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ON THE DENSITY AND NET WEIGHT OF REGULAR SPACES

BY

ARMANDO ROMERO MORALES (Puebla)

Abstract. We use the cardinal functions ac and lc, due to Fedeli, to establish bounds on the density and net weight of regular spaces which improve some well known bounds. In particular, we use the language of elementary submodels to establish that $d(X) \leq \pi\chi(X)^{ac(X)}$ for every regular space X. This generalizes the following result due to Shapirovskiĭ: $d(X) \leq \pi\chi(X)^{c(X)}$ for every regular space X.

1. Introduction. Among the best known theorems on cardinal functions are those which give an upper bound on the cardinality of a space in terms of other cardinal invariants. In [3], Fedeli introduced three cardinal functions which are useful to generalize two of the best known inequalities in the theory of cardinal functions: (1) Hajnal–Juhász's inequality [5]: for $X \in \mathcal{T}_2$, $|X| \leq 2^{c(X)\chi(X)}$; and (2) Arkhangel'skii's inequality [5]: for $X \in \mathcal{T}_2$, $|X| \leq 2^{L(X)t(X)\psi(X)}$.

The main aim of this paper is to use the cardinal functions ac and lc, due to Fedeli, to establish:

(1) If X is regular, then $d(X) \leq \pi\chi(X)^{ac(X)}$. The proof of this result uses the language of elementary submodels.
(2) If X is T_3 space, then $|X| \leq \pi\chi(X)^{lc(X)\psi(X)}$.

These cardinal inequalities generalize the following results due to Shapirovskiĭ ([1] and [5], respectively): (a) If X is regular, then $d(X) \leq \pi\chi(X)^{c(X)}$; and (b) for $X \in \mathcal{T}_3$, $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$. Later we will give an example to show that our result can give better estimates than Shapirovskiĭ's inequalities above.

Moreover, we will establish that:

(3) If X is regular, then $nw(X) \leq \pi w(X)^{lc(X)}$.

2. Notations and definitions. We refer the reader to [5] and [2] for definitions and terminology not explicitly given. Let πw, c, χ, ψ, $\pi\chi$ and nw denote the following standard cardinal functions: π-weight, cellularity, character, pseudocharacter, π-character and net weight respectively.

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Let X be a topological space and let Y be a subspace of X. Here and in what follows $\overline{Y}$ is the closure of Y in X. For any set X and cardinal κ, $[X]^{\leq\kappa}$ denotes the collection of all subsets of X with cardinality $\leq \kappa$; $[X]^{<\kappa}$ is defined analogously.

The following cardinal functions are due to Fedeli [3].

Definition 2.1. Let X be a topological space.

(a) $ac(X)$ is the smallest infinite cardinal κ such that there is a subset S of X such that $|S| \leq 2^\kappa$ and for every open collection $\mathcal{U}$ in X, there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ with $\bigcup\mathcal{U} \subseteq S \cup \overline{\bigcup\{V : V \in \mathcal{V}\}}$.
(b) $lc(X)$ is the smallest infinite cardinal κ such that there is a closed subset F of X such that $|F| \leq 2^\kappa$ and for every open collection $\mathcal{U}$ in X, there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ with $\bigcup\mathcal{U} \subseteq F \cup \overline{\bigcup\{V : V \in \mathcal{V}\}}$.

Clearly $ac(X) \leq lc(X) \leq c(X)$ for any topological space X.

3. Main results. Our first result is a generalization of Shapirovskiĭ's inequality [1] (see Corollary 3.2 below). The proof makes use of elementary submodels; the reader is referred to [4] for the use of elementary submodels in the theory of cardinal functions.

Theorem 3.1. *If X is regular, then $d(X) \leq \pi\chi(X)^{ac(X)}$.*

Proof. Let $\lambda = \pi\chi(X)$, $\gamma = ac(X)$ and $\kappa = \lambda^\gamma$. Let $S \in [X]^{\leq 2^\gamma}$ witness that $ac(X) = \gamma$. For every $x \in X$ let $\mathcal{B}_x$ be a local π-base in X at x with $|\mathcal{B}_x| \leq \lambda$. Let τ be the topology of X; and let $f : X \to \mathcal{P}(\tau)$ be the map defined by $f(x) = \mathcal{B}_x$ for every $x \in X$.

