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ABSOLUTE CONTINUITY FOR JACOBI MATRICES WITH POWER-LIKE WEIGHTS

BY

WOJCIECH MOTYKA (Kraków)

Abstract. This work deals with a class of Jacobi matrices with power-like weights. The main theme is spectral analysis of matrices with zero diagonal and weights $\lambda_n := n^\alpha(1+\Delta_n)$ where $\alpha \in (0,1]$. Asymptotic formulas for generalized eigenvectors are given and absolute continuity of the matrices considered is proved. The last section is devoted to spectral analysis of Jacobi matrices with $q_n = n+1+(-1)^n$ and $\lambda_n = \sqrt{q_{n-1}q_n}$.

1. Introduction. Jacobi matrices with power-like weights have been studied in several papers (for example [4, 3, 2]). We continue the study of [3] concerning Jacobi matrices with zero diagonal and weights of the form $n^\alpha(1+\Delta_n)$, where $1/2 < \alpha < 1$. Our main concern is to extend the results of [3] to α ranging over the whole interval $(0,1]$. If $(\Delta_n)_{n\in\mathbb{N}}$ satisfies some special assumptions, which depend on α, then the Jacobi matrices have purely absolutely continuous spectrum.

A *Jacobi matrix* is a tridiagonal infinite matrix of the form

$$J = \begin{pmatrix} q_1 & \lambda_1 & 0 & 0 & \cdots \\ \lambda_1 & q_2 & \lambda_2 & 0 & \cdots \\ 0 & \lambda_2 & q_3 & \lambda_3 & \cdots \\ 0 & 0 & \lambda_3 & q_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which induces an operator $\mathcal{J}$ acting in $l^2(\mathbb{N};\mathbb{C})$. This operator is defined on its maximal domain

$$D(\mathcal{J}) := \{u \in l^2(\mathbb{N};\mathbb{C}) : \mathcal{J}u \in l^2(\mathbb{N};\mathbb{C})\}$$

by the formula

$$(\mathcal{J}u)_n := \lambda_{n-1}u_{n-1} + q_n u_n + \lambda_n u_{n+1}, \qquad u \in D(\mathcal{J}); \tag{1}$$

we put $u_n = \lambda_n := 0$ for $n < 1$. We always assume that the weights $(\lambda_n)_{n\in\mathbb{N}}$

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and the diagonal $(q_n)_{n\in\mathbb{N}}$ satisfy

$$q_n, \lambda_n \in \mathbb{R}, \quad \lambda_n > 0, \quad n \in \mathbb{N}.$$

In this work we consider the operators of the above form with zero diagonal and the weights given by

$$\lambda_n := n^\alpha(1 + \Delta_n), \quad \alpha \in (0, 1], n \in \mathbb{N}. \tag{2}$$

We are interested in spectral properties of such operators, depending on the perturbation (Δ_n). In [3] J. Janas and S. Naboko proved that for (Δ_n) fulfilling two conditions:

$$(\varepsilon_n \Delta_n) \in l^1, \tag{3}$$

$$(\varepsilon_{2n} - \varepsilon_{2n-1}) \in l^1 \ \vee \ (\varepsilon_{2n+1} - \varepsilon_{2n}) \in l^1, \tag{4}$$

where $\varepsilon_n := \Delta_n - \Delta_{n-1}$, the associated Jacobi operator $\mathcal{J}$ has absolutely continuous spectrum covering the whole real line. They used the so called *grouping in blocks approach* and the *subordinacy theory* of Khan and Pearson. Thanks to the grouping in blocks method they were able to find the asymptotics of the solution of the generalized eigenequation

$$\lambda_{n-1}u_{n-1} + q_n u_n + \lambda_n u_{n+1} = \lambda u_n, \quad n \in \mathbb{N}, \lambda \in \mathbb{C}. \tag{5}$$

More precisely, let

$$\vec{u}_n := \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix} \quad \text{and} \quad B_n := \begin{pmatrix} 0 & 1 \\ -\lambda_{n-1}/\lambda_n & (\lambda - q_n)/\lambda_n \end{pmatrix}. \tag{6}$$

B_n is called the *transfer matrix*. Then we can write the equation $\mathcal{J}u = \lambda u$ as

$$\vec{u}_{n+1} = B_n \vec{u}_n, \quad n = 2, 3, \ldots. \tag{7}$$

As a consequence we also have

$$\vec{u}_{n+1} = B_n B_{n-1} \cdots B_{N_{s+1}-1} \cdots B_{N_s} \cdots B_{n_0} \vec{u}_{n_0}, \quad n = 2, 3, \ldots.$$

In [4, 3] one can find a detailed analysis of products $B_{N_{s+1}-1} \cdots B_{N_s}$ over the blocks $\Omega_s = [N_s, N_{s+1})$ of natural numbers such that $\mathbb{N} \setminus \{1\} = \bigcup_s \Omega_s$. The length of the block Ω_s is determined by the requirement

$$\sum_{t \in \Omega_s} \frac{1}{\lambda_t} = 2 + \mathcal{O}(s^{-\alpha/(1-\alpha)}). \tag{8}$$

This is forced by the necessity to obtain Fourier series with frequencies insignificantly different from integer numbers. For $\alpha \in (1/2, 1)$ the term $\mathcal{O}(s^{-\alpha/(1-\alpha)})$ is in l^1, so it can be neglected in further calculations. In [3] there are asymptotic formulas for generalized eigenvectors $(\vec{u}_n)_{n\in\mathbb{N}}$, which imply the absolute continuity of the spectrum of the relevant Jacobi matrices on the whole real line.

