-k}} x^\eta L(1/x)\,dx \leq \sum_{k=m}^\infty 2^{-k(\eta+1)-1}L(2^{k+1}) \\ &\leq 2^{-m(\eta+1)-1}L(2^m)\left[1 + \frac{C}{2^{\eta+1}} + \frac{C^2}{2^{2(\eta+1)}} + \cdots\right].\end{aligned} \tag{3.17}$$

Due to (3.6), the geometric series in brackets is convergent. Thus, from (3.17) it follows that

$$\int_0^{2^{-m}} x^\eta L(1/x)\,dx = O(2^{-m(\eta+1)}L(2^m)), \qquad m \in \mathbb{N}. \tag{3.18}$$

In case $-1 < \eta < 0$ an analogous estimation gives

$$\int_0^{2^{-m}} x^\eta L(1/x)\,dx \leq \sum_{k=m}^\infty 2^{-(k-1)(\eta+1)}L(2^{k+1}),$$

which also results in the same estimate (3.18), as $\eta + 1$ is still positive.

By (1.4) again, (3.16) is a simple consequence of (3.18). ∎

LEMMA 6. *If L satisfies condition $(*)$ and $\eta > -1$, then*

$$\int_0^h \frac{x^\eta}{L(1/x)}\,dx = O\left(\frac{h^{\eta+1}}{L(1/h)}\right), \qquad 0 < h < 1. \tag{3.19}$$

Proof. Clearly, it is enough to prove (3.19) in the special case $h := 2^{-m}$, $m \in \mathbb{N}$. It is easy to check that

$$\begin{aligned}\int_0^{2^{-m}} \frac{x^\eta}{L(1/x)}\,dx &= \sum_{k=m}^\infty \int_{2^{-k-1}}^{2^{-k}} \frac{x^\eta}{L(1/x)}\,dx \leq \sum_{k=m}^\infty \frac{2^{-k(\eta+1)-1}}{L(2^k)} \\ &\leq \frac{1}{2L(2^m)} \sum_{k=m}^\infty 2^{-k(\eta+1)} = O\left(\frac{2^{-m(\eta+1)}}{L(2^m)}\right), \qquad m \in \mathbb{N},\end{aligned} \tag{3.20}$$

provided that $\eta \geq 0$ (the case $-1 < \eta < 0$ can be treated analogously). In view of (1.4), (3.19) is a simple consequence of (3.20). ∎

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. (i) Suppose (2.1) is satisfied for some $0 < \alpha \leq 1$. By (1.1) and (1.2), we may write

$$\begin{aligned} |f(x+h) - f(x)| &= \Big|\sum_{k\in\mathbb{Z}} c_k e^{ikx}(e^{ikh} - 1)\Big| \\ &\leq \Big\{\sum_{|k|\leq n} + \sum_{|k|>n}\Big\}|c_k|\,|e^{ikh} - 1| =: S_n + R_n, \end{aligned} \tag{4.1}$$

say, where

$$n := [1/h], \qquad h > 0, \tag{4.2}$$

and $[\cdot]$ means the integer part.

We will use the inequality

$$|e^{ikh} - 1| = \Big|2\sin\frac{kh}{2}\Big| \leq \min\{2, |kh|\}, \qquad k \in \mathbb{Z}. \tag{4.3}$$

By (2.1) and (4.2), we obtain

$$|S_n| \leq h\sum_{|k|\leq n} |kc_k| = hO(n^{1-\alpha}L(n)) = O(h^\alpha L(1/h)). \tag{4.4}$$

On the other hand, by (4.2) and Lemma 3 (applied with $\gamma := 1 - \alpha$ and $\delta := 1$ in the case of (2.1)) we find that

$$|R_n| \leq 2\sum_{|k|>n} |c_k| = 2O(n^{-\alpha}L(n)) = O(h^\alpha L(1/h)). \tag{4.5}$$

Combining (4.1), (4.4) and (4.5) yields $f \in \mathrm{Lip}(\alpha, L)$.

