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SUMS OF RECIPROCALS OF ADDITIVE FUNCTIONS RUNNING OVER SHORT INTERVALS

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Abstract. Letting $f(n) = A\log n + t(n)$, where $t(n)$ is a small additive function and A a positive constant, we obtain estimates for the quantities $\sum_{x\le n\le x+H} 1/f(Q(n))$ and $\sum_{x\le p\le x+H} 1/f(Q(p))$, where $H = H(x)$ satisfies certain growth conditions, p runs over prime numbers and Q is a polynomial with integer coefficients, whose leading coefficient is positive, and with all its roots simple.

1. Introduction. Let $t(n)$ be an additive function for which there exist two positive constants c and $\xi > 0$ such that

$$|t(p^\alpha)| \le \frac{c}{p^\xi} \quad \text{for all prime powers } p^\alpha, \tag{1}$$

and let $A > 0$ be a fixed number; then let

$$f(n) := A\log n + t(n). \tag{2}$$

Additive functions of the type (2) include the family of additive functions f for which Ivić [3] obtained estimates of $\sum_{n\le x,\ f(n)\ne 0} 1/f(n)$; the same is true for the family of additive functions studied by Brinitzer [1].

Let Q be a polynomial with integer coefficients, whose leading coefficient is positive, and such that all its roots are simple. Our goal here is to provide good estimates for each of the two sums

$$\sum_{x\le n\le x+H} \frac{1}{f(Q(n))} \quad\text{and}\quad \sum_{x\le p\le x+H} \frac{1}{f(Q(p))},$$

where $H = H(x)$ satisfies certain growth conditions and p runs over prime numbers. Let D be the discriminant of Q; for each prime p dividing D, we shall assume that there exists a positive integer $\beta_0 = \beta_0(p)$ such that $\tau(p^\beta) = \tau(p^{\beta+1}) = \cdots$ for each integer $\beta \ge \beta_0$.

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From these estimates will follow good estimates for the more classical expressions

$$\sum_{x\le n\le x+H}\frac{1}{f(n)}\quad\text{and}\quad\sum_{x\le p\le x+H}\frac{1}{f(p+1)}.$$

2. Main results

THEOREM 1. *Let f be defined by* (2). *Let $\varepsilon<1$ be a fixed positive number and let $H=H(x)$ be an increasing function satisfying $x^{\varepsilon}\le H\le x^{1-\varepsilon}$ for all $x\ge x_0$ for a certain $x_0>0$. Moreover, let Q be as in Section* 1. *Then, given any positive integer r, there exist computable constants $e_1>0,e_2,\dots,e_r$ such that*

$$\sum_{x\le n\le x+H}\frac{1}{f(Q(n))}=H\sum_{j=1}^{r}\frac{e_j}{\log^j x}+O\biggl(\frac{H}{\log^{r+1}x}\biggr).$$

As usual, we define the logarithmic integral as follows:

$$\mathrm{li}(x)=\int_2^x\frac{du}{\log u}.$$

THEOREM 2. *Let f be defined by* (2). *Let $\varepsilon<1$ be a fixed positive number and let $H=H(x)$ be an increasing function satisfying $x^{7/12+\varepsilon}\le H\le x^{1-\varepsilon}$ for all $x\ge x_0$ for a certain $x_0>0$. Moreover, let Q be as in Section* 1. *Then, given any positive integer r, there exist computable constants $f_1>0$, $f_2,\dots,f_r$ such that*

$$\sum_{x\le p\le x+H}\frac{1}{f(Q(p))}=(\mathrm{li}(x+H)-\mathrm{li}(x))\sum_{j=1}^{r}\frac{f_j}{\log^j x}+O\biggl(\frac{H}{\log^{r+2}x}\biggr).$$

The following results are then consequences of the proofs of the above theorems.

