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ON THE NEUMANN PROBLEM WITH L^1 DATA

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Abstract. We investigate the solvability of the linear Neumann problem (1.1) with L^1 data. The results are applied to obtain existence theorems for a semilinear Neumann problem.

1. Introduction. The aim of this paper is twofold. In the first part of this paper we are concerned with the existence of solutions of the linear Neumann problem

$$\begin{cases} -\Delta u = \lambda u + f(x) & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and ν denotes the outward normal to $\partial\Omega$ and $\lambda \in \mathbb{R}$. We consider the case $f \in L^1(\Omega)$. Additional assumptions will be introduced later. In the second part of this work the existence results for (1.1) with $f \in L^1(\Omega)$ will be used to investigate the solvability of the Neumann problem for semilinear equations.

It is well known that problem (1.1) has a unique solution in $W^{1,2}(\Omega)$ if $f \in L^2(\Omega)$ whenever λ is not an eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We denote by λ_i the sequence of eigenvalues of (1.2). It is known that $\lambda_1 = 0$ and the corresponding eigenfunctions are constant. If $\lambda = \lambda_i$ then problem (1.2) has a solution provided $f \in E_i^\perp$, where E_i is the eigenspace corresponding to λ_i. In this case (1.1) has a family of solutions that can be represented as

$$u = \overline{u} + \varphi, \tag{1.3}$$

where $\overline{u}$ is a uniquely determined function in $E_i^\perp$ and $\varphi \in E_i$. From now on, by a solution of (1.1) with $\lambda = \lambda_i$ and $f \in E_i^\perp$ we mean the unique element $\overline{u} \in E_i^\perp$ in the representation (1.3).

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We recall that $W^{1,p}(\Omega)$, where $p \geq 1$, is the Sobolev space equipped with the norm

$$\|u\|_{W^{1,p}}^p = \int_\Omega (|\nabla u|^p + |u|^p)\, dx.$$

Throughout this paper, in a given Banach space X, we denote strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^p(\Omega)$ are denoted by $\|\cdot\|_{L^p}$.

Finally, we recall that a C^1 functional $\Phi:\ X \to \mathbb{R}$ on a Banach space X satisfies the *Palais–Smale condition* at a level c if each sequence $\{x_n\} \subset X$ such that $\Phi(x_n) \to c$ and $\Phi'(x_n) \to 0$ in X^* is relatively compact in X.

If $f \in L^1(\Omega)$ then solutions to problem (1.1) will be obtained by approximating f by $L^2(\Omega)$ functions. For this we need some estimates in an appropriate Sobolev norm of a solution by an L^1 norm of a data in L^2. These estimates will be discussed in Section 2. The results of Section 2 are used in Section 3 to obtain the existence theorem (see Theorem 3.1) for semilinear elliptic equations under the Landesman–Lazer conditions on the nonlinear term. Section 4 contains the extension of Theorem 3.1 to the semilinear Neumann problem involving the Hardy potential (see Theorem 4.4). In the final Section 5 we discuss the existence of global minimizers for a semilinear Neumann problem with a nonhomogeneous term in $L^1(\Omega)$. We point out here that the Dirichlet problem with L^1 data has been considered in [6].

2. Estimates of solutions of (1.1) in $W^{1,p}$ norm, $1 \leq p < 2$. If $f \in L^2(\Omega)$ and $\lambda < 0$ then for a solution $u \in W^{1,2}(\Omega)$ of (1.1) we have the estimate

$$\|u\|_{L^1} \leq \frac{1}{|\lambda|}\, \|f\|_{L^1}. \tag{2.1}$$

This easily follows by testing (1.1) with $u/(\varepsilon + u^2)^{1/2}$. We then have

$$\varepsilon \int_\Omega \frac{|\nabla u|^2}{(\varepsilon + u^2)^{3/2}}\, dx - \lambda \int_\Omega \frac{u^2}{(\varepsilon + u^2)^{1/2}}\, dx = \int_\Omega f\, \frac{u}{(\varepsilon + u^2)^{1/2}}\, dx \leq \int_\Omega |f|\, dx.$$

Letting $\varepsilon \to 0$ yields (2.1).

First we derive estimates of solutions of (1.1) in the case $\lambda < 0$.

Lemma 2.1. *Suppose that $N > 2$, $1 < m < 2N/(N+2)$ and $\lambda < 0$. Let $q = m^* = Nm/(N-m)$. If u is a solution of* (1.1) *with $f \in L^2(\Omega)$, then*

$$\begin{aligned}\int_\Omega |u|^{q^*}\, dx &\leq C_1 \Big(\int_\Omega (|\nabla u|^q + |u|^q)\, dx\Big)^{q^*/q} \\ &\leq C_2 \|f\|_{L^m(\Omega)}^{q^*/2} \Big(\int_\Omega |u|^{q^*}\, dx\Big)^{(1-r)/2} \Big(\int_\Omega (1+u^2)^{q^*/2}\, dx\Big)^{r/2},\end{aligned}$$

where $q^* = Nq/(N-q)$, $r = N(2-q)/(N-q)$ *and the constants* $C_1, C_2 > 0$ *are independent of* u *and* f.

Proof. Put $\varphi(x) = u/(1+u^2)^{r/2}$, $0 < r < 1$, where r will be determined later. Testing (1.1) with φ we obtain

$$(2.2)\qquad (1-r)\int_\Omega \frac{|\nabla u|^2}{(1+u^2)^{r/2}}\,dx + \int_\Omega \frac{|\lambda| u^2}{(1+u^2)^{r/2}}\,dx \le \int_\Omega \frac{|f|\,|u|}{(1+u^2)^{r/2}}\,dx \le \|f\|_{L^m(\Omega)}\Big(\int_\Omega |u|^{(1-r)m'}\,dx\Big)^{1/m'},$$

where $m' = m/(m-1)$. In what follows, we denote by C a constant which is independent of u and f and may vary from line to line. By the Sobolev inequality we have, for $1 < q < 2$,

$$(2.3)\qquad \Big(\int_\Omega |u|^{q^*}\,dx\Big)^{q/q^*} \le C\int_\Omega (|\nabla u|^q + |u|^q)\,dx$$
$$= C\int_\Omega \frac{|\nabla u|^q}{(1+u^2)^{rq/4}}(1+u^2)^{rq/4}\,dx + C\int_\Omega \frac{|u|^q}{(1+u^2)^{rq/4}}(1+u^2)^{rq/4}\,dx$$
$$\le C\Big(\int_\Omega \frac{|\nabla u|^2}{(1+u^2)^{r/2}}\,dx\Big)^{q/2}\Big(\int_\Omega (1+u^2)^{rq/2(2-q)}\,dx\Big)^{(2-q)/2}$$
$$+ C\Big(\int_\Omega \frac{u^2}{(1+u^2)^{r/2}}\,dx\Big)^{q/2}\Big(\int_\Omega (1+u^2)^{rq/2(2-q)}\,dx\Big)^{(2-q)/2}.$$