For $\alpha \in (0,1]$, the right hand side of (8) cannot be approximated by 2, because the term $\mathcal{O}(s^{-\alpha/(1-\alpha)})$ is no longer in l^1, for every α. That is why the spectral analysis of the Jacobi matrices with zero diagonal and weights (2) requires a different approach than that in [4, 3]. Just like there, we analyse (see the proof of Theorem 1) a product of transfer matrices to find the asymptotics of generalized eigenvectors, and applying the subordinacy theory we are able to show that $\sigma_{\mathrm{ac}}(\mathcal{J}) = \mathbb{R}$. The main difference is that we analyse only a pair of transfer matrices, not the whole block. In our case (Theorem 1) the spectral analysis of $\mathcal{J}$ can be obtained by using Levinson-type theorems found in [2], which apply to systems of linear equations of the form $x(n+1) = A(n)x(n)$ where $A(n)$ are $d \times d$ nonsingular matrices, for $n \geq n_0$.

The organization of the paper is as follows. In Section 2 we quote the definition of the Stolz classes $\mathcal{D}^k$ and prove Lemma 1 which will be used in the next section.

Section 3 contains two theorems which are our main results. The first theorem gives the asymptotics of the solutions of the generalized eigenequation for a class of Jacobi matrices with $q_n = 0$ and $\lambda_n = n^{\alpha}(1 + \Delta_n)$ where $\alpha \in (0,1]$ and $n \geq 1$. Some assumptions on the strength of the perturbation allow us to prove a result which holds for all α in $(0,1]$.

In Section 4 we discuss Remark 2.4 from [8]. Using a Levinson-type theorem from [2] we prove the absolute continuity of the Jacobi operators considered there.

2. Assumptions on the perturbation $(\Delta_n)_{n\in\mathbb{N}}$. First, we recall the definition of the Stolz classes $\mathcal{D}^k$. It will often be convenient to use this notion to state conditions on the weights and the diagonal of a Jacobi matrix. Günter Stolz introduced the $\mathcal{D}^k$ classes in [7] to investigate a discrete version of the Schrödinger operator. In what follows we always assume that $\mathbb{X}$ is one of the spaces $\mathbb{R}, \mathbb{C}, \mathbb{R}^d, \mathbb{C}^d, M_d(\mathbb{R})$ or $M_d(\mathbb{C})$.

Definition 1. Let Δ be the forward difference operator, i.e.

$$(\Delta a)(n) := a(n+1) - a(n), \qquad a \in l(\mathbb{N};\mathbb{X}),\ n = 1, 2, \ldots.$$

For every $k \in \mathbb{N}$ let $\mathcal{D}^k$ be the set of bounded sequences $a \in l(\mathbb{N};\mathbb{X})$ such that

$$\Delta^j a \in l^{k/j}(\mathbb{N};\mathbb{X}), \qquad j = 1, \ldots, k.$$

We finally set $\mathcal{D} := \bigcup_{k\in\mathbb{N}} \mathcal{D}^k$.

Stolz introduced the $\mathcal{D}^k$ classes for numerical sequences only. We use the above general form to simplify some notations.

To use Khan and Pearson's theory we need to know the asymptotic behavior of the solutions of the generalized eigenequation. That behavior can be obtained by applying Theorem 1.7 of [2] concerning asymptotic behavior for systems of linear difference equations of the form

$$\vec{u}(n+1) = A(n)\vec{u}(n), \quad n \geq n_0.$$

The idea is to apply that result to the system (7) but not in its original form. Namely (see the proof of Theorem 1), we will analyse the system

$$\vec{u}(2n+1) = B_{2n}B_{2n-1}\vec{u}(2n-1), \quad n \in \mathbb{N}.$$

Of course the sequence $B_{2n}B_{2n-1}$ must satisfy the assumptions of Theorem 1.7 in [2]. This requires some restrictions on the sequence $(\Delta_n)_{n\in\mathbb{N}}$, described by the following lemma.

