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ASTHENO-KÄHLER STRUCTURES ON CALABI–ECKMANN MANIFOLDS

BY

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Dedicated to Professor Kentaro Mikami on his sixtieth birthday

Abstract. We show that there exist astheno-Kähler structures on Calabi–Eckmann manifolds.

1. Introduction. A Hermitian metric g on a complex manifold M of complex dimension m is called *astheno-Kähler* if its Kähler form Ω satisfies $\partial\overline{\partial}\Omega^{m-2} = 0$ (cf. [4], [5], [9]), where ∂ and $\overline{\partial}$ are the complex exterior differentials. It is known that every holomorphic 1-form on a compact astheno-Kähler manifold is closed. We note that the condition $\partial\overline{\partial}\Omega^{m-2} = 0$ is automatically satisfied for $m = 2$.

The author [7] showed that there exist non-trivial examples of compact astheno-Kähler manifolds. Namely, let M_i be a 3-dimensional compact Sasakian manifold with the structure tensor fields $(\phi_i, \xi_i, \eta_i, g_i)$ for each $i = 1, 2$. On the product manifold $M = M_1 \times M_2$, the Riemannian product metric $g = g_1 + g_2$ is compatible with A. Morimoto's complex structure [8] defined by

$$J = \phi_1 - \eta_2 \otimes \xi_1 + \phi_2 + \eta_1 \otimes \xi_2. \tag{1.1}$$

Then the Kähler form Ω satisfies $dd^c\Omega = 0$, which is equivalent to $\partial\overline{\partial}\Omega = 0$, that is, the metric g is astheno-Kähler. Moreover, it was also shown in [7] that there exists a similar astheno-Kähler structure on the product manifold of a 3-dimensional compact Sasakian manifold and a compact cosymplectic manifold of dimension ≥ 3. In these examples, the dimensions of Sasakian manifolds are restricted to 3. For instance, the Calabi–Eckmann manifold $S^3 \times S^3$ is one of these astheno-Kähler manifolds.

In [10], K. Tsukada introduced a family of complex structures on the Calabi–Eckmann manifold $S^{2m_1+1} \times S^{2m_2+1}$ containing Morimoto's complex structure (1.1) and defined Hermitian metrics compatible with the complex

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structures. In this paper, we show that there exist astheno-Kähler structures among Tsukada's Hermitian structures on Calabi–Eckmann manifolds.

2. Preliminaries. Let (M, J, g) be a Hermitian manifold of complex dimension $m \geq 3$ with complex structure J and Hermitian metric g. The Kähler form Ω on M is defined by $\Omega(X, Y) = g(X, JY)$ for all vector fields X, Y on M. Extend the complex structure J to p-forms φ on M as follows:

$$\begin{aligned} J\varphi &= \varphi & &\text{for } p = 0, \\ (J\varphi)(X_1, \ldots, X_p) &= (-1)^p \varphi(JX_1, \ldots, JX_p) & &\text{for } p > 0, \end{aligned}$$

where $X_1, \cdots, X_p$ are vector fields on M. The real differential operator d^c (cf. [1]) is then defined by

$$d^c\varphi = -J^{-1}dJ\varphi = (-1)^p JdJ\varphi \quad \text{for any } p\text{-form } \varphi \text{ on } M.$$

Since it is well-known that $dd^c = 2\sqrt{-1}\,\partial\bar{\partial}$, an astheno-Kähler manifold (M, J, g) may be defined by the condition $dd^c\Omega^{m-2} = 0$.

3. Hermitian structures on Calabi–Eckmann manifolds

3.1. *Almost contact metric structures.* Let N be a differentiable manifold of dimension $2n+1$. An *almost contact structure* on N is a triple (ϕ, ξ, η), where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, and η is a 1-form on N satisfying the following conditions (cf. [2]):

$$\eta(\xi) = 1, \tag{3.1}$$

$$\phi^2 = -I + \eta \otimes \xi, \tag{3.2}$$

where I denotes the identity transformation on each tangent space of N. Endowed with (ϕ, ξ, η), N is called an *almost contact* manifold. Then we also have the following equalities:

$$\phi\xi = 0, \tag{3.3}$$

$$\eta \circ \phi = 0. \tag{3.4}$$

Moreover, if there is a Riemannian metric g on an almost contact manifold N satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3.5}$$

for any vector fields X, Y on N, then N is said to have an *almost contact metric structure* (ϕ, ξ, η, g) and N endowed with this structure is called an *almost contact metric manifold.* Then, from (3.1)–(3.5), we immediately get

$$\eta(X) = g(X, \xi) \quad \text{and} \quad g(X, \phi Y) = -g(Y, \phi X)$$

for any vector fields X, Y on N. The 2-form Φ defined by $\Phi(X, Y) = g(X, \phi Y)$ is called the *fundamental 2-form* on the almost contact metric manifold N. We have $\eta \wedge \Phi^n \neq 0$. If $\Phi = d\eta$, then N is, by definition, a contact manifold. Such an almost contact metric structure is called a *contact metric structure.*

An almost contact structure (ϕ, ξ, η) is said to be *normal* if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ denotes the Nijenhuis tensor field of ϕ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y]$$

for all vector fields X, Y on N. A normal contact metric structure is called a *Sasakian structure.* It is well-known (cf. [2], [10]) that there is a standard Sasakian structure on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$.