Let $A = \kappa \cup \{\kappa, X, \tau, S, f\}$ and take a set $\mathcal{M}$ such that $A \subseteq \mathcal{M}$, $|\mathcal{M}| = \kappa$ and which reflects enough formulas to carry out our argument. To be more precise we ask that $\mathcal{M}$ reflects enough formulas so that the following conditions are satisfied ([1]):

(1) $C \in \mathcal{M}$ for every $C \in [\mathcal{M}]^{\leq\gamma}$.
(2) $\mathcal{B}_x \in \mathcal{M}$ for every $x \in X \cap \mathcal{M}$.
(3) If $B \subseteq X$ and $B \in \mathcal{M}$, then $\overline{B} \in \mathcal{M}$.
(4) If $\mathcal{A} \in \mathcal{M}$, then $\bigcup\mathcal{A} \in \mathcal{M}$.
(5) If $B \subseteq X$ with $X \cap \mathcal{M} \subseteq B$ and $B \in \mathcal{M}$, then $X = B$.
(6) If $E \in \mathcal{M}$ and $|E| \leq \kappa$, then $E \subseteq \mathcal{M}$.

Observe that by (2) and (6), $\mathcal{B}_y \subset \mathcal{M}$ for every $y \in X \cap \mathcal{M}$. The proof is complete if $X \cap \mathcal{M}$ is dense in X. Suppose not and take $p \in X \setminus \overline{X \cap \mathcal{M}}$.

([1]) The existence of such a set $\mathcal{M}$ follows from Proposition 3 in [4]. Note for instance that (1) and (6) follow if $\mathcal{M}$ is closed under γ-sequences and from Lemma 4 in [4], respectively.

Since X is regular, there exists an open neighborhood R of p in X such that $\overline{R} \cap (X \cap \mathcal{M}) = \emptyset$. Let $\mathcal{U} = \{V \in \mathcal{B}_x : x \in X \cap \mathcal{M} \text{ and } V \cap R = \emptyset\}$.

Note $\overline{\bigcup \mathcal{U}} \cap R = \emptyset$ and $X \cap \mathcal{M} \subseteq \overline{\bigcup \mathcal{U}}$. On the other hand, as $\gamma = ac(X)$, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq \gamma}$ such that $\bigcup \mathcal{U} \subseteq S \cup \overline{\bigcup \mathcal{V}}$, hence $X \cap \mathcal{M} \subseteq \overline{\bigcup \mathcal{U}} \subseteq \overline{S \cup \overline{\bigcup \mathcal{V}}} =: B$.

Note that $p \notin B$. Indeed, $R \cap S \subseteq R \cap (X \cap \mathcal{M}) \subseteq \overline{R} \cap (X \cap \mathcal{M}) = \emptyset$. Moreover, $R \cap \overline{\bigcup \mathcal{V}} \subseteq R \cap \overline{\bigcup \mathcal{U}} = \emptyset$. Thus $R \cap B = \emptyset$.

We claim that $B \in \mathcal{M}$. Clearly $S \in \mathcal{M}$ and for each $V \in \mathcal{V}$, there exists $x \in X \cup \mathcal{M}$ such that $V \in \mathcal{B}_x \subseteq \mathcal{M}$, hence $V \in \mathcal{M}$, therefore $V \subseteq \mathcal{M}$ and since $|\mathcal{V}| \leq \gamma < \kappa$, we have $\mathcal{V} \in \mathcal{M}$, by (1); hence, by (4), $\bigcup \mathcal{V} \in \mathcal{M}$. On the other hand, $\bigcup \mathcal{V} \subseteq X$, and so by (3), $\overline{\bigcup \mathcal{V}} \in \mathcal{M}$. Therefore $\{S, \overline{\bigcup \mathcal{V}}\} \subseteq \mathcal{M}$, hence, by (4), $S \cup \overline{\bigcup \mathcal{V}} \in \mathcal{M}$. Since $S \cup \overline{\bigcup \mathcal{V}} \subseteq X$ and by (3), $\overline{S \cup \overline{\bigcup \mathcal{U}}} \in \mathcal{M}$, it follows that $B \in \mathcal{M}$.