Inserting (2.2) into (2.3) we obtain

$$\Big(\int_\Omega |u|^{q^*}\,dx\Big)^{q/q^*} \le C\int_\Omega (|\nabla u|^q + |u|^q)\,dx \le C\|f\|_{L^m}^{q/2}\Big(\int_\Omega |u|^{(1-r)m'}\,dx\Big)^{q/2m'}\Big(\int_\Omega (1+u^2)^{rq/2(2-q)}\,dx\Big)^{(2-q)/2}.$$

We now choose r so that $rq/(2-q) = q^*$ and $(1-r)m' = q^*$. This yields $r = N(2-q)/(N-q)$ and $q = m^* = Nm/(N-m)$. Since $1 < m < 2N/(N+2)$, we have $0 < r < 1$ and $1 < q < 2$. With this choice of r the above inequality becomes

$$\int_\Omega |u|^{q^*}\,dx \le C\Big(\int_\Omega (|\nabla u|^q + |u|^q)\,dx\Big)^{q^*/q} \le C\|f\|_{L^m}^{q^*/2}\Big(\int_\Omega |u|^{q^*}\,dx\Big)^{q^*/2m'}\Big(\int_\Omega (1+u^2)^{q^*/2}\,dx\Big)^{(2-q)q^*/2q}.$$

Since $q^*/2m' = (1-r)/2$ and $(2-q)q^*/2q = r/2$, the result follows. ∎

Corollary 2.2. *Suppose that $N > 2$, $1 < m < 2N/(N+2)$ and $\lambda < 0$. Let $\{f_n\}$ be a sequence in $L^2(\Omega)$ bounded in $L^m(\Omega)$. For each $n \geq 1$ let u_n be a solution of* (1.1) *with $f = f_n$. Then the sequence $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$, where $q = m^* = Nm/(N-m)$.*

We now consider the case $\lambda = 0$. If $f \in L^2(\Omega)$ and $\int_\Omega f(x)\,dx = 0$, then problem (1.1) has a solution of the form

$$u = \overline{u} + t, \tag{2.4}$$

where $t \in \mathbb{R}$ and $(*)$ $\int_\Omega \overline{u}(x)\,dx = 0$. According to the comments made in the introduction, by a solution of (1.1) we mean the $\overline{u}$ from representation (2.4). Functions u in $W^{1,2}(\Omega)$ satisfying $(*)$ obey the inequality

$$\Big(\int_\Omega |u|^{q^*}\,dx\Big)^{1/q^*} \leq C\Big(\int_\Omega |\nabla u|^q\,dx\Big)^{1/q}, \tag{2.5}$$

where $C > 0$ is a constant independent of u (see [5, p. 66]). Therefore repeating the proof of Lemma 2.1 we obtain

Lemma 2.3. *Suppose that $N > 2$, $1 < m < 2N/(N+2)$ and $\lambda = 0$. Let $q = m^* = Nm/(N-m)$. If u is solution of* (1.1) *with $f \in L^2(\Omega)$, then*

$$\begin{aligned}\int_\Omega |u|^{q^*}\,dx &\leq C_1\Big(\int_\Omega |\nabla u|^q\,dx\Big)^{q^*/q}\\ &\leq C_2\|f\|_{L^m}^{q^*/2}\Big(\int_\Omega |u|^{q^*}\,dx\Big)^{(1-r)/2}\Big(\int_\Omega (1+u^2)^{q^*/2}\,dx\Big)^{r/2},\end{aligned}$$

where $C_1, C_2 > 0$ are constants independent of f and u, and $0 < r < 1$ is the constant from Lemma 2.1.

Corollary 2.4. *Suppose that $N > 2$, $1 < m < 2N/(N+2)$ and $\lambda = 0$. Let $\{f_n\}$ be sequence in $L^2(\Omega)$ bounded in $L^m(\Omega)$. For each $n \geq 1$ let u_n be a solution of* (1.1) *with $f = f_n$. Then the sequence $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$, where $q = m^* = Nm/(N-m)$.*

Lemma 2.5. *Suppose that $1 \leq q < N/(N-1)$, $f \in L^2(\Omega)$ and $\lambda \in \mathbb{R}$. If $u \in W^{1,2}(\Omega)$ is a solution of problem* (1.1) *then*

$$\begin{aligned}\int_\Omega |u|^{q^*}\,dx &\leq C_1\Big(\int_\Omega (|\nabla u|^q + |u|^q)\,dx\Big)^{q^*/q}\\ &\leq C_2\Big(\int_\Omega (1+|u|)^{q^*}\,dx\Big)^{(2-q)q^*/2q}[\|f\|_{L^1}^{q^*/2} + \|u\|_{L^1}^{(2-r)q^*/2} + \|u\|_{L^1}^{q^*/2}],\end{aligned} \tag{2.6}$$

where $r = N(2-q)/(N-q)$ and $C_1, C_2 > 0$ are constants independent of f and u.