LEMMA 1. *Let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence as in* (2) *with* $|\Delta_n| \leq r < 1$ *for* $n \in \mathbb{N}$. *Then*

(i) $\left(\dfrac{\Delta_{n-1} - \Delta_n}{n^\alpha}\right)_{n\in\mathbb{N}} \in l^1 \;\Rightarrow\; \left(\dfrac{1}{\lambda_n}\right)_{n\in\mathbb{N}} \in \mathcal{D}^1,$

(ii) $(\Delta_n)_{n\in\mathbb{N}} \in \mathcal{D}^2 \;\Rightarrow\; \left(\dfrac{\lambda_{n-1}}{\lambda_n}\right)_{n\in\mathbb{N}} \in \mathcal{D}^1.$

Proof. According to the assumption $|\Delta_n| \leq r < 1$ for $n \in \mathbb{N}$, all the denominators of the terms below are different from zero and the sequence $(1+\Delta_n)$ is bounded for all $n \in \mathbb{N}$. We will use the Taylor expansion for x^α repeatedly in this proof.

(i) We have to show that

$$\Delta\frac{1}{\lambda_n} = \frac{(n-1)^\alpha(1+\Delta_{n-1}) - n^\alpha(1+\Delta_n)}{n^\alpha(1+\Delta_n)(n-1)^\alpha(1+\Delta_{n-1})} = \frac{(n-1)^\alpha - n^\alpha}{n^\alpha(n-1)^\alpha(1+\Delta_{n-1})} + \frac{\Delta_{n-1} - \Delta_n}{n^\alpha(1+\Delta_n)(1+\Delta_{n-1})} \tag{9}$$

is in l^1. By assumption the second term on the right hand side in (9) belongs to l^1, and since

$$\frac{(n-1)^\alpha - n^\alpha}{n^\alpha(n-1)^\alpha(1+\Delta_{n-1})} = \frac{n^\alpha - \alpha n^{\alpha-1} + \mathcal{O}(n^{\alpha-2}) - n^\alpha}{n^\alpha(n-1)^\alpha(1+\Delta_{n-1})} = \frac{-\alpha}{n(n-1)^\alpha(1+\Delta_{n-1})} + \mathcal{O}(n^{-2}),$$

so does the first term.

(ii) By definition we have

$$(\Delta_n)_{n\in\mathbb{N}} \in \mathcal{D}^2 \;\Leftrightarrow\; (\Delta\Delta_n)_{n\in\mathbb{N}} \in l^2 \;\wedge\; (\Delta^2\Delta_n)_{n\in\mathbb{N}} \in l^1. \tag{10}$$

Obviously, the first condition in (10) is equivalent to $(\Delta\Delta_n)^2 \in l^1$. Now

$$\begin{aligned}
\Delta\frac{\lambda_{n-1}}{\lambda_n} &= \frac{n^{2\alpha}(1+\Delta_n)^2 - (n^2-1)^\alpha(1+\Delta_{n-1})(1+\Delta_{n+1})}{n^\alpha(n+1)^\alpha(1+\Delta_n)(1+\Delta_{n+1})} \\
&= \frac{n^{2\alpha}\big((1+\Delta_n)^2 - (1+\Delta_{n-1})(1+\Delta_{n+1})\big)}{n^\alpha(n+1)^\alpha(1+\Delta_n)(1+\Delta_{n+1})} \\
&\quad + \frac{\alpha n^{2\alpha-2}(1+\Delta_{n-1})(1+\Delta_{n+1}) + \mathcal{O}(n^{2\alpha-4})}{n^\alpha(n+1)^\alpha(1+\Delta_n)(1+\Delta_{n+1})} \\
&= \frac{(1+\Delta_n)^2 - (1+\Delta_{n-1})(1+\Delta_{n+1})}{(1+\frac{1}{n})^\alpha(1+\Delta_n)(1+\Delta_{n+1})} + \mathcal{O}(n^{-2}) \\
&= \frac{(\Delta\Delta_n)^2}{(1+1/n)^\alpha(1+\Delta_n)(1+\Delta_{n+1})} - \frac{\Delta^2\Delta_{n-1}}{(1+1/n)^\alpha(1+\Delta_n)} + \mathcal{O}(n^{-2}).
\end{aligned}$$

The sequences in the above two denominators are bounded. Now using (10) we have $(\lambda_{n-1}/\lambda_n)_{n\in\mathbb{N}} \in \mathcal{D}^1$. ∎

This lemma will be used to prove Theorem 1 below. The conditions (i) and (ii) from Lemma 1 are sufficient for a Jacobi matrix with zero diagonal and weights (2) to be absolutely continuous. Those conditions are quite different from the conditions (3) and (4) used in [4] and [3], and there are no implications between them, despite the fact that the class of matrices considered here is larger than the class analysed in [4] and [3].

3. Absolute continuity of Jacobi matrices. This section contains two theorems on the asymptotics of the solutions of the system (7), and on the spectral properties of the operator defined by (1) and (2). To state these theorems we need some preparations. Let

$$S(n) := \begin{pmatrix} 0 & \lambda \\ -\lambda\lambda_{2n-2}/\lambda_{2n} & \lambda^2/\lambda_{2n-1}\lambda_{2n} \end{pmatrix}, \qquad S_\infty := \lim_{n\to\infty} S(n), \tag{11}$$

and $\mu_m(j)$ and $\mu_{m\infty}$ be the eigenvalues of $S(j)$ and S_∞, $m = 1, 2$.