THEOREM 3. *Let f and A be as in* (2). *Let $\varepsilon<1$ be a fixed positive number and let $H=H(x)$ be an increasing function satisfying $x^{\varepsilon}\le H\le x^{1-\varepsilon}$ for all $x\ge x_0$ for a certain $x_0>0$. Then, given any positive integer r, there exist computable constants $b_1>0,b_2,\dots,b_r$, independent of A, such that*

$$\sum_{x\le n\le x+H}\frac{1}{f(n)}=H\sum_{j=1}^{r}\frac{b_j}{(A\log x)^j}+O\biggl(\frac{H}{\log^{r+1}x}\biggr).$$

COROLLARY. *Let g be either one of the following multiplicative functions*:

$$g(n)=\sigma_k(n):=\sum_{d|n}d^k,\quad g(n)=\varphi(n)\quad(\textit{Euler function}),\quad g(n)=\tau^{(e)}(n),$$

where $\tau^{(e)}(n) = \tau^{(e)}(p_1^{\alpha_1} \cdots p_s^{\alpha_s}) := \tau(\alpha_1) \cdots \tau(\alpha_s)$ *stands for the number of exponential divisors of* n, *that is*, *those divisors* $d = p_1^{\beta_1} \dots p_s^{\beta_s}$ *of* n *such that* $\beta_i \mid \alpha_i$ *for* $i = 1, \dots, r$. *Let* $H = H(x)$ *be as in Theorem* 1. *Then*, *given any positive integer* r, *there exist computable constants* $b_j = b_j(g)$, $1 \le j \le r$, *such that*

$$\sum_{x \le n \le x+H} \frac{1}{\log g(n)} = H \sum_{j=1}^{r} \frac{b_j}{\log^j x} + O\biggl(\frac{H}{\log^{r+1} x}\biggr).$$

THEOREM 4. *Let* f, A, ε *and* $H = H(x)$ *be as in Theorem* 2. *Then*, *given any positive integer* r, *there exist computable constants* $d_1 > 0, d_2, \dots, d_r$, *independent of* A, *such that*

$$\sum_{x \le p \le x+H} \frac{1}{f(p+1)} = (\mathrm{li}(x+H) - \mathrm{li}(x)) \sum_{j=1}^{r} \frac{d_j}{(A \log x)^j} + O\biggl(\frac{H}{\log^{r+2} x}\biggr).$$

3. Preliminary results

LEMMA 1. *Let* t *be as in Section* 1. *Then*

$$|t(n)| \ll \frac{(\log n)^{\beta}}{\log\log n} \quad (n \ge 3), \tag{3}$$

where $\beta = \max(1-\xi, 1/4)$.

Proof. First, consider the case where $0 < \xi \le 3/4$. Then, let $t^{(1)}$ be the additive function defined on prime powers p^α by

$$t^{(1)}(p^\alpha) = \frac{c}{p^\xi}.$$

One can easily establish that

$$\max_{3 \le n \le x} t^{(1)}(n) \ll \frac{(\log x)^{1-\xi}}{\log\log x},$$

which clearly implies (3). On the other hand, if $\xi > 3/4$, then we have $|t(p^\alpha)| \le c/p^\xi < c/p^{3/4}$, so that the argument for the first case may be used again, and then case (3) follows once more.

LEMMA 2. *Let* $Q \in \mathbb{Z}[x]$ *have all roots simple. Let* $\varrho(m)$ *be the number of solutions of* $Q(n) \equiv 0 \pmod{m}$. *Let* D *be the discriminant of* Q. *Then for each prime number* p *such that* $(p, D) = 1$, *we have* $\varrho(p^\beta) = \varrho(p)$ *for each positive integer* β.

Proof. It is known that, for some positive integer x, $D = u(x)Q(x) + v(x)Q'(x)$ for some polynomials $u, v \in \mathbb{Z}[x]$. Now, given $\beta \ge 1$, let $x_1, \dots, x_t \pmod{p^\beta}$ be the solutions of $Q(x) \equiv 0 \pmod{p^\beta}$. If $Q(y) \equiv 0 \pmod{p^{\beta+1}}$, then $y = x_i + tp^\beta \pmod{p^{\beta+1}}$ for some $t \in \{0, 1, \dots, p-1\}$, so that $Q(x_i + tp^\beta) \equiv Q(x_i) + tp^\beta Q'(x_i) \pmod{p^{\beta+1}}$. Therefore, since $p \mid Q(x_i)$ and $p \nmid Q'(x_i)$, we infer that exactly one t is appropriate, which means that $\varrho(p^{\beta+1}) = \varrho(p^{\beta})$, thus completing the proof of Lemma 2.

Let $\pi(x,k,l)$ denote the number of primes $p \le x$ such that $p \equiv l \pmod{k}$.