Proof. First we establish (2.6) in the case $\lambda < 0$. We assume that $f \geq 0$ on Ω. By the maximum principle $u > 0$ on Ω. As a test function we take $\varphi(x) = (1+u)^{1-r}$, $r > 1$. Since $\varphi(x) \leq 1$ on Ω, we have

$$\int_\Omega \frac{|\nabla u|^2}{(1+u)^r}\, dx \leq \frac{1}{r-1}\Big(|\lambda| \int_\Omega |u|\, dx + \int_\Omega |f|\, dx\Big). \tag{2.7}$$

By the Sobolev inequality we have

$$\begin{aligned}\Big(\int_\Omega |u|^{q^*}\, dx\Big)^{q/q^*} &\leq C \int_\Omega (|\nabla u|^q + |u|^q)\, dx \\ &= C\int_\Omega \frac{|\nabla u|^q}{(1+u)^{rq/2}}\,(1+u)^{rq/2}\, dx + C\int_\Omega \frac{u^q}{(1+u)^{rq/2}}\,(1+u)^{rq/2}\, dx \\ &\leq C\Big(\int_\Omega \frac{|\nabla u|^2}{(1+u)^r}\, dx\Big)^{q/2}\Big(\int_\Omega (1+u)^{rq/(2-q)}\, dx\Big)^{(2-q)/2} \\ &\quad + C\Big(\int_\Omega \frac{u^2}{(1+u)^r}\, dx\Big)^{q/2}\Big(\int_\Omega (1+u)^{rq/(2-q)}\, dx\Big)^{(2-q)/2}.\end{aligned} \tag{2.8}$$

We now choose $rq/(2-q) = q^*$. This yields $r = N(2-q)/(N-q)$. So $r > 1$ if and only if $q < N/(N-1)$. Since $N > 2$, we have $r < 2$. Using (2.7) we rewrite (2.8) as

$$\begin{aligned}\Big(\int_\Omega |u|^{q^*}\, dx\Big)^{q/q^*} &\leq C \int_\Omega (|\nabla u|^q + |u|^q)\, dx \\ &\leq C\Big(\int_\Omega (1+u)^{q^*}\, dx\Big)^{(2-q)/2}\Big(\int_\Omega |u|\, dx + \int_\Omega |f|\, dx\Big)^{q/2} \\ &\quad + C\Big(\int_\Omega (1+u)^{q^*}\, dx\Big)^{(2-q)/2}\Big(\int_\Omega |u|^{2-r}\, dx\Big)^{q/2} \\ &\leq C\Big(\int_\Omega (1+u)^{q^*}\, dx\Big)^{2-q/2}[\|u\|_{L^1}^{q/2} + \|f\|_{L^1}^{q/2} + \|u\|_{L^1}^{q(2-r)/2}].\end{aligned} \tag{2.9}$$

From this we derive estimate (2.6) in the case $f \geq 0$ on Ω.

If $f \leq 0$ on Ω then $u \leq 0$ on Ω and we use as a test function $\varphi(x) = (1-u)^{1-r}$. By similar estimates we arrive at

$$\begin{aligned}\Big(\int_\Omega |u|^{q^*}\, dx\Big)^{q/q^*} &\leq C \int_\Omega (|\nabla u|^q + |u|^q)\, dx \\ &\leq C\Big(\int_\Omega (1-u)^{q^*}\, dx\Big)^{(2-q)/2}[\|u\|_{L^1}^{q/2} + \|f\|_{L^1}^{q/2} + \|u\|_{L^1}^{2(2-r)/2}],\end{aligned} \tag{2.10}$$

which yields (2.6) in this case.

If f changes sign and $\lambda < 0$ we represent a solution u as $u = u_1 - u_2$, where u_1 is a solution of (1.1) with $f = f^+$ and u_2 is a solution of (1.1) with $f = f^-$. Combining (2.9) and (2.10) we obtain estimate (2.6).

If $\lambda \geq 0$, we consider the problem

$$\begin{cases} -\Delta v + v = f + (\lambda+1)u & \text{in } \Omega, \\ \partial v/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

By the uniqueness of a solution we obviously have $v = u$ on Ω. Applying to v estimate (2.6) with f replaced by $f + (\lambda+1)u$ gives the result follows. ∎

REMARK 2.6. If $\lambda < 0$, then we always have $\int_\Omega |u|\,dx \leq (1/|\lambda|)\int_\Omega |f|\,dx$. Therefore in this case inequality (2.6) from Lemma 2.5 takes the form

$$(2.11)\qquad \int_\Omega |u|^{q^*}\,dx \leq C_1\Big(\int_\Omega (|\nabla u|^q + |u|^q)\,dx\Big)^{q^*/q} \leq C_2\Big(\int_\Omega (1+|u|)^{q^*}\,dx\Big)^{(1-q)q^*/2q}[\|f\|_{L^1}^{q^*/2} + \|f\|_{L^1}^{(2-r)q^*/2}].$$

To construct solutions by approximation we need the following result:

PROPOSITION 2.7. *Let $1 \leq q < N/(N-1)$ and $\{f_n\} \subset L^2(\Omega)$ be such that for every $n \in \mathbb{N}$ there exists a solution v_n of problem* (1.1) *with $f = f_n$, where $\lambda \in \mathbb{R}$. Suppose that $f_n \to 0$ in $L^1(\Omega)$ and that $\{v_n\}$ is bounded in $L^1(\Omega)$. Then $v_n \rightharpoonup 0$ in $W^{1,q}(\Omega)$.*

Proof. We follow the argument used in the proof of Lemma 2.3 in [6]. We choose $k \in \mathbb{N}$ such that

$$2k \leq N < 2(k+1).$$

We then have

$$\frac{N}{N-k+1} \leq \frac{2N}{N+2} < \frac{N}{N-k}.$$

For every $j \in \mathbb{N}$ we define a Sobolev exponent

$$p(j)^* = \frac{Np}{N-jp} \quad \text{whenever} \quad jp < N.$$

We put $p(0)^* = p$. We now choose $1 \leq p < N/(N-1)$ so that

$$\frac{2N}{N+2} \leq p(k-1)^* < \frac{N}{N-k}.$$

This inequality is valid if and only if

$$(2.12)\qquad \frac{2N}{N+2k} \leq p < \frac{N}{N-1}.$$

Since $N < 2(k+1)$ yields $2N/(N+2k) < N/(N-1)$, the choice of p satisfying (2.12) is possible. Moreover, if $k > 1$, since $p(k-1)^* < N/(N-k)$, we also have

$$p(k-2)^* < \frac{N}{N-k+1} < \frac{2N}{N+2}.$$

We now put $v_n^{(1)} = v_n$. Then $v_n^{(1)}$ satisfies the equation

$$-\Delta v_n^{(1)} + v_n^{(1)} = \lambda v_n^{(1)} + v_n^{(1)} + f_n \quad \text{in } \Omega. \tag{2.13}$$