THEOREM 1. *Let $\mathcal{J}$ be a Jacobi operator of the form* (1) *with zero diagonal and weights* (2) *for which we have*

(i) $|\Delta_n| \le r < 1$, $n \in \mathbb{N}$;
(ii) $((\Delta_{n-1} - \Delta_n)/n^\alpha)_{n\in\mathbb{N}} \in l^1$;
(iii) $(\Delta_n)_{n\in\mathbb{N}} \in \mathcal{D}^2$.

Then for any $\lambda \in \mathbb{R}\setminus\{0\}$ there exists a basis $\vec{x}_1 = \vec{x}_1(n,\lambda)$, $\vec{x}_2 = \vec{x}_2(n,\lambda)$ of the space of solutions of (7) *of the form*

$$\vec{x}_m(n,\lambda) = \left(\prod_{j=1}^{n-1}\left(-\frac{\lambda_{2j-1}}{\lambda_{2j}} + \frac{1}{\lambda_{2j-1}}\mu_m(j)\right)\right)(\vec{s}_{m\infty} + o(\mathbf{1})), \qquad m = 1, 2,$$

where $\mu_m(j) \to \mu_{m\infty}$, $\vec{s}_{m\infty}$ *is an eigenvector of* S_∞ *for* $\mu_{m\infty}$, *and* $o(\mathbf{1})$ *is a* C^2 *vector with* $o(1)$ *norm.*

Computing all the eigenvalues and eigenvectors mentioned in this theorem we get formulas for the basis vectors of the space of solutions of (7):

$$\tag{12} \vec{x}_1(n,\lambda) = \prod_{j=1}^{n-1}\Bigg(-\frac{\lambda_{2j-1}}{\lambda_{2j}} + \frac{\lambda^2}{2\lambda_{2j-1}^2\lambda_{2j}} + \frac{1}{2\lambda_{2j-1}}\sqrt{\frac{\lambda^4}{\lambda_{2j-1}^2\lambda_{2j}^2} - 4\frac{\lambda^2\lambda_{2j-2}}{\lambda_{2j}}}\Bigg)(\vec{s}_{1\infty} + o(\mathbf{1})),$$

$$\tag{13} \vec{x}_2(n,\lambda) = \prod_{j=1}^{n-1}\Bigg(-\frac{\lambda_{2j-1}}{\lambda_{2j}} + \frac{\lambda^2}{2\lambda_{2j-1}^2\lambda_{2j}} - \frac{1}{2\lambda_{2j-1}}\sqrt{\frac{\lambda^4}{\lambda_{2j-1}^2\lambda_{2j}^2} - 4\frac{\lambda^2\lambda_{2j-2}}{\lambda_{2j}}}\Bigg)(\vec{s}_{2\infty} + o(\mathbf{1})),$$

where

$$\vec{s}_{1\infty} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \vec{s}_{2\infty} = \begin{pmatrix} -1 \\ i \end{pmatrix},$$

Proof. The idea of the proof is to represent the product of the transfer matrices $B_{2n}B_{2n-1}$, given by the formula (6), as $a(n)I + p(n)S(n) + R(n)$, where I is the identity matrix, $(S(n))_{n\in\mathbb{N}} \in \mathcal{D}^1$, $(R(n))_{n\in\mathbb{N}} \in l^1$, $a(n) = -\lambda_{2n-1}/\lambda_{2n}$ and $p(n) = 1/\lambda_{2n-1}$. Next, using Theorem 1.7 in [2] we will get our theorem.

Since $(\lambda_{n-1}/\lambda_n)_{n\in\mathbb{N}} \in \mathcal{D}^1$, using Lemma 1 we have

$$\tag{14} \frac{\lambda_{2n-2}}{\lambda_{2n-1}} - \frac{\lambda_{2n-1}}{\lambda_{2n}} = r_n \quad \text{and} \quad (r_n)_{n\in\mathbb{N}} \in l^1.$$

By (6) we obtain

$$B_{2n}B_{2n-1} = \begin{pmatrix} -\frac{\lambda_{2n-2}}{\lambda_{2n-1}} & \frac{\lambda}{\lambda_{2n-1}} \\ -\frac{\lambda}{\lambda_{2n}}\frac{\lambda_{2n-2}}{\lambda_{2n-1}} & \frac{\lambda^2}{\lambda_{2n-1}\lambda_{2n}} - \frac{\lambda_{2n-1}}{\lambda_{2n}} \end{pmatrix}.$$