Finally, by (5), $B = X$. A contradiction. ∎

COROLLARY 3.2 (Shapirovskiĭ [1]). *If X is regular, then $d(X) \leq \pi\chi(X)^{c(X)}$.*

COROLLARY 3.3. *If X is regular, then $d(X) \leq \pi\chi(X)^{lc(X)}$.*

The space in Example 3.5 below shows that Theorem 3.1 can give a better estimate than Corollary 3.2.

By Theorem 3.1 and Efimov's inequality [5]: for any space X, $|RO(X)| \leq \pi w(X)^{c(X)}$, we have the following remarkable theorem due to Shapirovskiĭ [1].

THEOREM 3.4 (Shapirovskiĭ). *If X is regular, then*

$$w(X) \leq |RO(X)| \leq \pi\chi(X)^{c(X)}.$$

Proof. It is enough to note that $\pi w(X) = d(X)\pi\chi(X)$ for every topological space X (see [5, Theorem 3.8]). ∎

Of course, it is natural to ask if c can be replaced by ac or lc in Theorem 3.4. The answer is, in general, no!

EXAMPLE 3.5. Let $X = D(2^\kappa)$ be a discrete space of cardinality 2^κ for some cardinal $\kappa \geq \omega$. Note that $\pi\chi(X) = \omega$, $c(X) = 2^\kappa$, $ac(X) \leq \kappa$ and $lc(X) \leq \kappa$; hence $|RO(X)| = \pi\chi(X)^{c(X)} = 2^{2^\kappa}$, $\pi\chi(X)^{ac(X)} = \pi\chi(X)^{lc(X)} = 2^\kappa$ and therefore $|RO(X)| > \pi\chi(X)^{ac(X)} = \pi\chi(X)^{lc(X)}$. Moreover, $|RO(X)| > \pi w(X)^{ac(X)}$; hence, it is not possible to replace c by ac or lc in Efimov's inequality.

Another natural question is:

QUESTION 3.6. *If X is a regular space, is it true that $w(X) \leq \pi\chi(X)^{ac(X)}$ or $w(X) \leq \pi\chi(X)^{lc(X)}$?*

It is well known that if X is a dyadic space or if X is a topological group, then $w(X) = \pi w(X)$ (see [2] and [8], respectively). Thus, in this case, the answer to Question 3.6 is affirmative. However, in general Question 3.6 seems to be open. We will prove that if X is a compact space, then $w(X) \leq \pi\chi(X)^{lc(X)}$. This result is a consequence of the following theorem.

THEOREM 3.7. *If X is a regular space, then $nw(X) \leq \pi w(X)^{lc(X)}$.*

Proof. Let $\lambda = \pi w(X)$, $\gamma = lc(X)$, let F be a closed subset of X with $|F| \leq 2^\gamma$ witnessing that $lc(X) = \gamma$, and let $\mathcal{B}$ be a π-base of X with $|\mathcal{B}| = \lambda$. Define $\mathcal{F} = \{\overline{\bigcup\{V : V \in \mathcal{V}\}} : \mathcal{V} \in [\mathcal{B}]^{\leq\gamma}\}$ and $\mathcal{N} = \mathcal{F} \cup \{\{x\} : x \in F\}$. It is clear that $|\mathcal{N}| \leq \lambda^\gamma$; it remains to see that $\mathcal{N}$ is a net of X. Let U be an open subset of X and let $p \in U$. We have two cases:

(1) If $p \in F$, then $p \in \{p\} \subseteq U$.
(2) Assume $p \in X \setminus F$. Since X is regular, there exists an open neighborhood V of p in X such that $\overline{V} \subseteq U$. Let $\mathcal{V} = \{W \in \mathcal{B} : W \subseteq V\}$ and note that $p \in \overline{\bigcup \mathcal{V}}$; since $lc(X) = \gamma$, there is $\mathcal{W} \in [\mathcal{V}]^{\leq\gamma}$ such that $\mathcal{V} \subseteq F \cup \overline{\bigcup\{\overline{W} : W \in \mathcal{W}\}}$. Clearly $p \in \overline{\bigcup\{\overline{W} : W \in \mathcal{W}\}}$.

Therefore $nw(X) \leq \pi w(X)^{lc(X)}$. ∎

Using Theorem 3.7, we can derive several corollaries.