Since $\{v_n^{(1)}\}$ is bounded in $L^1(\Omega)$, we deduce from inequality (2.6) in Lemma 2.5 applied to $u = v_n^{(1)}$ and $f = \lambda v_n^{(1)} + v_n^{(1)} + f_n$ that $\{v_n^{(1)}\}$ is bounded in $W^{1,q}(\Omega)$ for every $1 \le q < N/(N-1)$. In particular, the sequence $v_n^{(1)}$ is bounded in $W^{1,p}(\Omega)$ with p chosen above. Therefore, up to a subsequence, $v_n^{(1)} \rightharpoonup v^{(1)}$ in $W^{1,p}(\Omega)$. If $k > 1$ (this holds in the case $N > 2$) we define for every $j = 2, \ldots, k$ a sequence of solutions of the following Neumann problem:

$$\begin{cases} -\Delta v_n^{(j)} + v_n^{(j)} = \lambda v_n^{(j-1)} + v_n^{(j-1)} & \text{in } \Omega, \\ \partial v_n^{(j)}/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

Since $\{v_n^{(1)}\}$ is bounded in $L^p(\Omega)$ with $p < 2N/(N+2)$, we can apply Lemma 2.1 with $u = v_n^{(2)}$, $f_n = \lambda v_n^{(1)} + v_n^{1)}$ and $m = p$ to get the boundedness of $v_n^{(2)}$ in $W^{1,p(1)^*}(\Omega)$ and consequently in $L^{p(1)^*}(\Omega)$. We can now repeat this procedure for the sequence $\{v_n^{(j)}\}$ using the fact that $\{v_n^{(j-1)}\}$ is bounded in $L^{p(j-2)^*}(\Omega)$ and that $p(j-2)^* \le p(k-2)^* < 2N/(N+2)$. In the final step we deduce that $\{v_n^{(k)}\}$ is bounded in $W^{1,p(k-1)^*}(\Omega)$, so by the Sobolev embedding theorem $\{v_n^{(k)}\}$ is bounded in $L^{p(k)^*}(\Omega)$. By the hypothesis on p we have $p(k)^* \ge 2$ and so $\{v_n^{(k)}\}$ is bounded in $L^2(\Omega)$. We now consider the sequence $\{v_n^{(k+1)}\}$ as solutions of the Neumann problem

$$\begin{cases} -\Delta v_n^{(k+1)} + v_n^{(k+1)} = \lambda v_n^{(k)} + v_n^{(k)} & \text{in } \Omega, \\ \partial v_n^{(k+1)}/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Testing this equation with $v_n^{(k+1)}$ we obtain

$$\int_\Omega (|\nabla v_n^{(k+1)}|^2 + (v_n^{(k+1)})^2)\,dx = (\lambda+1)\int_\Omega v_n^{(k)} v_n^{(k+1)}\,dx.$$

Since $\{v_n^{(k)}\}$ is bounded in $L^2(\Omega)$, with the aid of the Hölder inequality we show that $\{v_n^{(k+1)}\}$ is bounded in $W^{1,2}(\Omega)$. Therefore we may assume that $v_n^{(k+1)} \rightharpoonup v^{(k+1)}$ in $W^{1,2}(\Omega)$. Subtracting these equations successively we obtain

$$-\Delta(v_n^{(2)} - v_n^{(1)}) + (v_n^{(2)} - v_n^{(1)}) = f_n,$$
$$-\Delta(v_n^{(3)} - v_n^{(2)}) + (v_n^{(3)} - v_n^{(2)}) = (\lambda+1)(v_n^{(2)} - v_n^{(1)}),$$
$$\dots$$
$$-\Delta(v_n^{(k+1)} - v_n^{(k)}) + (v_n^{(k+1)} - v_n^{(k)}) = (\lambda+1)(v_n^{(k)} - v_n^{(k-1)}).$$

Applying Lemma 2.5 (see Remark 2.6) to the first equation we deduce that $v_n^{(2)} - v_n^{(1)} \to 0$ in $W^{1,p}(\Omega)$ since $f_n \to 0$ in $L^1(\Omega)$ and $\int_\Omega |v_n^{(2)} - v_n^{(1)}|\,dx \le \int_\Omega |f_n|\,dx$. From the second equation and from Lemma 2.1 we deduce that $v_n^{(3)} - v_n^{(2)} \to 0$ in $W^{1,p(1)^*}(\Omega)$. Repeating this argument successively we show that $v_n^{(k+1)} - v_n^{(k)} \to 0$ in $W^{1,p(k-1)^*}(\Omega)$. This shows that every sequence $v_n^{(j)}$ has the same limit in $W^{1,p}(\Omega)$ and that this limit is $v^{(1)}$. Hence $v^{(k+1)} = v^{(1)}$. Letting $n \to \infty$ in (2.13) we obtain

$$\begin{cases} -\Delta v^{(1)} = \lambda v^{(1)} & \text{in } \Omega, \\ \partial v^{(1)}/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

We now distinguish two cases. If λ is not an eigenvalue, then obviously $v^{(1)} = 0$. If $\lambda = \lambda_i$ for some i then $v^{(1)} \in E_i$. Since $v_n^{(1)} \in E_i^\perp$, it is easy to show that also $v^{(1)} \in E_i^\perp$. Hence in this case also $v^{(1)} = 0$. Therefore up to a subsequence, $v_n^{(1)} \rightharpoonup 0$ in $W^{1,p}(\Omega)$. Since $\{v_n^{(1)}\}$ is also bounded in $W^{1,q}(\Omega)$ for $1 \le q < N/(N-1)$, it is easy to show that up to a subsequence $v_n^{(1)} \rightharpoonup 0$ in $W^{1,q}(\Omega)$ for every $1 \le q < N/(N-1)$. Since every subsequence of v_n has the same limit, namely 0, we see that the sequence v_n converges to 0 weakly in $W^{1,q}(\Omega)$ for every $1 \le q < N/(N-1)$. ∎

As an application of Proposition 2.7 we establish the following theorem.

THEOREM 2.8. *Suppose that $f \in L^1(\Omega)$ and $1 \le q < N/(N-1)$.*

(i) *If $\lambda \ne \lambda_i$ for every i, then* (1.1) *has a unique solution $u \in W^{1,q}(\Omega)$* (*obtained by approximation*).