Using (14) we can write the above matrix as

$$B_{2n}B_{2n-1} = -\frac{\lambda_{2n-1}}{\lambda_{2n}}\, I + \frac{1}{\lambda_{2n-1}}\, S(n) + R(n),$$

where $S(n)$ is as in (11) and

$$R(n) = \begin{pmatrix} -r_n & 0 \\ 0 & 0 \end{pmatrix}.$$

By (14) we have $(R(n))_{n\in\mathbb{N}} \in l^1$. A product of two sequences in $\mathcal{D}^1$ is again in $\mathcal{D}^1$, so since

$$-\lambda\cdot\frac{\lambda_{2n-2}}{\lambda_{2n}} = -\lambda\cdot\frac{\lambda_{2n-2}}{\lambda_{2n-1}}\cdot\frac{\lambda_{2n-1}}{\lambda_{2n}}, \qquad \frac{\lambda^2}{\lambda_{2n-1}\lambda_{2n}} = \lambda^2\cdot\frac{1}{\lambda_{2n-1}}\cdot\frac{1}{\lambda_{2n}},$$

Lemma 1 implies $(S(n))_{n\in\mathbb{N}} \in \mathcal{D}^1$. Now, by assumption (i) it follows that

$$\det B_{2n}B_{2n-1} = \frac{\lambda_{2n-2}}{\lambda_{2n}} \neq 0, \qquad n\in\mathbb{N},$$

$$\det\biggl(-\frac{\lambda_{2n-1}}{\lambda_{2n}}\,I + \frac{1}{\lambda_{2n-1}}S(n)\biggr) = \biggl(\frac{\lambda_{2n-1}}{\lambda_{2n}}\biggr)^2 \neq 0, \qquad n\in\mathbb{N}.$$

Using the form of the weights λ_n, we obtain

$$\lim_{n\to\infty} -\frac{\lambda_{2n-1}}{\lambda_{2n}} = 1, \qquad \lim_{n\to\infty}\frac{1}{\lambda_{2n-1}} = 0.$$

The matrix S_∞ is

$$S_\infty = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix},$$

for which we have $\operatorname{discr} S_\infty = -4\lambda^2 < 0$, with $\lambda\in\mathbb{R}\setminus\{0\}$.

Now all that needs to be done is to apply Theorem 1.7 of [2] for $A(n) = B_{2n}B_{2n-1}$ and $V(n) = -a(n)I + p(n)S(n)$, where $a(n) = -\lambda_{2n-1}/\lambda_{2n}$ and $p(n) = 1/\lambda_{2n-1}$ for all $n\in\mathbb{N}$. This completes the proof. ■

Thanks to the asymptotic formulas (12) and (13) we are able to describe spectral properties of the operators considered. By the analysis of (12) and (13) we show as in [4, 3] that for any real λ there are no subordinate solutions. The subordinacy theory shows that the absolutely continuous spectrum of the operator $\mathcal{J}$ covers the whole $\mathbb{R}$.

Applying Lemma 1 and Theorem 1 we obtain

THEOREM 2. *The operator $\mathcal{J}$ as in Theorem* 1 *is absolutely continuous and $\sigma_{\mathrm{ac}}(\mathcal{J}) = \mathbb{R}$.*

Proof. We need to show that there are no subordinate solutions. Using Theorem 3 of [6] will then complete the (folklore) proof.

Any solution of (7) with $\lambda\neq 0$ is a linear combination of the vectors (12) and (13). In those formulas, for n large enough, the scalar factors are conjugate. We can thus rewrite them as

$$\vec{x}_1(n) = \prod_{j=1}^{n-1}(a_j + b_j i)(\vec{s}_{1\infty} + o(\mathbf{1})) = R_n e^{i\Phi_n}(\vec{s}_{1\infty} + o(\mathbf{1})),$$

$$\vec{x}_2(n) = \prod_{j=1}^{n-1}(a_j - b_j i)(\vec{s}_{2\infty} + o(\mathbf{1})) = R_n e^{-i\Phi_n}(\vec{s}_{2\infty} + o(\mathbf{1})).$$

So, the solutions of (7) are as follows:

$$\vec{u}(n) = c_1\vec{x}_1(n) + c_2\vec{x}_2(n) = R_n(c_1 e^{i\Phi_n}\vec{s}_{1,n} + c_2 e^{-i\Phi_n}\vec{s}_{2,n}),$$
$$\vec{v}(n) = d_1\vec{x}_1(n) + d_2\vec{x}_2(n) = R_n(d_1 e^{i\Phi_n}\vec{s}_{1,n} + d_2 e^{-i\Phi_n}\vec{s}_{2,n}),$$

where we put $\vec{s}_{i,n}$ for $\vec{s}_{i\infty} + o(\mathbf{1})$ for $i = 1, 2$.