(ii) Conversely, suppose that $kc_k \geq 0$ for all k and $f \in \mathrm{Lip}(\alpha, L)$ for some $0 < \alpha \leq 1$. Then there exists a constant C such that

$$|f(x) - f(0) = \Big|\sum_{k\in\mathbb{Z}} c_k(e^{ikx} - 1)\Big| \leq Cx^\alpha L(1/x), \qquad x > 0. \tag{4.6}$$

Taking the imaginary part of the above series, we have

$$\Big|\sum_{k\in\mathbb{Z}} c_k \sin kx\Big| \leq Cx^\alpha L(1/x), \qquad x > 0.$$

By uniform convergence, due to (1.1), the series $\sum c_k \sin kx$ may be integrated term by term on any interval $(0, h)$. By Lemma 5, we obtain

$$\begin{aligned} \Big|\sum_{|k|\geq 1} \frac{c_k}{k}\, 2\sin^2\frac{kh}{2}\Big| &= \Big|\sum_{|k|\geq 1} c_k\,\frac{1 - \cos kh}{k}\Big| \\ &= O(h^{\alpha+1}L(1/h)), \qquad h > 0. \end{aligned} \tag{4.7}$$

Making use of the well-known inequality

$$\sin t \geq \frac{2}{\pi} t \quad \text{for } 0 \leq t \leq \frac{\pi}{2},$$

and the fact that $kc_k \geq 0$ for all k, we conclude that

$$2 \sum_{|k| \leq n} kc_k \frac{h^2}{\pi^2} \leq 2 \sum_{|k| \geq 1} \frac{c_k}{k} \sin^2 \frac{kh}{2} = O(h^{\alpha+1} L(1/h)), \qquad h > 0, \tag{4.8}$$

where n is defined in (4.2). Now, from (1.4) and (4.8) it follows that

$$\sum_{|k| \leq n} kc_k = O(h^{\alpha-1} L(1/h)) = O(n^{1-\alpha} L(n)),$$

which is (2.1). ∎

Proof of Theorem 2. (i) Suppose (2.2) is satisfied for some $0 \leq \alpha < 1$. We start with (4.1), where n is defined in (4.2). Making use of the first inequality in (4.4) and applying Lemma 4 (with $\gamma := 1 - \alpha$ and $\delta := 1$ in the case of (2.3)) yields

$$|S_n| \leq h \sum_{|k| \leq n} |kc_k| = hO\left(\frac{n^{1-\alpha}}{L(n)}\right) = O\left(\frac{h^\alpha}{L(1/h)}\right). \tag{4.9}$$

On the other hand, it follows from (2.3) and (4.2) that

$$|R_n| \leq 2 \sum_{|k| > n} |c_k| = O\left(\frac{n^{-\alpha}}{L(n)}\right) = O\left(\frac{h^\alpha}{L(1/h)}\right). \tag{4.10}$$

Combining (4.1), (4.9) and (4.10) yields $f \in \operatorname{Lip}(\alpha, 1/L)$.

(ii) Conversely, suppose that $c_k \geq 0$ for all k and $f \in \operatorname{Lip}(\alpha, 1/L)$ for some $0 \leq \alpha < 1$. Similarly to (4.6), this time we have

$$|f(x) - f(0)| = \Big|\sum_{k \in \mathbb{Z}} c_k (e^{ikx} - 1)\Big| = O\left(\frac{x^\alpha}{L(1/x)}\right), \qquad x > 0. \tag{4.11}$$

Taking the real part of the above series, we have

$$\sum_{k \in \mathbb{Z}} c_k (1 - \cos kx) = \Big|\sum_{k \in \mathbb{Z}} c_k (\cos kx - 1)\Big| = O\left(\frac{x^\alpha}{L(1/x)}\right),$$

where we took into account that $c_k \geq 0$ for all k. By uniform convergence, due to (1.1), the series $\sum c_k (1 - \cos kx)$ may be integrated term by term on any interval $(0, h)$. Applying Lemma 6 gives

$$\sum_{|k| \geq 1} c_k \left(h - \frac{\sin kh}{k}\right) \leq \frac{Ch^{\alpha+1}}{L(1/h)}, \qquad h > 0,$$

where C is a constant. Substituting $h := 1/n$, we have

$$\sum_{|k|\ge 2n} c_k\left(\frac{1}{n}-\frac{\sin(k/n)}{k}\right) \le \frac{Cn^{-\alpha-1}}{L(n)}, \qquad n\in\mathbb{N}.$$