THEOREM A. *Let E be an arbitrary positive number and let $H = H(x)$ be as in Theorem 2. If $(k,l) = 1$, then uniformly for $k \le \log^E x$,*

$$\pi(x+H,k,l) - \pi(x,k,l) = \frac{\mathrm{li}(x+H) - \mathrm{li}(x)}{\varphi(k)}(1 + O(\exp\{-c_1\sqrt{\log x}\}))$$

for some positive constant c_1.

Proof. This follows directly from the Siegel–Walfisz Theorem (see Prachar [4, Chap. IX, Theorem 3.1]), according to which, uniformly for $k \le \log^E x$,

$$\psi(x+H,k,l) - \psi(x,k,l) = \frac{H}{\varphi(k)}(1 + O(\exp\{-c_1\sqrt{\log x}\})),$$

where

$$\psi(x,k,l) := \sum_{\substack{n \le x \\ n \equiv l \pmod{k}}} \Lambda(n),$$

with Λ standing for the von Mangoldt function.

REMARK. The exponent $7/12$ tied to the conditions on $H(x)$ (see statement of Theorem 2) comes from a result of Huxley [2].

4. The proof of Theorem 1. We may clearly assume that $r+1$ is even. Let $x > 0$ be a large number. Let k be the degree of the polynomial Q and let E be its leading coefficient. Let $\mathcal{J} = [x, x+H]$, $Y = Y(x) = \log^{\eta} x$, where η is a large number to be chosen later. Let also t_Y be the additive function defined on prime powers p^{α} by

$$t_Y(p^{\alpha}) = \begin{cases} t(p^{\alpha}) & \text{if } p^{\alpha} \le Y, \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$\kappa_Y(n) = t(n) - t_Y(n).$$

Finally, let $\varrho(m)$ be the number of solutions of $Q(n) \equiv 0 \pmod{m}$ and set

$$f_1(Q(n)) = A\log Q(x) + t(Q(n)), \qquad f_2(Q(n)) = A\log Q(x) + t_Y(Q(n)).$$

Since

$$\frac{Q(x+j)}{Q(x)} = 1 + O\left(\frac{j}{x}\right) \qquad \text{for } 1 \le j \le H,$$

it follows that

$$\sum_{n\in\mathcal{J}} \left|\frac{1}{f(Q(n))} - \frac{1}{f_1(Q(n))}\right| \ll \frac{1}{\log^2 x}\sum_{n\in\mathcal{J}} \log\left|\frac{Q(n)}{Q(x)}\right| \ll \frac{H^2}{x\log^2 x}. \tag{4}$$

Moreover,

$$\sum_{n\in\mathcal{J}}\left|\frac{1}{f_1(Q(n))}-\frac{1}{f_2(Q(n))}\right|\ll\frac{T}{\log^2 x}, \tag{5}$$

where

$$T:=\sum_{n\in\mathcal{J}}|\kappa_Y(Q(n))|.$$

In order to estimate T, we first observe that it follows from (1) that

$$\begin{aligned}|\kappa_Y(Q(n))|&\le\sum_{\substack{p^\alpha\|Q(n)\\ p^\alpha>Y}}|t(p^\alpha)|=\sum_{\substack{p^\alpha\|Q(n)\\ Y<p^\alpha\le H}}|t(p^\alpha)|+\sum_{\substack{p^\alpha\|Q(n)\\ p^\alpha>H}}|t(p^\alpha)|\\ &=\kappa_Y^{(1)}(Q(n))+\kappa_Y^{(2)}(Q(n)),\end{aligned}$$

say. Furthermore,

$$\begin{aligned}|\kappa_Y^{(2)}(Q(n))|&\le\sum_{\substack{p^\alpha\|Q(n)\\ p\le\sqrt{H},\,p^\alpha>H}}\frac{c}{p^\xi}+\sum_{\substack{p|Q(n)\\ \sqrt{H}<p\le H}}\frac{c}{p^\xi}+\frac{c}{H^\xi}\sum_{\substack{p|Q(n)\\ p>H}}1\\ &=K_1(Q(n))+K_2(Q(n))+K_3(Q(n)),\end{aligned} \tag{6}$$

say. Now let

$$T_j:=\sum_{n\in\mathcal{J}}\kappa_Y^{(j)}(Q(n))\qquad(j=1,2).$$

On the one hand,

$$T_1\ll H\sum_{p\ge Y}\frac{1}{p^{1+\xi}}+H\sum_{p^2>Y}\frac{1}{p\cdot p^{1+\xi}}+H\sum_{\substack{p^\alpha\ge Y\\ \alpha\ge 3}}\frac{1}{p^{\alpha-1}}\frac{1}{p^{1+\xi}}\ll\frac{H}{Y^\xi}+\frac{H}{\sqrt{Y}}. \tag{7}$$

On the other hand, it follows from (6) that

$$T_2\le M_1+M_2+M_3, \tag{8}$$

where $M_l=\sum_{n\in\mathcal{J}}K_l(Q(n))$ for $l=1,2,3$.