To show that there are no subordinate solutions we will prove that

$$\frac{\sum_{n=1}^{N} \|\vec{u}(n)\|^2}{\sum_{n=1}^{N} \|\vec{v}(n)\|^2} \ge \varrho > 0. \tag{15}$$

Indeed,

$$\begin{aligned}\frac{\sum_{n=1}^{N} \|\vec{u}(n)\|^2}{\sum_{n=1}^{N} \|\vec{v}(n)\|^2} &= \frac{\sum_{n=1}^{N} |R_n|^2 \|c_1 e^{i\Phi_n}\vec{s}_{1,n} + c_2 e^{-i\Phi_n}\vec{s}_{2,n}\|^2}{\sum_{n=1}^{N} |R_n|^2 \|d_1 e^{i\Phi_n}\vec{s}_{1,n} + d_2 e^{-i\Phi_n}\vec{s}_{2,n}\|^2} \\ &= \frac{\sum_{n=1}^{N} |R_n|^2 \|c_1 \vec{s}_{1,n} + c_2 e^{-2i\Phi_n}\vec{s}_{2,n}\|^2}{\sum_{n=1}^{N} |R_n|^2 \|d_1 \vec{s}_{1,n} + d_2 e^{-2i\Phi_n}\vec{s}_{2,n}\|^2}.\end{aligned} \tag{16}$$

The sequence $(e^{-2i\Phi_n})_{n\in\mathbb{N}} \subset \{z \in \mathbb{C} : |z| = 1\}$ has a convergent subsequence $(e^{-2i\Phi_{n_k}})_{k\in\mathbb{N}}$. Hence there are numbers $\underline{\Phi}$ and $\overline{\Phi}$ such that for any complex numbers a and b we have

$$\liminf_{n\to\infty} \|a\vec{s}_{1,n} + be^{-2i\Phi_n}\vec{s}_{2,n}\| = \|a\vec{s}_{1\infty} + be^{-2i\underline{\Phi}}\vec{s}_{2\infty}\|,$$
$$\limsup_{n\to\infty} \|a\vec{s}_{1,n} + be^{-2i\Phi_n}\vec{s}_{2,n}\| = \|a\vec{s}_{1\infty} + be^{-2i\overline{\Phi}}\vec{s}_{2\infty}\|.$$

It follows that for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for $n \ge n_0$,

$$\|c_1\vec{s}_{1,n} + c_2 e^{-2i\Phi_n}\vec{s}_{2,n}\|^2 \ge (\|c_1\vec{s}_{1\infty} + c_2 e^{-2i\underline{\Phi}}\vec{s}_{2\infty}\| - \varepsilon)^2, \tag{17}$$

$$\|d_1\vec{s}_{1,n} + d_2 e^{-2i\Phi_n}\vec{s}_{2,n}\|^2 \le (\|d_1\vec{s}_{1\infty} + d_2 e^{-2i\overline{\Phi}}\vec{s}_{2\infty}\| + \varepsilon)^2. \tag{18}$$

Now using (17) and (18) we can estimate the quotient in (15). First,

$$\begin{aligned}&\frac{\sum_{n=n_0}^{N} |R_n|^2 \|c_1\vec{s}_{1,n} + c_2 e^{-2i\Phi_n}\vec{s}_{2,n}\|^2}{\sum_{n=n_0}^{N} |R_n|^2 \|d_1\vec{s}_{1,n} + d_2 e^{-2i\Phi_n}\vec{s}_{2,n}\|^2} \\ &\qquad \ge \frac{(\|c_1\vec{s}_{1\infty} + c_2 e^{-2i\underline{\Phi}}\vec{s}_{2\infty}\| - \varepsilon)^2}{(\|d_1\vec{s}_{1\infty} + d_2 e^{-2i\overline{\Phi}}\vec{s}_{2\infty}\| + \varepsilon)^2} > 0.\end{aligned} \tag{19}$$

If $n < n_0$ then clearly

$$\frac{\sum_{n=1}^{n_0-1} |R_n|^2 \|c_1\vec{s}_{1,n} + c_2 e^{-2i\Phi_n}\vec{s}_{2,n}\|^2}{\sum_{n=1}^{n_0-1} |R_n|^2 \|d_1\vec{s}_{1,n} + d_2 e^{-2i\Phi_n}\vec{s}_{2,n}\|^2} > 0. \tag{20}$$

by the linear independence of $\vec{s}_{1\infty}$ and $\vec{s}_{2\infty}$. To show that (15) is true, we need one more fact:

$$\forall a, b, c, d > 0 \qquad \frac{a+b}{c+d} \ge \min\left\{\frac{a}{c}, \frac{b}{d}\right\}. \tag{21}$$

Now using (16), (19), (20) and (21) we can prove (15):

$$\frac{\sum_{n=1}^{N}\|\vec{u}(n)\|^2}{\sum_{n=1}^{N}\|\vec{v}(n)\|^2} \geq \min\{\zeta,\xi\} > 0, \tag{22}$$

where

$$\zeta := \frac{\sum_{n=1}^{n_0-1}|R_n|^2\|c_1\vec{s}_{1,n}+c_2e^{-2i\Phi_n}\vec{s}_{2,n}\|^2}{\sum_{n=1}^{n_0-1}|R_n|^2\|d_1\vec{s}_{1,n}+d_2e^{-2i\Phi_n}\vec{s}_{2,n}\|^2},$$
$$\xi := \frac{(\|c_1\vec{s}_{1\infty}+c_2e^{-2i\underline{\Phi}}\vec{s}_{2\infty}\|-\varepsilon)^2}{(\|d_1\vec{s}_{1\infty}+d_2e^{-2i\overline{\Phi}}\vec{s}_{2\infty}\|+\varepsilon)^2}.$$

(22) implies that for any $\lambda \in \mathbb{R}\setminus\{0\}$ there are no subordinate solutions of the generalized eigenequation.

