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ON THE FINITENESS OF THE FUNDAMENTAL GROUP OF A COMPACT SHRINKING RICCI SOLITON

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Abstract. Myers's classical theorem says that a compact Riemannian manifold with positive Ricci curvature has finite fundamental group. Using Ambrose's compactness criterion or J. Lott's results, M. Fernández-López and E. García-Río showed that the finiteness of the fundamental group remains valid for a compact shrinking Ricci soliton. We give a self-contained proof of this fact by estimating the lengths of shortest geodesic loops in each homotopy class.

Myers's classical theorem says that a compact Riemannian manifold with positive Ricci curvature has finite fundamental group. Fernández-López and García-Río [3] provided two methods to generalize this result to the Ricci soliton case: one used Ambrose's criterion for the compactness of a manifold under some Ricci curvature conditions [1], and the other used Lott's results [4]. On the other hand, Derdziński [2] proved the finiteness of the first homology group of a compact shrinking Ricci soliton by estimating the lengths of closed geodesics. Here we optimize Derdziński's method to give another proof of the finiteness of the fundamental group of a compact shrinking Ricci soliton.

Recall that a Riemannian manifold (M, g) is a *shrinking Ricci soliton* if there exist $c > 0$ and a C^∞ vector field X such that

$$\mathrm{Ric} + \mathcal{L}_X g = cg, \tag{1}$$

where Ric is the Ricci tensor of g and $\mathcal{L}_X$ is the Lie derivative in direction X. Fernández-López and García-Río proved

THEOREM. *Any compact shrinking Ricci soliton has finite fundamental group.*

In the following, we give a different proof of this theorem.

Proof. Fix a base point $p \in M$. It is known that in each homotopy class $\alpha \in \pi_1(M, p)$, there is a shortest geodesic loop γ representing α, which is smooth except at p. We assume that all geodesic loops are of unit velocities,

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and denote by T their tangent vector fields. By the second variation formula, for any piecewise smooth vector field V along γ with $V(p)=0$, we have

$$\int_\gamma \langle R_{TV}V,T\rangle - \int_\gamma |\nabla_T V|^2 \le 0. \tag{2}$$

Now fix one such shortest geodesic loop and let $\{e_1=T,e_2,\dots,e_n\}$ be an orthonormal basis of T_pM. Translate the basis along γ to get parallel vector fields $\{e_i(t)\}_{i=1}^n$. Denote by $L=L(\gamma)$ the length of γ. Define $f:[0,\infty)\to[0,1]$ and $g:[0,L]\to[0,1]$, with any chosen $r\ge 2/L$, by

$$f(t)=\begin{cases} rt, & 0\le t\le r^{-1},\\ 1, & r^{-1}\le t,\end{cases}\qquad g(t)=\begin{cases} r(L-t), & L-r^{-1}\le t\le L,\\ 1, & t\le L-r^{-1}.\end{cases}$$

Define vector fields V_i $(i=2,\dots,n)$ along γ by

$$V_i(t)=\begin{cases} fe_i, & 0\le t\le r^{-1},\\ e_i, & r^{-1}\le t\le L-r^{-1},\\ ge_i, & L-r^{-1}\le t\le L.\end{cases}$$

Substituting each V_i for V in (2) and summing the resulting inequalities over $i=2,\dots,n$, one has

$$\begin{aligned}\int_0^{r^{-1}} f^2\operatorname{Ric}(T,T)+\int_{r^{-1}}^{L-r^{-1}}\operatorname{Ric}(T,T)+\int_{L-r^{-1}}^{L} g^2\operatorname{Ric}(T,T)&\\ \le \int_0^{r^{-1}}(n-1)|f'|^2+\int_{L-r^{-1}}^{L}(n-1)|g'|^2.&\end{aligned}\tag{3}$$

By (1), the left side of (3) is

$$\begin{aligned}\int_0^L\operatorname{Ric}(T,T)+\int_0^{r^{-1}}(f^2-1)\operatorname{Ric}(T,T)+\int_{L-r^{-1}}^{L}(g^2-1)\operatorname{Ric}(T,T)&\\ \ge cL-Dr^{-1}-\int_0^L(\mathcal{L}_Xg)(T,T)&\\ = cL-Dr^{-1}-2\langle X,T\rangle|_{t=0}^{t=L}\ge cL-Dr^{-1}-D,&\end{aligned}$$

where $D>0$ is a constant satisfying $\sup_{x\in M}|\operatorname{Ric}|(x)\le D/2$ and $|X|(p)\le D/4$. Noting that the right side of (3) equals $2(n-1)r$, one has

$$cL-Dr^{-1}-D\le 2(n-1)r. \tag{4}$$

Suppose $E=rL$ is sufficiently large, say $E\ge 2D/c$; then (4) implies that $(cL-DL/E-D)L\le 2(n-1)E$, and consequently one gets a uniform upper bound for L by some constant depending on n,c and D. Now the finiteness of $\pi_1(M,p)$ follows from the classical Arzelà–Ascoli compactness theorem. ∎

Remark. From the above proof, even more information can be obtained. First note that the condition $\mathrm{Ric} + \mathcal{L}_X > 0$ is enough to prove the finiteness of $\pi_1(M, p)$, if M is supposed to be compact. If we do not know whether M is compact or not, two alternative subcases are remarkable, given $\mathrm{Ric} + \mathcal{L}_X \geq cg$ with $c > 0$. One is that if Ric is bounded above, say $\mathrm{Ric} \leq Cg$ over M for another constant $C \in \mathbb{R}$, then $\pi_1(M, p)$ remains finite. Note that there is no restriction on the vector field X, which can be checked along the above proof. Hence any complete shrinking Ricci soliton with Ricci curvature bounded above also has finite fundamental group. The other is that if we suppose X is bounded over M, but without restrictions on Ric, then M is compact. For this aspect, see [3].

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REFERENCES

[1] W. Ambrose, *A theorem of Myers*, Duke Math. J. 24 (1957), 345–348.

[2] A. Derdziński, *A Myers-type theorem and compact Ricci solitons*, Proc. Amer. Math. Soc. 134 (2006), 3645–3648.

[3] M. Fernández-López and E. García-Río, *A remark on compact Ricci solitons*, preprint.

[4] J. Lott, *Some geometric properties of the Bakry–Émery–Ricci tensor*, Comment. Math. Helv. 78 (2003), 865–883.

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