(ii) *If $\lambda = \lambda_i$ for some i and $\int_\Omega f\varphi\,dx = 0$ for every $\varphi \in E_i$, then* (1.1) *has a family of solutions in $W^{1,q}(\Omega)$ having a representation $u = \overline{u} + \varphi$, where $\overline{u}$ is uniquely determined by approximation and $\varphi \in E_i$. Moreover $\int_\Omega \overline{u}\varphi\,dx = 0$ for every $\varphi \in E_i$.*

Proof. We follow the argument used in the proof of Theorem 1.1 in [6]. We need the following statement. Suppose that $f \in L^1(\Omega)$ and $\int_\Omega f\varphi_i^j\,dx = 0$ for some i and every $j = 1,\dots,k$, where $\varphi_i^1,\dots,\varphi_i^k$ form a basis of E_i. Then there exists a sequence $\{f_n\}$ of functions in $L^2(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and $(*)$ $\int_\Omega f_n\varphi_i^j\,dx = 0$ for $j = 1,\dots,k$.

To prove our theorem we take any sequence $\{f_n\} \subset L^2(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$. If $\lambda = \lambda_i$, $\{f_n\}$ is chosen so that relation $(*)$ is satisfied. Then for every $n \in \mathbb{N}$ there exists a solution $u_n \in W^{1,2}(\Omega)$ of problem (1.1) with $f = f_n$. In the next step we show that $\{u_n\}$ is bounded in $L^1(\Omega)$. This can be established arguing by contradiction and using Proposition 2.7. It then follows from Lemma 2.5 that $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$. Thus, up to a subsequence, $u_n \rightharpoonup u$ in $W^{1,q}(\Omega)$, where u is a solution of (1.1). For details we refer to the paper [6]. ∎

3. Neumann problem with Landesman–Lazer conditions. We consider the semilinear Neumann problem

$$\begin{cases} -\Delta u + g(x,u) = f(x) & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We assume that

(G) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$\lim_{s\to\infty} g(x,s) = g^+(x) \quad \text{and} \quad \lim_{s\to-\infty} g(x,s) = g^-(x)$$

exist, $g^+, g^- \in L^\infty(\Omega)$ and $g^-(x) \le g(x,s) \le g^+(x)$ a.e. on $\Omega \times \mathbb{R}$.

THEOREM 3.1. *Let $f \in L^1(\Omega)$ and assume that*

$$\int_\Omega g^-(x)\,dx < \int_\Omega f(x)\,dx < \int_\Omega g^+(x)\,dx.$$

Then problem (3.1) *has at least one solution belonging to $W^{1,q}(\Omega)$ for every $1 \le q < N/(N-1)$.*

Proof. First we assume that $f \in L^2(\Omega)$. In this case the result is well known and it has been proved using degree theory. We offer a different proof inspired by the paper [3]. The method that we shall use will also be applied in the case where $f \in L^1(\Omega)$. For every $n \in \mathbb{N}$ we consider the semilinear problem

$$\begin{cases} -\Delta u + \frac{1}{n}u = -g(x,u) + f(x) & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Since g is a bounded function, with the aid of the Schauder fixed point theorem we obtain the existence of a solution $u_n \in W^{1,2}(\Omega)$.

We now show that the sequence $\{u_n\}$ is bounded in $L^2(\Omega)$. If $\|u_n\|_{L^2} \to \infty$, then we put $v_n = u_n/\|u_n\|_{L^2}$. It is clear that

$$\int_\Omega |\nabla v_n|^2\,dx + \frac{1}{n} = \int_\Omega f\,\frac{v_n}{\|u_n\|_{L^2}}\,dx - \int_\Omega \frac{g(x,u_n)v_n}{\|u_n\|_{L^2}}\,dx.$$

Since the right-hand side converges to 0 we deduce that $\lim_{n\to 0}\int_\Omega |\nabla v_n|^2\,dx = 0$. By the Sobolev embedding theorem $v_n \to v$ in $L^2(\Omega)$. Obviously v is a nonzero constant function. Assume $v = t > 0$. Then $u_n = v_n\|u_n\|_{L^2} \to \infty$

a.e. on Ω. We now observe that

$$\frac{1}{n}\int_\Omega u_n\,dx = \int_\Omega f(x)\,dx - \int_\Omega g(x, v_n\|u_n\|_{L^2})\,dx.$$

Letting $n \to \infty$ we obtain

$$\int_\Omega g^+(x)\,dx \le \int_\Omega f(x)\,dx,$$

which is impossible. If $t < 0$ by the same argument we obtain

$$\int_\Omega f(x)\,dx \le \int_\Omega g^-(x)\,dx,$$

which is again impossible. Thus $\{u_n\}$ is bounded in $L^2(\Omega)$. It is easy to see that $\{u_n\}$ is also bounded in $W^{1,2}(\Omega)$. Therefore we may assume that $u_n \rightharpoonup u$ in $W^{1,2}(\Omega)$ and u is a solution of (3.1).

If $f \in L^1(\Omega)$ we approximate f in $L^1(\Omega)$ by a sequence of functions $\{f_n\} \subset L^2(\Omega)$. By the first part of the proof, problem (3.1) with $f = f_n$ has a solution in $u_n \in W^{1,2}(\Omega)$. Repeating the argument used in the first part of the proof we show that $\{u_n\}$ is bounded in $L^1(\Omega)$. We now apply Lemma 2.5 with f replaced by $f_n - g(x, u_n)$ to deduce that $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$. Therefore, up to a subsequence, $u_n \rightharpoonup u$ in $W^{1,q}(\Omega)$ and u is a solution of (3.1). ■

4. Problems involving Hardy potentials. We commence by extending Lemma 2.1 to the following problem:

$$\begin{cases} -\Delta u - \mu u/|x|^2 = \lambda u + f & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

It is assumed that $0 \in \overline{\Omega}$ and $N \ge 3$. We put

$$0 < \mu^* = \inf\left\{ \int_\Omega (|\nabla u|^2 + u^2)\,dx :\ u \in W^{1,2}(\Omega),\ \int_\Omega \frac{u^2}{|x|^2}\,dx = 1 \right\}.$$

Testing μ^* with constant functions we obtain

$$\mu^* \le \frac{|\Omega|}{\int_\Omega \frac{dx}{|x|^2}}.$$

For every $0 < \mu < \mu^*$ we consider the eigenvalue problem

$$\begin{cases} -\Delta u - \mu u/|x|^2 = \lambda u & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