If $\lambda = 0$, then we have explicit formulas for the norms of solutions of the generalized eigenequation with the boundary conditions $(u(1),u(2)) = (0,1)$ or $(1,0)$:

$$\|\vec{x}_1(n)\| = \prod_{j=1}^{\lfloor n/2\rfloor-1}\left|\frac{\lambda_{2j}}{\lambda_{2j+1}}\right|, \qquad \|\vec{x}_2(n)\| = \prod_{j=1}^{\lfloor (n-1)/2\rfloor}\left|\frac{\lambda_{2j-1}}{\lambda_{2j}}\right|. \tag{23}$$

We will show that every solution $\vec{u}(n)$ (being a linear combination of $\vec{x}_1(n)$ and $\vec{x}_2(n)$ given by (23)) satisfies

$$\|\vec{u}(n)\|^2 \leq \frac{C}{\lambda_{n-1}}. \tag{24}$$

(24) implies that there are no subordinate solutions (for details see Lemma 2.2 in [3]) and this yields $\lambda = 0 \in \sigma_{\mathrm{ac}}(\mathcal{J})$. To prove (24), we will show that

$$\frac{\lambda_n}{\lambda_{n+1}} = \frac{\lambda_{n-1}}{\lambda_n}(1+r_n), \qquad n \geq 2, \tag{25}$$

where $(r_n)_{n\in\mathbb{N}} \in l^1$.

Assuming that (25) holds, we prove (24) for $\vec{u}(n) = \vec{x}_1(n)$, the proof for $\vec{x}_2(n)$ being similar:

$$\|\vec{x}_1(n)\|^2 = \prod_{j=1}^{\lfloor n/2\rfloor-1}\left|\frac{\lambda_{2j}}{\lambda_{2j+1}}\right|^2 = \prod_{j=1}^{\lfloor n/2\rfloor-1}\left|\frac{\lambda_{2j}}{\lambda_{2j+1}}\right|\left|\frac{\lambda_{2j-1}}{\lambda_{2j}}\right|\left|1+r_n\right|$$
$$= \frac{\lambda_1}{\lambda_{n-1}}\prod_{j=1}^{\lfloor n/2\rfloor-1}|1+r_n| \leq \frac{\lambda_1}{\lambda_{n-1}}e^{\|(r_n)\|_{l^1}} = \frac{C}{\lambda_{n-1}}.$$

To end the proof we show (25) (omitting some details):

$$\frac{\lambda_n}{\lambda_{n+1}} = \frac{(n-1)^\alpha\big(1+\frac{1}{n-1}\big)^\alpha(1+\Delta_n)}{n^\alpha\big(1+\frac{1}{n}\big)^\alpha(1+\Delta_n)} = \frac{\lambda_{n-1}}{\lambda_n}\bigg(1+\frac{1}{n^2-1}\bigg)^\alpha \frac{(1+\Delta_n)^2}{(1+\Delta_{n-1})(1+\Delta_{n+1})}. \tag{26}$$

The second factor of this product is equal to

$$1+\frac{\alpha}{n^2+1}+\mathcal{O}(n^{-4}) = 1+r_n^{(1)}, \tag{27}$$

where $(r_n^{(1)}) \in l^1$. The third factor equals

$$\bigg(1+\frac{\Delta_n-\Delta_{n-1}}{1+\Delta_{n-1}}\bigg)\bigg(1+\frac{\Delta_n-\Delta_{n+1}}{1+\Delta_{n+1}}\bigg). \tag{28}$$

After some calculations we can rewrite (28) as

$$1+\frac{(\Delta\Delta_{n-1})^2-(1+\Delta_{n-1})(\Delta^2\Delta_{n-1})}{(1+\Delta_{n-1})(1+\Delta_{n+1})} = 1+r_n^{(2)}. \tag{29}$$

Using the assumptions of the theorem we find that $(r_n^{(2)})_{n\in\mathbb{N}}$ is an l^1 sequence. Combining (26) with (27) and (29) one can see that (25) holds. It implies that there are no subordinate solutions for $\lambda = 0$.

Concluding our considerations, we have proved that there can be no subordinate solutions for all $\lambda \in \mathbb{R}$, which finishes the proof. ∎

4. An example of Jacobi matrices with irregular entries. In this section we consider a Jacobi matrix defined by

$$q_n = n+1+(-1)^n, \qquad \lambda_n = \sqrt{q_{n-1}q_n}, \qquad n \geq 1. \tag{30}$$

which was introduced by R. Szwarc in [8] (Remark 2.4). Because the sequence $(q_n/\lambda_n)_{n\in\mathbb{N}}$ does not belong to $\mathcal{D}^1$, the author claims that the Levinson-type theorems in [2] are not valid in this case. On the contrary, Theorem 1.6 in [2] does apply to this example. To see this, we recall that theorem below. It concerns systems of equations of the form

$$u(n+1) = A(n)u(n), \qquad (A(n))_{n\geq n_0} \in l(M_2(\mathbb{C})). \tag{31}$$

THEOREM 3. *Assume that* $A(n) = V(n)+R(n)$ *for* $n \geq n_0$ *and*

- (i) $\det A(n), \det V(n) \neq 0$ *for* $n \geq n_0$,
- (ii) $(V(n))_{n\geq n_0} \in \mathcal{D}^1$,
- (iii) $(R(n))_{n\geq n_0} \in l^1$.