In order to estimate M_1, observe that the conditions $p^\alpha\parallel Q(n)$, $p<\sqrt{H}$, $p^\alpha>H$ imply that there is a divisor p^β of p^α for which $\sqrt{H}\le p^\beta<H$ with $\beta\ge 2$. Consequently,

$$M_1\ll H\sum_{p^\beta>\sqrt{H}}\frac{1}{p^\beta}\ll H^{3/4}, \tag{9}$$

say. Similarly, and by using Lemma 2, we infer that

$$M_2\ll\frac{H}{H^{\xi/2}}\sum_{\sqrt{H}<p<H}\frac{\varrho(p)}{p}\ll H^{1-\xi/2}\log\log H, \tag{10}$$

and that, since $\sum_{p|Q(n),\ p>H} 1$ is bounded,

$$M_3 \ll H^{1-\xi}. \tag{11}$$

Inserting (9), (10) and (11) in (8) shows that

$$T_2 \ll H^{3/4} + H^{1-\xi/2}. \tag{12}$$

Substituting (7) and (12) into (5), we obtain

$$\sum_{n\in\mathcal{J}} \left| \frac{1}{f_1(Q(n))} - \frac{1}{f_2(Q(n))} \right| \ll \frac{H}{\log^{\xi\eta+2} x} + \frac{H}{\log^{\eta/2+2} x}, \tag{13}$$

provided $0 < \xi < 1$, which has indeed been assumed.

Then, letting $S(x,H) := \sum_{x\le n\le x+H} 1/f(Q(n))$ and

$$S^*(x,H) := \sum_{n\in\mathcal{J}} \frac{1}{f_2(Q(n))}, \tag{14}$$

we deduce from (4) and (13) that

$$S(x,H) - S^*(x,H) = O\left(\frac{H}{\log^{r+1} x}\right),$$

provided $\eta = \eta(r,\xi,\varepsilon)$ is chosen large enough. This means that in order to complete the proof of Theorem 1, it is sufficient to prove that

$$S^*(x,H) = H \sum_{j=1}^{r} \frac{e_j}{\log^j x} + O\left(\frac{H}{\log^{r+1} x}\right). \tag{15}$$

First observe that it follows from Lemma 1 that

$$|t_Y(Q(n))| < \frac{A}{2} \log Q(x) \qquad (n \in \mathcal{J}),$$

so that

$$\begin{aligned} \frac{1}{f_2(Q(n))} &= \frac{1}{A\log Q(x) + t_Y(Q(n))} \\ &= \frac{1}{A\log Q(x)} \left\{ 1 - \frac{t_Y(Q(n))}{A\log Q(x)} + \left(\frac{t_Y(Q(n))}{A\log Q(x)}\right)^2 + \cdots \right. \\ &\qquad \left. + (-1)^r \left(\frac{t_Y(Q(n))}{A\log Q(x)}\right)^r + O\left(\frac{|t_Y(Q(n))|^{r+1}}{\log^{r+1} Q(x)}\right) \right\}. \end{aligned} \tag{16}$$

Now let

$$R_j(\mathcal{J}) := \sum_{n\in\mathcal{J}} t_Y^j(Q(n)),$$

so that from (14) and (16), we have

$$S^*(x,H) = \sum_{j=0}^{r} (-1)^j \frac{R_j(\mathcal{J})}{(A\log Q(x))^{j+1}} + O\left(\frac{R_{r+1}(\mathcal{J})}{\log^{r+2} x}\right). \tag{17}$$