It is not difficult to show that for every $0 < \mu < \mu^*$ the first eigenvalue $\lambda_1(\mu) < 0$ is strictly negative and the corresponding principal eigenfunction ϕ_μ can be taken positive. This is an easy consequence of an obvious modification of the proof of Proposition 5.1 in [2]. Obviously $\lambda_1(\mu)$ is given by

$$\lambda_1(\mu)=\inf\left\{\int_\Omega\left(|\nabla u|^2-\mu\,\frac{u^2}{|x|^2}\right)dx : u\in W^{1,2}(\Omega),\ \int_\Omega u^2\,dx=1\right\}.$$

It is easy to show that $-1<\lambda_1(\mu)<-\mu|\Omega|^{-1}\int_\Omega|x|^{-2}\,dx$. Testing (4.1) with $\varphi=u/(1+u^2)^{r/2}$, $0<r<1$, we obtain

$$(4.3)\qquad (1-r)\int_\Omega\frac{|\nabla u|^2}{(1+u^2)^{r/2}}\,dx-\lambda\int_\Omega\frac{u^2}{(1+u^2)^{r/2}}\,dx \le \mu\int_\Omega\frac{u^2}{|x|^2(1+u^2)^{r/2}}\,dx+\int_\Omega\frac{uf}{(1+u^2)^{r/2}}\,dx.$$

Let $\psi=u/(1+u^2)^{r/4}$. Since $|\nabla\psi|^2\le|\nabla u|^2/(1+u^2)^{r/2}$, from the definition of μ^* we obtain

$$\mu^*\int_\Omega\frac{u^2}{|x|^2(1+u^2)^{r/2}}\,dx\le\int_\Omega\frac{|\nabla u|^2}{(1+u^2)^{r/2}}\,dx+\int_\Omega\frac{u^2}{(1+u^2)^{r/2}}\,dx.$$

This combined with (4.3) gives

$$((1-r)\mu^*-\mu)\int_\Omega\frac{u^2}{|x|^2(1+u^2)^{r/2}}\,dx-(\lambda+1-r)\int_\Omega\frac{u^2}{(1+u^2)^{r/2}}\,dx \le\int_\Omega\frac{|fu|}{(1+u^2)^{r/2}}\,dx.$$

If $\mu<(1-r)\mu^*$ and $\lambda<-1+r$, then

$$(4.4)\qquad \int_\Omega\frac{u^2}{|x|^2(1+u^2)^{r/2}}\,dx\le\frac{1}{(1-r)\mu^*-\mu}\int_\Omega\frac{|fu|}{(1+u^2)^{r/2}}\,dx.$$

Estimate (4.3) allows us to repeat the proof of Lemma 2.1 and establish the following estimate for solutions of problem (4.1):

LEMMA 4.1. *Suppose that $N>2$, $1<m<2N/(N+2)$, $q=m^*=Nm/(N-m)$ and $r=N(2-q)/(N-q)$. If $0<\mu<\mu^*$, $\lambda<-1+r$ and u is a solution of* (4.1) *with $f\in L^2(\Omega)$, then there exist constants $C_1,C_2>0$ such that*

$$(4.5)\qquad \int_\Omega|u|^{q^*}\,dx\le C_1\left(\int_\Omega(|\nabla u|^q+|u|^q)\,dx\right)^{q^*/q} \le C_2\|f\|_{L^m}^{q^*/2}\left(\int_\Omega|u|^{q^*}\,dx\right)^{(1-r)/2}\left(\int_\Omega(1+u^2)^{q^*/2}\,dx\right)^{r/2}.$$

To proceed further, we need the following form of the maximum principle. If $f\ge0$ on Ω (resp. $f\le0$ on Ω) and $\lambda<\lambda_1(\mu)$, then a solution of (4.1) is non-negative (resp. non-positive) on Ω. Indeed, let $f \geq 0$ and u be a solution in $W^{1,2}(\Omega)$ of (4.1). Testing (4.1) with u^- we obtain

$$-\int_\Omega \left(|\nabla u^-|^2 - \frac{\mu}{|x|^2}(u^-)^2 - \lambda (u^-)^2 \right) dx = \int_\Omega u^- f\, dx. \tag{4.6}$$

Since $\lambda < \lambda_1(\mu)$, we have

$$\int_\Omega \left(|\nabla u^-|^2 - \frac{\mu}{|x|^2}(u^-)^2 - \lambda (u^-)^2 \right) dx \geq 0.$$

This combined with (4.6) yields $u^- = 0$ a.e. on Ω.

Lemma 4.2. *Suppose that $1 < q < N/(N-1)$, $f \in L^2(\Omega)$ and $\lambda \in \mathbb{R}$. If u is a solution of problem (4.1) then*

$$\begin{aligned}\int_\Omega |u|^{q^*}\, dx &\leq C_1 \Big(\int_\Omega (|\nabla u|^q + |u|^q)\, dx \Big)^{q^*/q} \\ &\leq C_2 \Big(\int_\Omega (1+|u|)^{q^*}\, dx \Big)^{(2-q)q^*/2q} (\|f\|_{L^1}^{q^*/2} + \|u\|_{L^1}^{(2-r)q^*/2} + \|u\|_{L^1}^{q^*/2}),\end{aligned} \tag{4.7}$$

where $r = N(2-q)/(N-q)$.