If $(V(n)) \in M_2(\mathbb{R})$ *for* $n \geq n_0$ *and* $\operatorname{discr} V_\infty < 0$, *where* $V_\infty := \lim_{n\to\infty} V(n)$, *then there exists a basis x_1, x_2 of the space of solutions of* (31) *of the form*

$$x_m = \Big(\prod_{j=n_0}^{n-1} \lambda_m(j)\Big)(v_{m\infty} + o(\mathbf{1})),$$

where $\lambda_m(j) \to \lambda_{m\infty}$, $\lambda_m(j)$ is an eigenvalue of $V(n)$, and $v_{m\infty}$ is an eigenvector of $V(n)$ for $\lambda_{m\infty}$, for $m = 1, 2$; $o(\mathbf{1})$ is a C^2 vector with $o(1)$ norm.

In Szwarc's example we have

$$B_{2n}B_{2n-1} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{n}\begin{pmatrix} 1/2 & (1+2\lambda)/4 \\ (1-2\lambda)/4 & 1-\lambda \end{pmatrix} + R(n)$$

with $(R(n)) \in l(\mathbb{N}; M_2(\mathbb{R}))$. In this case

$$V(n) := \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{n}\begin{pmatrix} 1/2 & (1+2\lambda)/4 \\ (1-2\lambda)/4 & 1-\lambda \end{pmatrix}$$

and it belongs to $\mathcal{D}^1$ because it is the sum of a constant matrix and another constant matrix multiplied by a sequence in $\mathcal{D}^1$. This is true for all $\lambda \in \mathbb{R}$. It remains to compute the discriminant of the matrix

$$V_\infty = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

which is equal to $(\operatorname{tr} V_\infty)^2 - 4 \det V_\infty = 1 - 4 = -3 < 0$, for all $\lambda \in \mathbb{R}$.

Now we can apply Theorem 3. We obtain the asymptotic behavior of the solutions of (7) which is determined by the product of the eigenvalues of $V(j)$ matrices. Next, by subordinacy theory we prove that $\sigma(\mathcal{J}) = \sigma_{\mathrm{ac}}(\mathcal{J}) = \mathbb{R}$. The eigenvalues of $V(j)$ are

$$\lambda_1(j) = -\frac{1}{2} + \frac{3-2\lambda}{4j} + \frac{1}{2}\sqrt{3}i\Big(1 - \frac{1+2\lambda}{6j}\Big) + \mathcal{O}(n^{-2}),$$
$$\lambda_2(j) = -\frac{1}{2} + \frac{3-2\lambda}{4j} - \frac{1}{2}\sqrt{3}i\Big(1 - \frac{1+2\lambda}{6j}\Big) + \mathcal{O}(n^{-2}).$$

Just as in the proof of Theorem 2, the eigenvalues are conjugate. So the proof of absolute continuity of the Jacobi operator given by (30) is essentially the same as that of Theorem 2.

REFERENCES

[1] Yu. M. Berezanskiĭ, *Expansions in Eigenfunctions of Selfadjoint Operators*, Naukova Dumka, Kiev, 1965 (in Russian).

[2] J. Janas and M. Moszyński, *Spectral properties of Jacobi matrices by asymptotic analysis*, J. Approx. Theory 120 (2003), 309–336.

[3] J. Janas and S. Naboko, *Jacobi matrices with power-like weights—grouping in blocks approach*, J. Funct. Anal. 166 (1999), 218–243.

[4] —, —, *Asymptotics of generalized eigenvectors for unbounded Jacobi matrices with power-like weights*, *Pauli matrices commutation relations and Cesàro averaging*, in: Oper. Theory Adv. Appl. 117, Birkhäuser, 2000, 165–186.

[5] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer, Berlin, 1980.

[6] S. Khan and D. B. Pearson, *Subordinacy and spectral theory for infinite matrices*, Helv. Phys. Acta 65 (1992), 505–527.

[7] G. Stolz, *Spectral theory for slowly oscillating potentials I. Jacobi matrices*, Manuscripta Math. 84 (1994), 245–260.

[8] R. Szwarc, *Absolute continuity of spectral measure for certain unbounded Jacobi matrices*, in: Internat. Ser. Numer. Math. 142, Birkhäuser, 2002, 255–262.

Institute of Mathematics
Polish Academy of Sciences
Św. Tomasza 30
31-027 Kraków, Poland
E-mail: namotyka@cyf-kr.edu.pl

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