MAXIMAL OPERATORS OF FEJÉR MEANS OF DOUBLE VILENKIN–FOURIER SERIES

BY

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Abstract. The main aim of this paper is to prove that the maximal operator $\sigma_0^* := \sup_n |\sigma_{n,n}|$ of the Fejér means of the double Vilenkin–Fourier series is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the additive group of integers modulo $m_k$. Define the group $G_m$ as the complete direct product of the groups $Z_{m_j}$, with the product of the discrete topologies of $Z_{m_j}$’s. The direct product $\mu$ of the measures $\mu_k(\{j\}) := \frac{1}{m_k} (j \in Z_{m_k})$ is the Haar measure on $G_m$ with $\mu(G_m) = 1$.

If the sequence $m$ is bounded, then $G_m$ is called a bounded Vilenkin group, otherwise it is an unbounded Vilenkin group. The elements of $G_m$ can be represented by sequences $x := (x_0, x_1, \ldots, x_j, \ldots) (x_j \in Z_{m_j})$. It is easy to give a base of neighborhoods of $x \in G_m$:

$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$

for $n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{N}_+$.

The generalized number system based on $m$ is defined in the following way: $M_0 := 1, M_{k+1} := m_k M_k$ ($k \in \mathbb{N}$). Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ ($j \in \mathbb{N}_+$) and only a finite number of $n_j$’s are not zero. We use the following notations. For $n > 0$ let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$), $n^{(k)} := \sum_{j=k}^{\infty} n_j M_j$ and $n_{(k)} := n - n^{(k)}$.

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Denote by $L^p(G_m)$ the usual (one-dimensional) Lebesgue spaces, with norms $\| \cdot \|_p (1 \leq p \leq \infty)$.

Next, we introduce on $G_m$ an orthonormal system which is called the Vilenkin system. First define the complex-valued functions $r_k : G_m \to \mathbb{C}$, called the generalized Rademacher functions, in this way:

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \quad (r^2 = -1, \ x \in G_m, \ k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_{nk}(x) \quad (n \in \mathbb{N}).$$

If $m = 2$, we call this system the Walsh–Paley system. The Vilenkin system is orthonormal and complete in $L^1(G_m)$ [8].

Now, we introduce analogues of the usual definitions of Fourier analysis. If $f \in L^1(G_m)$ we can make the following definitions:

- Fourier coefficients:
  $$\hat{f}(k) := \int_{G_m} f \overline{\psi}_k \, d\mu \quad (k \in \mathbb{N}),$$

- partial sums:
  $$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k \quad (n \in \mathbb{N}_+, \ S_0 f := 0),$$

- Fejér means:
  $$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_n f \quad (n \in \mathbb{N}_+),$$

- Dirichlet kernels:
  $$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}_+).$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n, \\ 0 & \text{if } x \in G_m \setminus I_n. \end{cases}$$ (1)

For $f \in L_1(G_m \times G_m)$, the rectangular partial sums of the double Vilenkin–Fourier series of $f$ are defined as follows:

$$S_{M,N}(f; x^1, x^2) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i,j) \psi_i(x^1) \psi_j(x^2),$$
where the number
\[ \hat{f}(i, j) = \int_{G_m \times G_m} f(x^1, x^2) \overline{\psi}_i(x^1) \overline{\psi}_j(x^2) \mu(x^1, x^2). \]
is said to be the \((i, j)\)th Vilenkin–Fourier coefficient of \(f\) (\(\mu\) is the product measure \(\mu \times \mu\)).

The norm (or quasinorm) of the space \(L_p(G_m \times G_m)\) is defined by
\[ \|f\|_p := \left( \int_{G_m \times G_m} |f(x^1, x^2)|^p \mu(x^1, x^2) \right)^{1/p} \quad (0 < p < \infty). \]

The space weak-\(L_p(G_m \times G_m)\) consists of all measurable functions \(f\) for which
\[ \|f\|_{\text{weak-}L_p(G_m \times G_m)} := \sup_{\lambda > 0} \lambda \mu(|f| > \lambda)^{1/p} < \infty. \]

Let
\[ I_{n,k}(x^1, x^2) := I_n(x^1) \times I_k(x^2). \]
The \(\sigma\)-algebra generated by the rectangles \(\{I_{n,k}(x^1, x^2) : (x^1, x^2) \in G_m \times G_m\}\) will be denoted by \(\mathcal{F}_{n,k} (n, k \in \mathbb{N})\).

