COLLOQUIUM MATHEMATICUM

VOL. 107 2007 NO. 2

DIVERGENT VECTOR SEQUENCES $\{y_n\}$ *WITH* $\Delta y_n \to 0$

BY

ROMAN WITUŁA (Gliwice)

Abstract. An approximation property of divergent sequences in normed vector spaces is discussed.

1. Introduction. We are interested in sequences $\{y_n\}$ of elements of a normed vector space $\mathbb{X}$ such that $\Delta y_n := y_{n+1} - y_n \to 0$ as $n \to \infty$. Each vector sequence $\{y_n\}$ of this kind can be approximated by a sequence of the form

$$x_n = \beta_n y + y_n, \quad y_n \in \mathbb{N}, \tag{1}$$

where $\|y\| = 1$ and $\{\beta_n\} \subset \mathbb{R}$ is a sequence convergent to zero. Then, of course, $\|x_n - y_n\| \to 0$ as $n \to \infty$. Moreover, the sequence $\{x_n\}$ can be chosen so that

$$\|\Delta x_n\| = \alpha_n, \quad n \in \mathbb{N},$$

is any given null sequence with $\alpha_n > \|y_{n+1} - y_n\|$, $n \in \mathbb{N}$.

Interesting results are obtained when $\{y_n\}$ is a divergent sequence with one of the properties: either $\limsup \|y_n\| = \infty$, or $\liminf \|y_n\| < \limsup \|y_n\| < \infty$, or $\{y_n\}$ is divergent and, at the same time, has no subsequence with convergent sequence $\{\|y_n\|\}$.

2. Main result

THEOREM 1. *Let* $(\mathbb{X}, \|\cdot\|)$ *be a normed linear space over either* $F = \mathbb{R}$ *or* $\mathbb{C}$ *and let* $\{y_n\} \subset \mathbb{X}$, $\Delta y_n := y_{n+1} - y_n \to 0$. *Then, for every* $\{\alpha_n\} \subset (0, \infty)$ *such that* $\alpha_n \to 0$ *and* $\|y_{n+1} - y_n\| < \alpha_n$, $n \in \mathbb{N}$, *and for every* $y \in \mathbb{X}$ *with* $\|y\| = 1$, *there exists a sequence* $\{\beta_n\} \subset \mathbb{R}$ *convergent to zero such that*

$$\|x_{n+1} - x_n\| = \alpha_n$$

where x_n *is defined in* (1).

Proof. Fix $\{\alpha_n\}$ and $y \in \mathbb{X}$ as in the statement. We define β_n by induction in such a way that

$$\|\beta_{n+1} y + y_{n+1} - \beta_n y - y_n\| = \alpha_n \tag{2}$$

2000 *Mathematics Subject Classification*: 40A05, 46B20.

Key words and phrases: divergent vector sequences.

and

$$(3)\qquad |\beta_{n+1}| < |\beta_n| \quad \text{whenever} \quad \beta_n\beta_{n+1} > 0$$

for every $n \in \mathbb{N}$. Set $\beta_1 = 0$. Suppose that $\beta_1, \dots, \beta_n$ satisfying (2) and (3) have already been defined. Clearly, each of the following two functions is continuous:

$$(-\infty, \beta_n] \ni \beta \overset{f}{\mapsto} \|\beta y + y_{n+1} - \beta_n y - y_n\|$$

and

$$[\beta_n, \infty) \ni \beta \overset{g}{\mapsto} \|\beta y + y_{n+1} - \beta_n y - y_n\|.$$

Moreover, we have $f(\beta_n) < \alpha_n$, $g(\beta_n) < \alpha_n$ and

$$\lim_{\beta\to\infty} f(\beta) = \lim_{\beta\to\infty} g(\beta) = \infty.$$

Hence, by the Darboux property, there exist

$$(4)\qquad \beta_{n+1}^{(1)} \in (-\infty, \beta_n) \quad \text{and} \quad \beta_{n+1}^{(2)} \in (\beta_n, \infty)$$

such that

$$f(\beta_{n+1}^{(1)}) = g(\beta_{n+1}^{(2)}) = \alpha_n.$$

Let $\beta_{n+1} = \beta_{n+1}^{(1)}$ if $\beta_n \ge 0$. In the opposite case let $\beta_{n+1} = \beta_{n+1}^{(2)}$. Then conditions (2) and (3) are obviously satisfied.

We show that the sequence $\{\beta_n\}$ is convergent. We distinguish two cases. First, assume that $\beta_n\beta_{n+1} > 0$ for sufficiently large $n \in \mathbb{N}$. This means, by (4), that

$$\text{either} \quad 0 < \beta_{n+1} < \beta_n, \quad \text{or} \quad \beta_n < \beta_{n+1} < 0$$

for sufficiently large $n \in \mathbb{N}$. Hence, the sequence $\{\beta_n\}$ is convergent. The second case, where $\beta_n\beta_{n+1} \le 0$ for infinitely many n, also implies the convergence of $\{\beta_n\}$. Indeed, since

$$|\beta_{n+1} - \beta_n| \le \|\beta_{n+1} y + y_{n+1} - \beta_n y - y_n\| + \|y_{n+1} - y_n\| < 2\alpha_n$$

it follows that

$$(5)\qquad \max\{|\beta_n|, |\beta_{n+1}|\} < 2\alpha_n$$

in the case of $\beta_n\beta_{n+1} \le 0$, i.e. for infinitely many positive integers n. From (3), (5) and the convergence of $\{\alpha_n\}$ to zero we can easily deduce that $\{\beta_n\}$ is convergent.