Proof. The proof is similar to that of Lemma 2.5. We only consider the case $\lambda < \lambda_1(\mu)$. If $f \geq 0$ we use $(1+u)^{1-r}$ as a test function to obtain

$$(r-1)\int_\Omega \frac{|\nabla u|^2}{(1+u)^r}\, dx + \mu \int_\Omega \frac{u}{|x|^2}(1+u)^{1-r}\, dx \leq |\lambda| \int_\Omega |u|\, dx + \int_\Omega |f|\, dx, \tag{4.8}$$

since $r > 1$. If $f \leq 0$, then $u \leq 0$ and we use as a test function $(1-u)^{1-r}$, $r > 1$. We then obtain

$$\begin{aligned}(r-1)\int_\Omega \frac{|\nabla u|^2}{(1-u)^r}\, dx - \mu \int_\Omega \frac{u}{|x|^2}(1-u)^{1-r}\, dx \\ \leq \lambda \int_\Omega u(1-u)^{1-r}\, dx + \int_\Omega f(1-u)^{1-r}\, dx.\end{aligned}$$

Since $u \leq 0$ on Ω we deduce from this that

$$\int_\Omega \frac{|\nabla u|^2}{(1-u)^r}\, dx \leq \frac{|\lambda|}{r-1} \int_\Omega |u|\, dx + \int_\Omega |f|\, dx. \tag{4.9}$$

Estimates (4.8) and (4.9) allow us to repeat the argument from the proof of Lemma 2.5. ∎

Proposition 4.3. *Let $1 \leq q < N/(N-1)$ and $\{f_n\} \subset L^2(\Omega)$ be such that for every $n \in \mathbb{N}$ there exists a solution $v_n \in W^{1,2}(\Omega)$ of the problem*

$$\begin{cases} -\Delta v_n - \mu v_n/|x|^2 = \lambda v_n + f_n & \text{in } \Omega, \\ \partial v_n/\partial \nu = 0 & \text{in } \partial\Omega, \end{cases}$$

where $0 < \mu < \mu^*$ *and* $\lambda \in \mathbb{R}$. *Suppose that* $f_n \to 0$ *in* $L^1(\Omega)$ *and that* $\{v_n\}$ *is bounded in* $L^1(\Omega)$. *Then* $v_n \rightharpoonup 0$ *in* $W^{1,q}(\Omega)$.

Proof. The proof is similar to that of Proposition 2.7. We only sketch some details. First we choose k and p as in the proof of Proposition 2.7. We put $v_n^{(1)} = v_n$; then $v_n^{(1)}$ satisfies the equation

$$-\Delta v_n^{(1)} + v_n^{(1)} - \frac{\mu}{|x|^2}\, v_n^{(1)} = \lambda v_n^{(1)} + v_n^{(1)} + f_n.$$

Since $v_n^{(1)}$ is bounded in $L^1(\Omega)$, Lemma 4.2 shows that $v_n^{(1)}$ is bounded in $W^{1,p}(\Omega)$ with p satisfying (2.12). Hence up to a subsequence $v_n^{(1)} \rightharpoonup v^{(1)}$ in $W^{1,p}(\Omega)$. We now define sequences of solutions for every $j = 2, \ldots, k$ of the following Neumann problems:

$$\begin{cases} -\Delta v_n^{(j)} + v_n^{(j)} - \mu v_n^{(j)}/|x|^2 = \lambda v_n^{(j-1)} + v_n^{(j-1)} & \text{in } \Omega, \\ \partial v_n^{(j)}/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\{v_n^{(k)}\}$ is bounded in $L^{p(k)^*}(\Omega)$ with $p(k)^* \geq 2$. Hence $\{v_n^{(k)}\}$ is bounded in $L^2(\Omega)$. In the next step we consider the sequence $\{v_n^{(k+1)}\}$ which is a solution of the Neumann problem:

$$\begin{cases} -\Delta v_n^{(k+1)} + v_n^{(k+1)} - \mu v_n^{(k+1)}/|x|^2 = \lambda v_n^{(k)} + v_n^{(k)} & \text{in } \Omega, \\ \partial v_n^{(k+1)}/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

From this we deduce the estimate

$$\left(1 - \frac{\mu}{\mu^*}\right) \int_\Omega (|\nabla v_n^{(k+1)}|^2 + (v_n^{(k+1)})^2)\, dx \leq (|\lambda| + 1)\|v_n^{(k)}\|_{L^2}\|v_n^{(k+1)}\|_{L^2}.$$

This implies that the sequence $\{v_n^{(k+1)}\}$ is bounded in $W^{1,2}(\Omega)$. Subtracting the equations satisfied by $v_n^{(k)}$ we obtain

$$-\Delta(v_n^{(2)} - v_n^{(1)}) + v_n^{(2)} - v_n^{(1)} - \frac{\mu}{|x|^2}\,(v_n^{(2)} - v_n^{(1)}) = -f_n,$$

$$-\Delta(v_n^{(3)} - v_n^{(2)}) + v_n^{(3)} - v_n^{(2)} - \frac{\mu}{|x|^2}\,(v_n^{(3)} - v_n^{(2)}) = (\lambda+1)(v_n^{(2)} - v_n^{(1)}),$$

$$\cdots$$

$$-\Delta(v_n^{(k+1)} - v_n^{(k)}) + v_n^{(k+1)} - v_n^{(k)} - \frac{\mu}{|x|^2}\,(v_n^{(k+1)} - v_n^{(k)}) = (\lambda+1)(v_n^{(k)} - v_n^{(k-1)}).$$

Lemmas 4.1 and 4.2 show that $v^{(j)} = v^{(1)}$ for $j = 1, \ldots, k+1$. To complete the proof we repeat the final part of the proof of Proposition 2.7. ∎

Theorem 2.8 can be extended in an obvious way to the eigenvalue problem (4.2). This allows us to establish the existence of solutions of a semilinear

Neumann problem

$$(4.10)\qquad \begin{cases} -\Delta u - \mu u/|x|^2 + g(x,u) = \lambda_1(\mu)u + f & \text{in } \Omega,\\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

with the Landesman–Lazer conditions. We denote by ϕ_μ the principal eigenfunction corresponding to $\lambda_1(\mu)$.

THEOREM 4.4. *Let $f \in L^1(\Omega)$ and suppose that the nonlinearity satisfies* (**G**). *If*

$$\int_\Omega g_-(x)\phi_\mu(x)\,dx < \int_\Omega f(x)\phi_\mu(x)\,dx < \int_\Omega g_+(x)\phi_\mu(x)\,dx,$$

then problem (4.10) *has at least one solution belonging to $W^{1,q}(\Omega)$ for every $1 \le q < N/(N-1)$.*

5. Solutions as global minimizers. In this section we investigate global minima corresponding to L^1 data. This situation occurs in the following nonlinear problem:

$$(5.1)\qquad \begin{cases} -\Delta u = f(x,u) + h(x) & \text{in } \Omega,\\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $h \in L^1(\Omega)$, $h \not\equiv 0$. It is assumed that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following condition:

(F) $|f(x,t)| \le C_1(|t|^\alpha + 1)$ for all $t \in \mathbb{R}$ and a.e. on $x \in \Omega$, where $C_1 > 0$ and $1/2 < \alpha < 1$.