On the other hand, an almost contact metric structure (ϕ, ξ, η, g) satisfying $d\Phi = 0$ and $d\eta = 0$ is called an *almost cosymplectic structure.* A normal almost cosymplectic structure is called a *cosymplectic structure.* The product of a unit circle and a compact Kähler manifold is the trivial example of compact cosymplectic manifolds. Non-trivial examples of compact cosymplectic manifolds are found in [3] and [6].

3.2. *Tsukada's Hermitian structures on the product of two Sasakian manifolds.* Let M_i be a $(2m_i + 1)$-dimensional Sasakian manifold with the structure tensor fields $(\phi_i, \xi_i, \eta_i, g_i)$ for each $i = 1, 2$. On the product manifold $M = M_1 \times M_2$, K. Tsukada [10] introduced an almost complex structure J defined by

$$J = \phi_1 - \left(\frac{a}{b}\,\eta_1 + \frac{a^2 + b^2}{b}\,\eta_2\right) \otimes \xi_1 + \phi_2 + \left(\frac{1}{b}\,\eta_1 + \frac{a}{b}\,\eta_2\right) \otimes \xi_2, \tag{3.6}$$

where $a, b \in \mathbb{R}$ and $b \neq 0$. In the case of $a = 0$ and $b = 1$, this almost complex structure coincides with A. Morimoto's complex structure (1.1). Since each almost contact structure is normal, we can prove, by the same method as A. Morimoto [8], that this almost complex structure J is integrable. Thus M endowed with J is a complex manifold of complex dimension $m = m_1 + m_2 + 1$.

K. Tsukada also introduced the following Hermitian metric g on the complex manifold (M, J):

$$g = g_1 + g_2 + a\,(\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1) + (a^2 + b^2 - 1)\,\eta_2 \otimes \eta_2. \tag{3.7}$$

Then the Kähler form Ω on the Hermitian manifold (M, J, g) is given by

$$\Omega = \Phi_1 + \Phi_2 - 2b\,\eta_1 \wedge \eta_2, \tag{3.8}$$

where Φ_i denotes the fundamental 2-form on M_i for each $i = 1, 2$. In particular, we can define this Hermitian structure on the Calabi–Eckmann manifold $S^{2m_1+1} \times S^{2m_2+1}$.

4. Astheno-Kähler structures on Calabi–Eckmann manifolds. In this section, we show that there exist astheno-Kähler structures among the Hermitian structures defined by (3.6) and (3.7) on the Calabi–Eckmann manifold $M = S^{2m_1+1} \times S^{2m_2+1}$, or more generally, on the product manifold $M = M_1 \times M_2$ of two Sasakian manifolds.

Since M_i is Sasakian, i.e., $\Phi_i = d\eta_i$ for each $i = 1, 2$, we have

$$d\Omega = -2b(\Phi_1 \wedge \eta_2 - \eta_1 \wedge \Phi_2). \tag{4.1}$$

We now show that Φ_1 is J-invariant, i.e., $J\Phi_1 = \Phi_1$. For any vector fields X, Y on M,

$$\begin{aligned}(J\Phi_1)(X,Y) &= \Phi_1(JX,JY) = g_1(JX,\phi_1 JY) = g_1(JX,\phi_1^2 Y_1)\\ &= g_1(\phi_1 X_1,\phi_1^2 Y_1) = g_1(X_1,\phi_1 Y_1) = \Phi_1(X_1,Y_1) = \Phi_1(X,Y).\end{aligned}$$

Of course, Φ_2 is also J-invariant. Similarly, we can show that η_1 and η_2 satisfy

$$J\eta_1 = \frac{a}{b}\,\eta_1 + \frac{a^2+b^2}{b}\,\eta_2, \qquad J\eta_2 = -\frac{1}{b}\,\eta_1 - \frac{a}{b}\,\eta_2.$$

Since, from (4.1), $d^c\Omega = JdJ\Omega = Jd\Omega = -2b(J\Phi_1 \wedge J\eta_2 - J\eta_1 \wedge J\Phi_2)$, we obtain

$$d^c\Omega = 2[\Phi_1 \wedge (\eta_1 + a\eta_2) + (a\eta_1 + (a^2+b^2)\eta_2) \wedge \Phi_2]. \tag{4.2}$$