Since

$$\frac{1}{n}-\frac{\sin(k/n)}{k} \ge \frac{1}{2n} \quad \text{for all } |k|\ge 2n, \tag{4.12}$$

it follows that

$$\frac{1}{2n}\sum_{|k|\ge 2n} c_k \le \frac{Cn^{-\alpha-1}}{L(n)}, \qquad n\in\mathbb{N}.$$

Due to (1.4), this inequality is equivalent to (2.3). ∎

5. Concluding remarks. We make the following supplements to parts (ii) of our Theorems 1 and 2.

THEOREM 3. *Suppose* $\{c_k\}$ *is a sequence of nonnegative real numbers satisfying* (1.1), *and* f *is defined in* (1.2). *If* $f\in \mathrm{Lip}(\alpha, L)$ *for some* $0<\alpha<1$ *and* L *satisfying condition* $(*)$, *then* (2.1) *holds.*

THEOREM 4. *Suppose* $\{c_k\}$ *is a sequence of real numbers satisfying* (1.1) *and such that* $kc_k\ge 0$ *for all* k, *and* f *is defined in* (1.2). *If* $f\in \mathrm{Lip}(\alpha, 1/L)$ *for some* $0<\alpha<1$ *and* L *satisfying condition* $(*)$, *then* (2.3) *holds.*

Before proving Theorems 3 and 4, we recall that the series

$$\sum_{k\in\mathbb{Z}}(-i\,\mathrm{sign}\,k)c_k e^{ikx} \tag{5.1}$$

is said to be the *conjugate series* of the trigonometric series in (1.2). It is well known (see, e.g., [7, Ch. 7, §§1–2]) that if $f\in L^1(\mathbb{T})$, then the conjugate function $\widetilde{f}$ defined by

$$\widetilde{f}(x) := \lim_{h\to 0+} -\frac{1}{\pi}\int_h^\pi \frac{f(x+t)-f(x-t)}{2\tan\frac{1}{2}t}\,dt$$

exists at almost every $x\in\mathbb{T}$. Furthermore, if (1.2) is the Fourier series of $f\in L^1(\mathbb{T})$ and if $\widetilde{f}\in L^1(\mathbb{T})$, then (5.1) is the Fourier series of $\widetilde{f}$.

After these preliminaries, the following corollary can be immediately deduced from the combination of Theorems 1 and 3, or Theorems 2 and 4, respectively.

COROLLARY. *Suppose* $\{c_k\}$ *is a sequence of real numbers satisfying* (1.1) *and one of the following conditions*:

$$c_k\ge 0 \quad \textit{for all } k\in\mathbb{Z},$$

or

$$kc_k \geq 0 \quad \textit{for all } k \in \mathbb{Z},$$

and let f *be defined in* (1.2). *If* $f \in \mathrm{Lip}(\alpha, L)$ *or* $f \in \mathrm{Lip}(\alpha, 1/L)$ *for some* $0 < \alpha < 1$ *and* L *satisfying condition* $(*)$, *then* $\widetilde{f} \in \mathrm{Lip}(\alpha, L)$ *or* $\widetilde{f} \in \mathrm{Lip}(\alpha, 1/L)$, *respectively*, *for the same* α *and* L.

Now we turn to the proofs of Theorems 3 and 4.

Proof of Theorem 3. We begin with inequality (4.6) in the proof of Theorem 1, with h in place of x. This time we take the real part of the relevant series to obtain

$$\sum_{k\in\mathbb{Z}} c_k(1-\cos kh) = \Big|\sum_{k\in\mathbb{Z}} c_k(\cos kh - 1)\Big| \leq Ch^\alpha L(1/h), \qquad h > 0,$$

where C is a constant and we used the assumption that $c_k \geq 0$ for all k.