Denote by \(f = (f^{(n,k)} : n, k \in \mathbb{N})\) a martingale with respect to \((\mathcal{F}_{n,k} : n, k \in \mathbb{N})\) (for details see, e.g., [9, 13]). The maximal function and the diagonal maximal function of a martingale \(f\) are defined by
\[ f^* = \sup_{n,k \in \mathbb{N}} |f^{(n,k)}|, \quad f^\square = \sup_{n \in \mathbb{N}} |f^{(n,n)}|, \]
respectively. In case \(f \in L_1(G_m \times G_m)\), the maximal functions are also given by
\[ f^*(x^1, x^2) = \sup_{n,k \in \mathbb{N}} \frac{1}{\mu(I_{n,k}(x^1, x^2))} \left| \int_{I_{n,k}(x^1, x^2)} f(u^1, u^2) \mu(u^1, u^2) \right|, \]
\[ f^\square(x^1, x^2) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_{n,n}(x^1, x^2))} \left| \int_{I_{n,n}(x^1, x^2)} f(u^1, u^2) \mu(u^1, u^2) \right|. \]
for \((x^1, x^2) \in G_m \times G_m\).

The Hardy martingale spaces \(H_p(G_m \times G_m)\) and \(H_p^\square(G_m \times G_m)\) \((0 < p < \infty)\) consist of all martingales for which
\[ \|f\|_{H_p} := \|f^*\|_p < \infty \quad \text{and} \quad \|f\|_{H_p^\square} := \|f^\square\|_p < \infty, \]
respectively.

If \(f \in L_1(G_m \times G_m)\) then it is easy to show that the sequence \((S_{M_n,M_k}(f) : n, k \in \mathbb{N})\) is a martingale. If \(f\) is a martingale, that is, \(f = (f^{(n,k)} : n, k \in \mathbb{N})\), then the Vilenkin–Fourier coefficients must be defined in a slightly different
way:
\[
\hat{f}(i, j) = \lim_{k,l \to \infty} \int_{G_m \times G_m} f(k,l)(x^1, x^2) \overline{\psi}_i(x^1) \overline{\psi}_j(x^2) \mu(x^1, x^2).
\]

The Vilenkin–Fourier coefficients of \( f \in L_1(G_m \times G_m) \) are the same as those of the martingale \((S_{M_n, M_k}(f) : n, k \in \mathbb{N}) \) obtained from \( f \).

For \( n, k \in \mathbb{N}_+ \) and a martingale \( f \) the Fejér mean of order \((n, k)\) of the double Vilenkin–Fourier series of \( f \) is given by
\[
\sigma_{n,k}(f; x^1, x^2) = \frac{1}{nk} \sum_{i=0}^{n-1} \sum_{j=0}^{k-1} S_{i,j}(f; x^1, x^2).
\]

For a martingale \( f \) the restricted and unrestricted maximal operators of the Fejér means are defined by
\[
\sigma^*_\lambda f(x^1, x^2) = \sup_{1/M_\lambda \leq n/k \leq M_\lambda} |\sigma_{n,k}(f; x^1, x^2)|,
\]
\[
\sigma^* f(x^1, x^2) = \sup_{n,k \in \mathbb{N}} |\sigma_{n,k}(f; x^1, x^2)|.
\]

In the one-dimensional case the weak type inequality
\[
\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1 \quad (\lambda > 0)
\]
can be found in Zygmund [15] for the trigonometric series, in Schipp [5] for Walsh series and in Pál and Simon [4] for bounded Vilenkin series. Again in one dimension, Fujii [2] and Simon [7] verified that \( \sigma^* \) is bounded from \( H_1 \) to \( L_1 \). Weisz [10, 12] generalized this by proving the boundedness of \( \sigma^* \) from the martingale Hardy space \( H_p \) to \( L_p \) for \( p > 1/2 \). Simon [6] gave a counterexample to show that this does not hold for \( 0 < p < 1/2 \). In the endpoint case \( p = 1/2 \) Weisz [14] proved that \( \sigma^* \) is bounded from \( H_{1/2} \) to weak-\( L_{1/2} \). By interpolation it follows that \( \sigma^* \) is not bounded from \( H_p \) to weak-\( L_p \) for any \( 0 < p < 1/2 \). It is an open question whether \( \sigma^* \) is bounded from \( H_{1/2} \) to \( L_{1/2} \) or not. (We think the answer is no.)