By taking the exterior differential of this equation, we get

$$dd^c\Omega = 2[\Phi_1^2 + 2a\Phi_1 \wedge \Phi_2 + (a^2+b^2)\Phi_2^2]. \tag{4.3}$$

From (4.1)and (4.2) we also obtain

$$d\Omega \wedge d^c\Omega = 4b[\Phi_1^2 + 2a\Phi_1 \wedge \Phi_2 + (a^2+b^2)\Phi_2^2] \wedge \eta_1 \wedge \eta_2. \tag{4.4}$$

We now assume that the complex dimension m of M is greater than 3. Then

$$\begin{aligned}dd^c\Omega^{m-2} &= d(d^c\Omega^{m-2}) = d(JdJ\Omega^{m-2}) = d(Jd\Omega^{m-2})\\ &= (m-2)d[J(d\Omega \wedge \Omega^{m-3})] = (m-2)d[(Jd\Omega) \wedge (J\Omega^{m-3})]\\ &= (m-2)d[d^c\Omega \wedge \Omega^{m-3}]\\ &= (m-2)[dd^c\Omega \wedge \Omega^{m-3} - d^c\Omega \wedge d\Omega^{m-3}\\ &= (m-2)[dd^c\Omega \wedge \Omega^{m-3} - (m-3)d^c\Omega \wedge d\Omega \wedge \Omega^{m-4}]\\ &= (m-2)[dd^c\Omega \wedge \Omega + (m-3)d\Omega \wedge d^c\Omega] \wedge \Omega^{m-4}.\end{aligned}$$

On the other hand, from (3.8) and (4.1)–(4.4) we have

$$\begin{aligned}&dd^c\Omega \wedge \Omega + (m-3)d\Omega \wedge d^c\Omega\\ &\quad = 2[\Phi_1^2 + 2a\Phi_1 \wedge \Phi_2 + (a^2+b^2)\Phi_2^2] \wedge [\Phi_1 + \Phi_2 + 2(m-4)b\eta_1 \wedge \eta_2].\end{aligned}$$

By the binomial theorem, we also have

$$\begin{aligned}\Omega^{m-4} &= (\Phi_1 + \Phi_2 - 2b\eta_1 \wedge \eta_2)^{m-4} \\ &= \sum_{i=0}^{m-4} \binom{m-4}{i} (\Phi_1 + \Phi_2)^{(m-4)-i} \wedge (-2b\eta_1 \wedge \eta_2)^i \\ &= (\Phi_1 + \Phi_2)^{m-4} - 2(m-4)b(\Phi_1 + \Phi_2)^{m-5} \wedge \eta_1 \wedge \eta_2 \\ &= [\Phi_1 + \Phi_2 - 2(m-4)b\eta_1 \wedge \eta_2] \wedge (\Phi_1 + \Phi_2)^{m-5}.\end{aligned}$$

Since $[\Phi_1 + \Phi_2 + 2(m-4)b\eta_1 \wedge \eta_2] \wedge [\Phi_1 + \Phi_2 - 2(m-4)b\eta_1 \wedge \eta_2] = (\Phi_1 + \Phi_2)^2$, we get

$$\begin{aligned}&[dd^c\Omega \wedge \Omega + (m-3)d\Omega \wedge d^c\Omega] \wedge \Omega^{m-4} \\ &\qquad = 2[\Phi_1^2 + 2a\Phi_1 \wedge \Phi_2 + (a^2 + b^2)\Phi_2^2] \wedge (\Phi_1 + \Phi_2)^{m-3}.\end{aligned}$$

Hence

$$\begin{aligned}dd^c\Omega^{m-2} &= 2(m-2)[\Phi_1^2 + 2a\Phi_1 \wedge \Phi_2 + (a^2 + b^2)\Phi_2^2] \wedge (\Phi_1 + \Phi_2)^{m-3} \\ &= 2(m-2) \sum_{k=0}^{m-3} \binom{m-3}{k} [\Phi_1^{(m-1)-k} \wedge \Phi_2^k \\ &\qquad + 2a\Phi_1^{(m-2)-k} \wedge \Phi_2^{k+1} + (a^2 + b^2)\Phi_1^{(m-3)-k} \wedge \Phi_2^{k+2}] \\ &= 2(m-2) \sum_{k=0}^{m-1} C(m,k)\Phi_1^{(m-1)-k} \wedge \Phi_2^k,\end{aligned}$$

where $C(m,k)$ are given as follows:

$$C(m,0) = 1, \quad C(m,1) = m - 3 + 2a,$$

$$C(m,m-2) = 2a + (m-3)(a^2 + b^2), \qquad C(m,m-1) = a^2 + b^2,$$

$$C(m,k) = \binom{m-3}{k} + 2\binom{m-3}{k-1}a + \binom{m-3}{k-2}(a^2 + b^2) \quad \text{for } 2 \le k \le m-3.$$