Moreover, we assume that

$$(5.2)\qquad \lim_{|t|\to\infty} |t|^{-2\alpha} \int_\Omega F(x,t)\,dx = -\infty,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$.

THEOREM 5.1. *Suppose* (**F**) *holds. If $h \in L^1(\Omega)$ and $h \not\equiv 0$, then problem* (5.1) *admits a solution.*

Proof. First we assume that $h \in L^2(\Omega)$. A solution will be obtained as a global minimizer of the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx - \int_\Omega h(x)u\,dx.$$

We commence by showing that J is a coercive functional on $W^{1,2}(\Omega)$. Since $W^{1,2}(\Omega)$ admits the decomposition $u = \overline{u} + t$, where $t \in \mathbb{R}$ and $\int_\Omega \overline{u}(x)\,dx = 0$, we introduce an equivalent norm on $W^{1,2}(\Omega)$ given by

$$\|u\|^2 = \int_\Omega |\nabla\overline{u}|^2\,dx + t^2.$$

By Sobolev inequalities we have, for $u \in W^{1,2}(\Omega)$,

$$\|\overline{u}\|_{L^2} \le C\|\nabla \overline{u}\|_{L^2} \quad \text{and} \quad \|\overline{u}\|_{L^{\alpha+1}} \le C\|\nabla \overline{u}\|_{L^2}, \tag{5.3}$$

where $C > 0$ is a constant independent of u. We now follow some estimates from the paper [7]. We have

$$\begin{aligned}
\Big|\int_\Omega F(x,u)\,dx - \int_\Omega F(x,t)\,dx\Big| &= \Big|\int_\Omega\int_0^1 f(x,t+s\overline{u})\overline{u}\,ds\,dx\Big| \\
&\le C_1 \int_\Omega\int_0^1 (|t+s\overline{u}|^\alpha + 1)|\overline{u}|\,ds\,dx \le C_1\int_\Omega(|\overline{u}|^\alpha + |t|^\alpha + 1)|\overline{u}|\,dx \\
&\le \int_\Omega\bigg(\frac{|\overline{u}|^2}{8C^2} + 2(CC_1)^2 t^{2\alpha}\bigg)\,dx + C_1\|\overline{u}\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} + C_1\|\overline{u}\|_{L^1(\Omega)} \\
&\le \frac{1}{8}\int_\Omega |\nabla\overline{u}|^2\,dx + C_2(|t|^{2\alpha} + \|\overline{u}\|_{L^1(\Omega)} + \|\overline{u}\|^{\alpha+1}_{L^{\alpha+1}(\Omega)})
\end{aligned}$$

for $u \in W^{1,2}(\Omega)$. Therefore the Young inequality yields

$$\Big|\int_\Omega F(x,u)\,dx - \int_\Omega F(x,t)\,dx\Big| \le \frac{1}{4}\int_\Omega |\nabla\overline{u}|^2\,dx + C_3(|t|^{2\alpha}+1) \tag{5.4}$$

for some constant $C_3 > 0$. Using (5.4) we derive

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \Big(\int_\Omega F(x,u)\,dx - \int_\Omega F(x,\overline{u})\,dx\Big) \\
&\quad - \int_\Omega F(x,\overline{u})\,dx - \int_\Omega hu\,dx \\
&\ge \frac{1}{4}\int_\Omega |\nabla\overline{u}|^2\,dx - \int_\Omega h\overline{u}\,dx - t\int_\Omega h\,dx - C_3|t|^{2\alpha} - \int_\Omega F(x,\overline{u})\,dx - C_3 \\
&\ge \frac{1}{4}\int_\Omega |\nabla u|^2\,dx - C_3 - \Big(\int_\Omega h^2\,dx\Big)^{1/2}\Big(\int_\Omega \overline{u}^2\,dx\Big)^{1/2} \\
&\quad - \frac{|t|^{2\alpha}}{4}\Big(C_3 - |t|^{1-2\alpha}\int_\Omega |h|\,dx + t^{-2\alpha}\int_\Omega F(x,t)\,dx\Big).
\end{aligned}$$

This implies that $\lim_{\|u\|\to\infty} J(u) = \infty$. Hence the functional J is coercive and bounded below. Since J satisfies the $(\mathrm{PS})_c$ condition there exists $v \in W^{1,2}(\Omega)$ such that $J(v) = \inf_{u\in W^{1,2}(\Omega)} J(u)$. The minimizer v is a solution of problem (5.1). If $h \in L^1(\Omega)$, we choose a sequence $\{h_n\} \subset L^2(\Omega)$ such that $h_n \to h$ in $L^1(\Omega)$. Let $u_n \in W^{1,2}(\Omega)$ be a solution of problem (5.1) with $h = h_n$. We show that $\{u_n\}$ is bounded in $L^1(\Omega)$. Arguing by contradiction assume $\|u_n\|_{L^1(\Omega)} \to \infty$ and put $v_n = u_n/\|u_n\|_{L^1(\Omega)}$. Then v_n satisfies

$$-\Delta v_n = g_n \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial v_n}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $g_n = \|u_n\|^{-1}_{L^1(\Omega)}(f(x,u_n) + h_n)$. Since $\alpha < 1$, we see that $g_n \to 0$ in $L^1(\Omega)$. By Proposition 2.7, $v_n \rightharpoonup 0$ in $W^{1,q}(\Omega)$ with $1 \le q < N/(N-1)$. By the Sobolev embedding theorem $v_n \to 0$ up to a subsequence in $L^1(\Omega)$, which is a contradiction. In the final step we apply Lemma 2.5. ∎

REMARK 5.2. Inspection of the proof of Theorem 5.1 shows that if $\int_\Omega h\,dx = 0$ we can assume that $\alpha \in (0,1)$. In this situation the lower bound of J takes the form

$$J(u) \ge \frac{1}{4}\int_\Omega |\nabla \overline{u}|^2\,dx - C_4 - \Big(\int_\Omega h^2\,dx\Big)^{1/2}\Big(\int_\Omega \overline{u}^2\,dx\Big)^{1/2} - |t|^{2\alpha}\Big(C_4 - t^{-2\alpha}\int_\Omega F(x,t)\,dx\Big).$$

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