COROLLARY 3.8. *If X is regular, then $nw(X) \leq \pi\chi(X)^{lc(X)}$.*

Proof. Since $\pi w(X) = d(X)\pi\chi(X)$ for every topological space X (see [5, 3.8]), by Theorem 3.7, $\pi w(X) \leq \pi\chi(X)^{lc(X)}\pi\chi(X) = \pi\chi(X)^{lc(X)}$; hence $nw(X) \leq (\pi\chi(X)^{lc(X)})^{lc(X)} = \pi\chi(X)^{lc(X)}$. ∎

COROLLARY 3.9. *For $X \in \mathcal{T}_3$, $|X| \leq \pi\chi(X)^{lc(X)\psi(X)}$.*

Proof. Since $|X| \leq nw(X)^{\psi(X)}$ for every T_1 space X (see [5, Theorem 4.1]), and since by Theorem 3.7, $nw(X)^{\psi(X)} \leq \pi\chi(X)^{lc(X)\psi(X)}$, the assertion follows. ∎

COROLLARY 3.10. *For $X \in \mathcal{T}_3$, $|X| \leq 2^{\pi\chi(X)lc(X)\psi(X)}$.*

COROLLARY 3.11 (Shapirovskiĭ). *For $X \in \mathcal{T}_3$, $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$.*

COROLLARY 3.12. *For $X \in \mathcal{T}_3$, $|X| \leq 2^{\pi\chi(X)c(X)\psi(X)}$.*

A generalization of the inequality in Corollary 3.11 has also been obtained by Sun [7] and Fedeli [3]. On the other hand, if X is as in Example 3.5, then $|X| = \pi\chi(X)^{lc(X)\psi(X)} < \pi\chi(X)^{c(X)\psi(X)}$.

COROLLARY 3.13. *If X is a Hausdorff compact space, then $w(X) \leq \pi\chi(X)^{lc(X)}$.*

Proof. Since $w(X) = nw(X)$ for every compact space X (see [5, 7.4]), the assertion follows from Corollary 3.8. ∎

Recently van Mill [6] proved that if X is compact and power homogeneous (i.e. X^κ is homogeneous for some κ), then $|X| \leq 2^{\pi\chi(X)c(X)}$. This is a consequence of the following more general result due to van Mill [6].

THEOREM 3.14. *If X is a compact and power homogeneous space, then $|X| \leq w(X)^{\pi\chi(X)}$.*

Now, it is natural to ask if c can be replaced by ac or lc in van Mill's inequality above. At the moment, the author does not know the answer for ac, but for lc the answer is "yes".

COROLLARY 3.15. *If X is a compact and power homogeneous space, then $|X| \leq 2^{lc(X)\pi\chi(X)}$.*

Proof. By Theorem 3.14, $|X| \leq w(X)^{\pi\chi(X)}$; and by Corollary 3.13, we have $w(X)^{\pi\chi(X)} \leq (\pi\chi(X)^{lc(X)})^{\pi\chi(X)} = \pi\chi(X)^{lc(X)\pi\chi(X)}$; hence $|X| \leq \pi\chi(X)^{lc(X)\pi\chi(X)} = 2^{lc(X)\pi\chi(X)}$. ∎

COROLLARY 3.16 (van Mill). *If X is a compact and power homogeneous space, then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

At the moment the author does not know the answers to the following questions.

QUESTION 3.17. *Is it true that $w(X) \leq \pi w(X)^{ac(X)}$ or $w(X) \leq \pi w(X)^{lc(X)}$ for every regular space X?*

QUESTION 3.18. *Is it true that $w(X) \leq \pi w(X)^{ac(X)}$ for every compact Hausdorff space X?*

QUESTION 3.19. *Is it true that $|X| \leq \pi\chi(X)^{ac(X)\psi(X)}$ for every regular space X?*

QUESTION 3.20. *Is it true that $|X| \leq 2^{\pi\chi(X)ac(X)\psi(X)}$ for every regular space X?*

QUESTION 3.21. *Is there an example of a compact space X such that $\omega < c(X) = ac(X) = lc(X)$?*

QUESTION 3.22. *Is there an example of a compact and power homogeneous space X such that $|X| \leq 2^{lc(X)\pi\chi(X)} < 2^{c(X)\pi\chi(X)}$?*

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Facultad de Ciencias Físico-Matemáticas
Universidad Autónoma de Puebla
Puebla, México
E-mail: est112@fcfm.buap.mx

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