At the same time, by (2), we have

$$\|x_{n+1} - x_n\| = \alpha_n$$

for every $n \in \mathbb{N}$, where $x_n = \beta_n y + y_n$. Since for the sequence

$$x_n^* = (\beta_n - \lim_{k\to\infty} \beta_k) y + y_n$$

we have $\|\Delta x_n\| = \|\Delta x_n^*\|$, it can be assumed that $\{\beta_n\}$ is convergent to zero and thus the proof is finished. ∎

REMARK 1. If $\sum \alpha_n < \infty$, then each sequence $\{\beta_n\}$ with $\beta_n \in \{\beta_n^{(1)}, \beta_n^{(2)}\}$ for every $n \in \mathbb{N}$ is convergent, because

$$\begin{aligned}|\beta_m - \beta_n| = \|\beta_m y - \beta_n y\| &\le \|y_m - y_n\| + \|\beta_m y + y_m - \beta_n y - y_n\| \\ &\le \|y_m - y_n\| + \sum_{k=m}^{n-1} \|\beta_{k+1} y + y_{k+1} - \beta_k y - y_k\| \\ &= \|y_m - y_n\| + \sum_{k=m}^{n-1} \alpha_k\end{aligned}$$

for every $m, n \in \mathbb{N}$, $m < n$.

REMARK 2. If $\{y_n\} \subset \mathbb{X}$ is such that $\lim_{n\to\infty} \|y_n\| = \infty$, then the proof of Theorem 1 can be easily visualized. For simplicity, let us draw the sequence $\{y_n\}$ on the horizontal axis. The element x_{n+1} belongs to the intersection of a straight line with a sphere of radius α_n; there are two possibilities of choosing such an element (see Remark 3 below) and one has to choose the appropriate one; Figure 1 illustrates this situation.

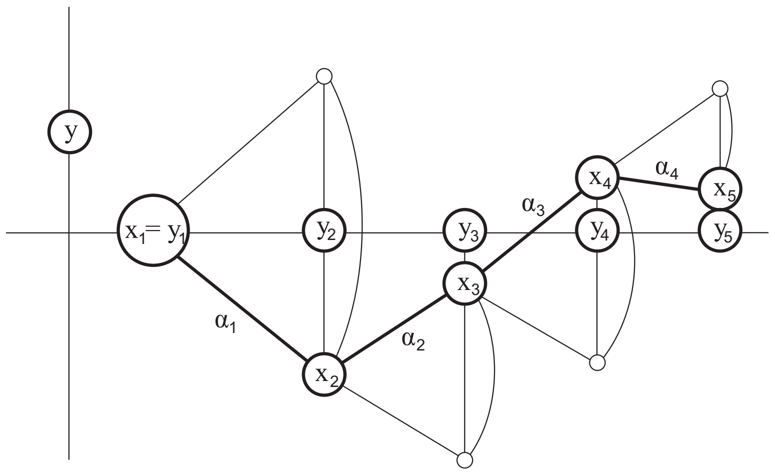

Fig. 1.

REMARK 3. As follows from the proof of Theorem 1, the intersection of a straight line with a sphere contains at least two points. Note that (see for example [1]) if $C \subset X$ is convex and $x_0 \in \operatorname{cl} C$, $x_1 \in \operatorname{int} C$ then $x_0 + \Theta(x_1 - x_0) \in \operatorname{int} C$ for every $\Theta \in (0, 1]$, so there are no more such points.

COROLLARY 1. *Let* $(\mathbb{X}, \|\cdot\|)$ *be an incomplete normed linear space. Then for every positive real* p *there exist divergent series* $\sum x_n$ *and* $\sum y_n$ *with*

$x_n, y_n \in \mathbb{X}$ *such that*

$$\{\|x_n\|\} \in l^p \setminus \bigcup_{q<p} l^q \quad \textit{and} \quad \{\|y_n\|\} \in \bigcup_{q>p} l^q \setminus l^p.$$

Proof. By Theorem 1, it is sufficient to prove that

$$l^p \setminus \bigcup_{q<p} l^q \neq \emptyset \quad \text{and} \quad \bigcup_{q>p} l^q \setminus l^p \neq \emptyset.$$

But an easy computation shows that

$$\left\{n^{-1-2\frac{\log\log(n+3)}{\log n}}\right\}_{n=1}^{\infty} \in l^1 \setminus \bigcup_{q<1} l^q$$

and

$$\{n^{-1}\}_{n=1}^{\infty} \in \bigcup_{q>1} l^q \setminus l^1,$$

which implies the above relationships. ∎

Acknowledgments. The author wishes to express his gratitude to Professor Piotr Biler and Mr. Jerzy Trzeciak for several helpful suggestions during the preparation of the final version of the paper.

REFERENCES

[1] P.-J. Laurent, *Approximation et optimisation*, Hermann, Paris, 1972.

Institute of Mathematics
Silesian University of Technology
Kaszubska 23
44-100 Gliwice, Poland
E-mail: r.witula@polsl.pl

Received 23 June 2005;
revised 7 June 2006 (4626)