We shall now estimate each $R_j(\mathcal{J})$ with good accuracy. Indeed,

$$(18)\qquad R_j(\mathcal{J}) = \sum_{l=1}^{j} \sum_{k_1+\cdots+k_l=j} \frac{j!}{k_1!\cdots k_l!} \sum_{p_1^{\alpha_1}<\cdots<p_l^{\alpha_l}\le Y} t(p_1^{\alpha_1})^{k_1}\ldots t(p_l^{\alpha_l})^{k_l}\Delta,$$

where $p_1,\ldots,p_l$ are any collection of distinct primes and $\alpha_1,\ldots,\alpha_l$ are positive integers such that $p_1^{\alpha_1}<\cdots<p_l^{\alpha_l}\le Y$ and

$$\Delta = \Delta(p_1^{\alpha_1},\ldots,p_l^{\alpha_l}) := \#\{n\in\mathcal{J} : p_j^{\alpha_j}\parallel Q(n),\ j=1,\ldots,l\}.$$

One easily sees that Δ may be written as

$$(19)\qquad \Delta = H\sum_{\delta\mid p_1\cdots p_l}\frac{\mu(\delta)}{\delta}\,\frac{\varrho(p_1^{\alpha_1}\cdots p_l^{\alpha_l}\delta)}{p_1^{\alpha_1}\cdots p_l^{\alpha_l}} + O\Big(\sum_{\delta\mid p_1\cdots p_l}\varrho(p_1^{\alpha_1}\cdots p_l^{\alpha_l}\delta)\Big).$$

Clearly the contribution of the error term in (19) to the right hand side of (18) is $O_j(1)$.

Now writing

$$\Sigma^*(Y\mid k_1,\ldots,k_l) = \sum_{p_1^{\alpha_1}<\cdots<p_l^{\alpha_l}\le Y} t(p_1^{\alpha_1})^{k_1}\cdots t(p_l^{\alpha_l})^{k_l}\sum_{\delta\mid p_1\cdots p_l}\frac{\mu(\delta)\varrho(p_1^{\alpha_1}\cdots p_l^{\alpha_l}\delta)}{\delta p_1^{\alpha_1}\cdots p_l^{\alpha_l}},$$

it follows from (18) and (19) that

$$(20)\qquad \begin{aligned} R_j(\mathcal{J}) &= H\sum_{l=1}^{j}\sum_{k_1+\cdots+k_l=j}\frac{j!}{k_1!\cdots k_l!}\,\Sigma^*(Y\mid k_1,\ldots,k_l) + O_j(1)\\ &= HD_j(Y) + O_j(1),\end{aligned}$$

say. We shall now manage to replace $D_j(Y)$ by

$$D_j := \sum_{l=1}^{j}\sum_{k_1+\cdots+k_l=j}\frac{j!}{k_1!\cdots k_l!}\,\Sigma^*(\infty\mid k_1,\ldots,k_l),$$

carefully monitoring the error term created by this substitution, that is, showing that

$$(21)\qquad |\Sigma^*(\infty\mid k_1,\ldots,k_l) - \Sigma^*(Y\mid k_1,\ldots,k_l)| \ll \frac{1}{\sqrt{Y}} + \frac{1}{Y^{\xi}},$$

thus enabling us, using (20), to replace (17) by

$$(22)\qquad S^*(x,H) = H\sum_{j=0}^{r}(-1)^j\frac{D_j}{(A\log Q(x))^{j+1}} + O\bigg(\frac{H}{\log^{r+2}x}\bigg),$$

provided η is chosen sufficiently large. Then, since

$$A\log Q(x) = A\log(Ex^k + O(x^{k-1})) = Ak\log x + A\log E + O(1/x),$$

it follows that

$$\frac{1}{(A\log Q(x))^{j+1}}=\sum_{\nu=j+1}^{r+1}\frac{u_{\nu,j}}{\log^\nu x}+O\left(\frac{1}{\log^{r+2}x}\right), \tag{23}$$

with suitable constants $u_{\nu,j}$. Using (23) in (22) yields (15).

Hence, it remains to prove (21). Indeed, by (1), it is clear that

$$\sum_{p_l^{\alpha_l}>Y}\frac{t(p_l^{\alpha_l})^{k_l}}{p_l^{\alpha_l}}\ll\frac{1}{\sqrt{Y}}+\frac{1}{Y^\xi} \tag{24}$$

and that

$$\sum_{p_1^{\alpha_1}<\cdots<p_{l-1}^{\alpha_{l-1}}}\prod_{j=1}^{l-1}\frac{t(p_j^{\alpha_j})^{k_j}}{p_j^{\alpha_j}}=O(1). \tag{25}$$

Since (21) clearly follows from (24) and (25), the proof of Theorem 1 is complete.