Analogously to (4.8), we conclude that

$$2\sum_{|k|\leq n} c_k \frac{k^2h^2}{\pi^2} \leq 2\sum_{k\in\mathbb{Z}} c_k \sin^2\frac{kh}{2} \leq Ch^\alpha L(1/h),$$

where n is defined in (4.2). Hence

$$\sum_{|k|\leq n} k^2 c_k \leq \frac{C\pi^2}{2}\, h^{\alpha-2}L(1/h) = O(n^{2-\alpha}L(n)). \tag{5.2}$$

Applying part (i) of Lemma 3 (with $\delta = 2$ and $\gamma = 1$) shows that (5.2) is equivalent to (2.2). Then part (ii) of Lemma 3 (with $\delta = 1$ and $\gamma = 1-\alpha$) implies that (2.2) is equivalent to (2.1), provided that $0 < \alpha < 1$ (because $\gamma = 1 - \alpha$ must be positive). ∎

Proof of Theorem 4. We begin with inequality (4.11) in the proof of Theorem 2. This time we take the imaginary part of the relevant series to obtain

$$\Big|\sum_{k\in\mathbb{Z}} c_k \sin kx\Big| = O\bigg(\frac{x^\alpha}{L(1/x)}\bigg), \qquad x > 0.$$

By uniform convergence, due to (1.1), the series $\sum c_k \sin kx$ may be integrated term by term on any interval $(0, y)$. Applying Lemma 6 yields

$$\sum_{|k|\geq 1} \frac{c_k}{k}\,(1-\cos ky) = O\bigg(\frac{y^{\alpha+1}}{L(1/y)}\bigg), \qquad y > 0, \tag{5.3}$$

where we have taken into account that $kc_k \geq 0$ for all k.

Again we may integrate the series in (5.3) term by term on any interval $(0, h)$. Applying Lemma 6 one more time, we find that

$$\sum_{|k|\geq 1} \frac{c_k}{k}\bigg(\frac{h - \sin kh}{k}\bigg) \leq C\,\frac{h^{\alpha+2}}{L(1/h)}, \qquad h > 0,$$

where C is a constant. Substituting $h := 1/n$, we have

$$\sum_{|k|\ge 2n} \frac{c_k}{k}\left(\frac{1}{n} - \frac{\sin(k/n)}{k}\right) \le C\,\frac{n^{-\alpha-2}}{L(n)}, \qquad n \in \mathbb{N}.$$

In view of inequality (4.12), it follows that

$$\frac{1}{2n}\sum_{|k|\ge 2n} \frac{c_k}{k} \le C\,\frac{n^{-\alpha-2}}{L(n)}.$$

Due to (1.4), this is equivalent to

$$\sum_{|k|\ge n} \frac{c_k}{k} = O\left(\frac{n^{-\alpha-1}}{L(n)}\right), \qquad n \in \mathbb{N}. \tag{5.4}$$

Applying part (ii) of Lemma 4 (with $\delta = 2$ and $\gamma = \alpha + 1$) shows that (5.4) is equivalent to (2.4). Then part (i) of Lemma 4 (with $\delta = 1$ and $\gamma = 1 - \alpha$) implies that (2.4) is equivalent to (2.3), provided that $0 < \alpha < 1$ (since $\gamma = 1 - \alpha$ must be less than $\delta = 1$). ∎

REFERENCES

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Cambridge Univ. Press, 1987.

[2] L. Y. Chan, *Generalized Lipschitz classes and asymptotic behavior of Fourier series*, J. Math. Anal. Appl. 155 (1991), 371–377.

[3] M. Izumi and S. Izumi, *Lipschitz classes and Fourier coefficients*, J. Math. Mech. 18 (1969), 857–870.

[4] L. Leindler, *Strong approximation and generalized Lipschitz classes*, in: Functional Analysis and Approximation (Oberwolfach, 1980), Int. Ser. Numer. Math. 60, Birkhäuser, Basel, 1981, 343–350.

[5] F. Móricz, *Absolutely convergent Fourier series and function classes*, J. Math. Anal. Appl. 324 (2006), 1168–1177.

[6] J. Németh, *Fourier series with positive coefficients and generalized Lipschitz classes*, Acta Sci. Math. (Szeged) 54 (1990), 291–304.

[7] A. Zygmund, *Trigonometric Series*, Vol. 1, Cambridge Univ. Press, 1959.

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