If $p > m_i$, then $\Phi_i^p = 0$ on M_i. Therefore, if $0 \le k < m_2$, then $\Phi_1^{(m-1)-k} = 0$ on M_1, and if $m_2 < k \le m-1$, then $\Phi_2^k = 0$ on M_2. Thus

$$\Phi_1^{(m-1)-k} \wedge \Phi_2^k = 0 \quad \text{on } M \quad \text{if } k \ne m_2,$$

and hence

$$dd^c\Omega^{m-2} = 2(m-2)C(m,m_2)\Phi_1^{m_1} \wedge \Phi_2^{m_2}.$$

Moreover, $C(m,m_2) = 0$ is a necessary and sufficient condition for the Hermitian structure defined by (3.6) and (3.7) on M to be astheno-Kähler. The condition

$$C(m,m_2) = \binom{m-3}{m_2} + 2\binom{m-3}{m_2-1}a + \binom{m-3}{m_2-2}(a^2 + b^2) = 0$$

implies

$$m_1(m_1-1)+2m_1m_2a+m_2(m_2-1)(a^2+b^2)=0.$$

We deduce the following.

THEOREM 4.1. *Let M_i be a $(2m_i+1)$-dimensional Sasakian manifold with the structure tensor fields $(\phi_i,\xi_i,\eta_i,g_i)$ for each $i=1,2$, and $m=m_1+m_2+1>3$. Then the Hermitian structure defined by* (3.6) *and* (3.7) *on the product manifold of $M=M_1\times M_2$ is astheno-Kähler if and only if the constants a and b satisfy*

$$m_1(m_1-1)+2m_1m_2a+m_2(m_2-1)(a^2+b^2)=0.$$

We note that, in the case of $m=3$, i.e., $m_1=m_2=1$, the astheno-Kähler condition $dd^c\Omega^{m-2}=dd^c\Omega=0$ is equivalent to $a=0$ because of (4.3). That is, the conclusion of Theorem 4.1 is also valid in the case of $m=3$.

By the last theorem, the Calabi–Eckmann manifold $S^{2m_1+1}\times S^{2m_2+1}$ can be an example of a compact astheno-Kähler manifold.

REMARK 4.1. Let M_1 be a $(2m_1+1)$-dimensional Sasakian manifold with the structure tensor fields $(\phi_1,\xi_1,\eta_1,g_1)$, and M_2 a $(2m_2+1)$-dimensional cosymplectic manifold with the structure tensor fields $(\phi_2,\xi_2,\eta_2,g_2)$. On $M=M_1\times M_2$, we can then consider Tsukada's Hermitian structure (3.6)–(3.7). Since $\Phi_1=d\eta_1$ and $d\Phi_2=0, d\eta_2=0$, we get

$$d\Omega=-2b\Phi_1\wedge\eta_2,\qquad d^c\Omega=2\Phi_1\wedge(\eta_1+a\eta_2),\qquad dd^c\Omega=2\Phi_1^2.$$

Therefore

$$dd^c\Omega\wedge\Omega+(m-3)d\Omega\wedge d^c\Omega=2\Phi_1^2\wedge[\Phi_1+\Phi_2+2(m-4)b\eta_1\wedge\eta_2],$$

and hence we obtain

$$\begin{aligned} dd^c\Omega^{m-2}&=2(m-2)\Phi_1^2\wedge(\Phi_1+\Phi_2)^{m-3}\\ &=2(m-2)\sum_{k=0}^{m-3}\binom{m-3}{k}\Phi_1^{(m-1)-k}\wedge\Phi_2^k. \end{aligned}$$

If $m_1=1$, then $\Phi_1^2=0$ on M_1, that is, each of Tsukada's Hermitian structures on M is astheno-Kähler.

If $m_1>1$, then $m-3\geq m_2$. Therefore, if $0\leq k<m_2$, then $\Phi_1^{(m-1)-k}=0$ on M_1, and if $m_2<k\leq m-3$, then $\Phi_2^k=0$ on M_2. Thus

$$\Phi_1^{(m-1)-k}\wedge\Phi_2^k=0\quad\text{on } M\quad\text{if } k\neq m_2,$$

and hence

$$dd^c\Omega^{m-2}=2(m-2)\binom{m-3}{m_2}\Phi_1^{m_1}\wedge\Phi_2^{m_2}\neq 0\quad\text{on } M.$$

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