For the two-dimensional Vilenkin–Fourier series Weisz [11] proved the following results:

**THEOREM A** (Weisz [11]). Let \( p > 1/2 \). Then the maximal operator \( \sigma^*_\lambda \) is bounded from \( H_p \square \) to \( L_p \).

**THEOREM B** (Weisz [11]). Let \( p > 1/2 \). Then the maximal operator \( \sigma^* \) is bounded from \( H_p \) to \( L_p \).

The main aim of this paper is to prove that for any bounded Vilenkin system the maximal operator \( \sigma^* \) (resp. \( \sigma^*_\lambda \)) is not bounded from \( H_{1/2} \) (resp. \( H_{1/2} \square \)) to weak-\( L_{1/2} \). Moreover, we prove that the following is true.
Theorem 1. For any bounded Vilenkin system the maximal operator $\sigma_0^*$ is not bounded from $H_{1/2}$ to weak-$L_{1/2}$.

Thus, as regards boundedness of $\sigma^*$ and $\sigma_\lambda^*$, the case of double Vilenkin–Fourier series differs from that of one-dimensional Vilenkin–Fourier series.

By Theorem 1 and interpolation it follows that $\sigma_0^*$ is not bounded from $H_p$ to weak-$L_p$ for any $0 < p < 1/2$. In particular, in Theorems A and B the assumption $p > 1/2$ is essential. On the other hand, it would be interesting to find a decent space to replace weak-$L_{1/2}$ in order to have the relevant boundedness. However, this question does not seem to be easy.

The Fejér kernel of order $n$ of the Vilenkin–Fourier series is defined by

$$K_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x).$$

Set

$$K_{s,t}(x) := \sum_{j=s}^{s+t-1} D_k(x).$$

In order to prove the theorem we need the following lemmas.

**Lemma 1** ([3]). Suppose that $s, t, n \in \mathbb{N}$ and $x \in I_t \setminus I_{t+1}$. If $t \leq s \leq |n|$, then

$$K_{n^{s+1},s}(x) = \begin{cases} M_t M_s \psi_n(x) \frac{1}{1 - r_t(x)} & \text{if } x - x_t e_t \in I_s, \\ 0 & \text{otherwise}. \end{cases}$$

**Lemma 2.** Let $2 < A \in \mathbb{N}_+$, $k \leq s < A$ and $n_A^* := M_{2A} + M_{2A-2} + \cdots + M_2 + M_0$. Then

$$n_{A-1}^*|K_{n_{A-1}}^*(x)| \geq M_{2k} M_{2s}/4$$

for $x \in I_{2A}(0, \ldots, 0, x_{2k} \neq 0, 0, \ldots, 0, x_{2s} \neq 0, x_{2s+1}, \ldots, x_{2A-1})$, $k = 0, 1, \ldots, A - 3$, $s = k + 2, k + 3, \ldots, A - 1$.

**Proof.** Let $n \in \mathbb{N}_+$. It is known [1] that

$$D_n(x) = \psi_n(x) \left( \sum_{j=0}^{\infty} D_{M_j}(x) \sum_{u=m_j-n_j}^{m_j-1} r_u(x) \right),$$