5. The proof of Theorem 2. The proof is very similar to that of Theorem 1. As in that proof, we may clearly assume that $r+1$ is even. Using the notation introduced in Section 4 and repeating the same argument, we obtain

$$\begin{aligned}\sum_{p\in\mathcal{J}}&\left|\frac{1}{f(Q(p))}-\frac{1}{f_2(Q(p))}\right|\\&\le\sum_{p\in\mathcal{J}}\left|\frac{1}{f(Q(p))}-\frac{1}{f_1(Q(p))}\right|+\sum_{p\in\mathcal{J}}\left|\frac{1}{f_1(Q(p))}-\frac{1}{f_2(Q(p))}\right|\\&\ll\frac{H^2}{x\log^2x}+\frac{T}{\log^2x}\ll\frac{H}{\log^{r+2}x},\end{aligned} \tag{26}$$

provided η is large enough. Then, proceeding as we did to obtain (16) and (17), we get

$$\sum_{p\in\mathcal{J}}\frac{1}{f_2(Q(p))}=\sum_{j=0}^{r}(-1)^j\frac{S_j(\mathcal{J})}{(A\log Q(x))^{j+1}}+O\left(\frac{S_{r+1}(\mathcal{J})}{\log^{r+2}x}\right), \tag{27}$$

where

$$S_j(\mathcal{J})=\sum_{p\in\mathcal{J}}t_Y^j(Q(p)).$$

Then

$$S_j(\mathcal{J})=\sum_{l=1}^{j}\sum_{k_1+\cdots+k_l=j}\frac{j!}{k_1!\cdots k_l!}\sum_{p_1^{\alpha_1}<\cdots<p_l^{\alpha_l}\le Y}t(p_1^{\alpha_1})^{k_1}\cdots t(p_l^{\alpha_l})^{k_l}\varDelta_*, \tag{28}$$

where again $p_1, \ldots, p_l$ are any collection of distinct primes and $\alpha_1, \ldots, \alpha_l$ are positive integers such that $p_1^{\alpha_1} < \cdots < p_l^{\alpha_l} \le Y$ and

$$\Delta_* = \Delta_*(p_1^{\alpha_1}, \ldots, p_l^{\alpha_l}) := \#\{p \in \mathcal{J} : p_j^{\alpha_j} \parallel Q(p),\ j = 1, \ldots, l\}.$$

Then, letting $\varrho^*(m)$ be the number of residue classes $s \pmod m$ such that $Q(s) \equiv 0 \pmod m$ and $(s, m) = 1$, and calling upon Theorem A, we obtain

$$\Delta_*(p_1^{\alpha_1}, \ldots, p_l^{\alpha_l}) = (\mathrm{li}(x+H) - \mathrm{li}(x)) \sum_{\delta \mid p_1 \ldots p_l} \frac{\mu(\delta)}{\delta} \frac{\varrho^*(p_1^{\alpha_1} \cdots p_l^{\alpha_l} \delta)}{\varphi(p_1^{\alpha_1} \cdots p_l^{\alpha_l})} + O\left(\frac{H}{\log x} \exp\{-c_1 \sqrt{\log x}\} \frac{1}{p_1^{\alpha_1} \cdots p_l^{\alpha_l}} \right). \tag{29}$$

Then, observing that the contribution of the error term in (29) to the sum in (28) is at most

$$O\left(\frac{H}{\log x} \exp\{-c_1 \sqrt{\log x}\} \right),$$

and continuing the proof as we did in Section 4, we find that

$$S_j(\mathcal{J}) = (\mathrm{li}(x+H) - \mathrm{li}(x)) K_j + O\left(\frac{H}{\log^{r+3} x} \right), \tag{30}$$

where

$$K_j := \sum_{l=1}^{j} \sum_{k_1 + \cdots + k_l = j} \frac{j!}{k_1! \cdots k_l!} \sum_{p_1^{\alpha_1} < \cdots < p_l^{\alpha_l}} t(p_1^{\alpha_1})^{k_1} \cdots t(p_l^{\alpha_l})^{k_l} \times \sum_{\delta \mid p_1 \cdots p_l} \frac{\mu(\delta)}{\delta} \frac{\varrho^*(p_1^{\alpha_1} \cdots p_l^{\alpha_l} \delta)}{\varphi(p_1^{\alpha_1} \cdots p_l^{\alpha_l})}.$$

Then substituting (30) in (27), and taking into account (26), completes the proof of Theorem 2.

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