thus

$$|D_n(x)| \leq \sum_{j=0}^{\infty} n_j D_{M_j}(x).$$

Since for $x \in I_l \setminus I_{l+1}$,

$$\sum_{j=0}^{\infty} n_j D_{M_j}(x) = \sum_{j=0}^{l} n_j M_j \leq m_l M_l = M_{l+1},$$
if \( s \leq l \) we obtain
\[
|K_{n^{s+1},M_s}(x)| = \left| \sum_{u=n^{s+1}}^{n^{s+1}+M_s-1} D_u(x) \right| \leq M_l+1M_s.
\]
From Lemma 1 we see that
\[
K_{n^{2l+1},M_{2l}}(x) = 0 \quad \text{for } l = s + 1, s + 2, \ldots, A - 1.
\]
If \( l < s \leq |n|, x \in I_l \setminus I_{l+1} \) and \( x - x_l e_l \in I_s \), then also from Lemma 1 we get
\[
1 \leq \frac{|K_{n^{s+1},M_s}(x)|}{M_lM_s} = \frac{1}{2|\sin(\pi x_l/m_l)|} \leq \frac{m_l}{\pi}.
\]
Using these facts, the equality from [3, p. 16]
\[
nK_n = \sum_{h=0}^{|n|} \sum_{j=0}^{n_h-1} K_{n^{h+1}+jM_h,M_h},
\]
and
\[
(n^*_A) = \begin{cases} 1 & \text{if } 2 | h, h < 2A, \\ 0 & \text{otherwise} \end{cases}
\]
we estimate
\[
n^*_A |K_{n^*_A}(x)| = \left| \sum_{h=0}^{2A-2} \sum_{j=0}^{\lfloor 2A/2h \rfloor} K_{(n^*_A)^{h+1},M_h}(x) \right| = \left| \sum_{l=0}^{s} K_{(n^*_A)^{2l+1},M_{2l}}(x) \right|
\]
\[
\geq |K_{(n^*_A)^{2s+2},M_{2s}}(x)| - \left| \sum_{l=0}^{s-1} K_{(n^*_A)^{2l+2},M_{2l}}(x) \right|
\]
\[
\geq M_{2s}M_{2k} - \sum_{l=0}^{s-1} |K_{(n^*_A)^{2l+2},M_{2l}}(x)| \geq M_{2s}M_{2k} - \sum_{l=0}^{s-1} M_{2l+1}M_{2k}.
\]
It is easy to see that
\[
\sum_{l=0}^{s-1} M_{2l+1} = \sum_{l=0}^{s-2} M_{2l+1} + M_{2s-1} \leq M_{2s-2} + M_{2s-1}
\]
\[
= \frac{M_{2s}}{m_{2s-1}m_{2s-2}} + \frac{M_{2s}}{m_{2s-1}} \leq \frac{3M_{2s}}{4}.
\]
Summarizing,
\[ n_{A-1}^* |K_{n_{A-1}^*}(x)| \geq \frac{M_{2A}M_{2k}}{4}. \]

**Proof of Theorem 1.** Let \( A \in \mathbb{N}_+ \) and
\[
f_A(x^1, x^2) := (D_{M_{2A+1}}(x^1) - D_{M_{2A}}(x^1))(D_{M_{2A+1}}(x^2) - D_{M_{2A}}(x^2)).
\]

It is evident that
\[
\hat{f}_A(i, k) = \begin{cases} 
1 & \text{if } i, k = M_{2A}, \ldots, M_{2A+1} - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Then we can write
\[
(2) \quad S_{i,j}(f_A; x^1, x^2) = \begin{cases} 
(D_i(x^1) - D_{M_{2A}}(x^1))(D_j(x^2) - D_{M_{2A}}(x^2)), & \text{if } i, j = M_{2A} + 1, \ldots, M_{2A+1} - 1, \\
f_A(x^1, x^2) & \text{if } i, j \geq M_{2A+1}, \\
0 & \text{otherwise}.
\end{cases}
\]

Since
\[
f_A^*(x^1, x^2) = \sup_{n, k \in \mathbb{N}} |S_{M_n, M_k}(f_A; x^1, x^2)| = |f_A(x^1, x^2)|,
\]
from (1) we get
\[
(3) \quad \|f_A\|_{H_p} = \|f_A^*\|_p = \|D_{M_{2A+1}} - D_{M_{2A}}\|_p^2
\]
\[
= \left( \left( \int_{I_{2A}}^{I_{2A+1}} M_{2A}^p + \int_{I_{2A+1}} M_{2A+1}^p - M_{2A}^p \right)^{1/p} \right)^2
\]
\[
= \left( \left( \frac{m_{2A} - 1}{M_{2A+1}} M_{2A}^p + \frac{(m_{2A} - 1)^p M_{2A}}{M_{2A+1}} \right)^{1/p} \right)^2
\]
\[
\leq 2^{2/p} m_{2A}^2 M_{2A}^{2-2/p} \leq cM_{2A}^{2-2/p}.
\]

Since
\[
D_{k+M_{2A}} - D_{M_{2A}} = \psi_{M_{2A}} D_k, \quad k = 1, \ldots, M_{2A},
\]
from (2) we obtain
\[
(4) \quad \sigma_0^* f_A(x^1, x^2) = \sup_{n \in \mathbb{N}} |\sigma_{n,n}(f_A; x^1, x^2)| \geq |\sigma_{n_A^*, n_A^*}(f_A; x^1, x^2)|
\]
\[
= \frac{1}{(n_{A}^*)^2} \sum_{i=0}^{n_{A}^*-1} \sum_{j=0}^{n_{A}^*-1} S_{i,j}(f_A; x^1, x^2)\]
\[
\begin{align*}
&= \frac{1}{(n_A^*)^2} \left| \sum_{i=M_2+1}^{n_A^*-1} \sum_{j=M_2+1}^{n_A^*-1} (D_i(x^1) - D_{M_2A}(x^1))(D_j(x^2) - D_{M_2A}(x^2)) \right| \\
&= \frac{1}{(n_A^*)^2} \left| \sum_{i=1}^{n_A^*-1-1} \sum_{j=1}^{n_A^*-1-1} (D_{i+M_2A}(x^1) - D_{M_2A}(x^1))(D_{j+M_2A}(x^2) - D_{M_2A}(x^2)) \right| \\
&= \frac{(n_A^*-1)^2}{(n_A^*)^2} |K_{n_A^*-1}(x^1)| |K_{n_A^*-1}(x^2)|.
\end{align*}
\]

Let \(q := \sup \{m_i : i \in \mathbb{N} \} \). For every \(l = 1, \ldots, \left[\frac{1}{4} \log q \sqrt{A} \right] - 1 \) (\(A \) is supposed to be large enough) let \(k_l^1 \) and \(k_l^2 \) be the smallest natural numbers for which

\[
M_2A\sqrt{A} \frac{1}{q^{4l}} \leq M_{2k_l^1}^2 < M_2A\sqrt{A} \frac{1}{q^{4l-4}},
\]

\[
M_2A\sqrt{A} q^{4l} \leq M_{2k_l^2}^2 < M_2A\sqrt{A} q^{4l+4}.
\]

Define

\[
I_{2A}^{k_l^1,k_l^2}(x) := I_{2A}(0, \ldots, 0, x_{2k} \neq 0, 0, \ldots, 0, x_{2s} \neq 0, x_{2s+1}, \ldots, x_{2A-1})
\]

and let

\[
(x^1, x^2) \in I_{2A}^{k_l^1,k_l^1+1}(x^1) \times I_{2A}^{k_l^2,k_l^2+1}(x^2).
\]

Then from Lemma 2 and (4) we obtain

\[
\sigma_0^* f_A(x^1, x^2) \geq c \frac{M_{2k_l^1}^2 M_{2k_l^2}^2}{M_{2A}^2} \geq cM_2A\sqrt{A} \frac{1}{q^{4l-4}} \frac{M_2A\sqrt{A} q^{4l}}{M_{2A}^2} \geq cA.
\]

On the other hand,

\[
\mu\{ (x^1, x^2) \in G_m \times G_m : |\sigma_0^* f_A(x^1, x^2)| \geq cA \}
\]

\[
\geq c \sum_{l=1}^{\left[\frac{1}{4} \log q \sqrt{A} \right]} \sum_x \mu(I_{2A}^{k_l^1,k_l^1+1}(x^1) \times I_{2A}^{k_l^2,k_l^2+1}(x^2))
\]

\[
\left( \sum_x := \sum_{x_{2k_l^1+3} = 0}^{m_{2k_l^1+3} - 1} \cdots \sum_{x_{2A-1} = 0}^{m_{2A-1} - 1} \sum_{x_{2k_l^2+3} = 0}^{m_{2k_l^2+3} - 1} \cdots \sum_{x_{2A-1} = 0}^{m_{2A-1} - 1} \right)
\]

\[
\geq c \sum_{l=1}^{\left[\frac{1}{4} \log q \sqrt{A} \right]} \frac{m_{2k_l^1+3} \cdots m_{2A-1} m_{2k_l^2+3} \cdots m_{2A-1}}{M_{2A}^2}
\]
\[
\frac{1}{M_{2k_l+2}^2} r \geq \sum_{l=1}^{\left\lfloor \frac{1}{4} \log \sqrt{A} \right\rfloor} \frac{1}{M_{2k_l}^2 M_{2k_l+2}^2}
\]

\[
\frac{1}{(M_{2A} \sqrt{A} q^{-4l+1})^{1/2} (M_{2A} \sqrt{A} q^{4l+4})^{1/2}} \geq \frac{c \log A}{M_{2A} \sqrt{A}}.
\]

Combining this with (3) we obtain

\[
\frac{cA}{\|f_A\|_{H^{1/2}}} \geq \frac{cA \log^2 A}{M_{2A}^2 A} \to \infty \quad \text{as } A \to \infty.
\]

The Theorem is proved.

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