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WHEN $\aleph_1$ MANY SETS ARE CONTAINED IN A COUNTABLY GENERATED σ-FIELD

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Abstract. We discuss the problem when $\aleph_1$ sets are contained in a σ-generated σ-field on some set X. This is related to a problem raised by K. P. S. Bhaskara Rao and Rae Michael Shortt [Dissertationes Math. 372 (1998)] which we answer. We also briefly discuss generating the family of all subsets from rectangles.

Introduction. We first make some remarks about notation and recall some basic facts.

MA and CH stand for Martin's Axiom and Continuum Hypothesis respectively. $f[A]$ denotes the image of A under a function f. If $\mathcal{G} \subseteq \mathcal{P}(X)$, then $\sigma(\mathcal{G})$ denotes the smallest σ-field on X containing $\mathcal{G}$, commonly called the σ-field *generated* by $\mathcal{G}$ (then $\mathcal{G}$ is a *generator* of $\sigma(\mathcal{G})$). If $\mathcal{G}$ is countable, then $\sigma(\mathcal{G})$ is *countably generated* or *σ-generated*. Let $\mathcal{A}$ be a σ-field on X. If a nonempty $A \in \mathcal{A}$ has the property that $A \subseteq B$ or $A \cap B = \emptyset$ for every $B \in \mathcal{A}$, then A is called an *atom* of $\mathcal{A}$. If the atoms of $\mathcal{A}$ form a partition of X, then $\mathcal{A}$ is called *atomic*. If for any $x, y \in X$, $x \neq y$, there exists $A \in \mathcal{A}$ such that $x \in A$ and $y \notin A$, then we say that $\mathcal{A}$ *separates points* of X. We will need the following two easy propositions. Every σ-generated σ-field is atomic and of cardinality $\leq \mathfrak{c}$. Any atomic σ-field separates points iff its atoms are singletons. These facts and most others we are using can be found in [2] and [8].

Let $\mathcal{A}$, $\mathcal{B}$ be σ-fields on X, Y respectively. Then $\mathcal{R}(\mathcal{A}, \mathcal{B}) = \{A \times B : A \in \mathcal{A} \wedge B \in \mathcal{B}\})$ will be called a *family of rectangles*. We write $\mathcal{A} \otimes_\sigma \mathcal{B}$ for $\sigma(\mathcal{R}(\mathcal{A}, \mathcal{B}))$. If $h : \mathcal{A} \to \mathcal{B}$ is such that $h(\bigcup \mathcal{F}) = \bigcup h[\mathcal{F}]$ and $h(X \setminus A) = Y \setminus h(A)$ for every countable $\mathcal{F} \subseteq \mathcal{A}$ and $A \in \mathcal{A}$, then we say that h is a *homomorphism*. If, in addition, h is a bijection, then we will call it an *isomorphism*. If $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism and $\mathcal{G} \subseteq \mathcal{A}$, then $h[\sigma(\mathcal{G})] = \sigma(h[\mathcal{G}])$. For example, if $f : X \to Y$ then $F : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by $F(B) = f^{-1}[B]$ for $B \subseteq Y$ is a homomorphism, and if f is a bijection, then F is an isomorphism.

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1. Families of cardinality $\aleph_1$

DEFINITION 1.1. For any set A, we define $*(A)$ to be the following statement: "For any $\mathcal{F} \subset \mathcal{P}(A)$ of cardinality $\leq \aleph_1$, there exists a countable $\mathcal{F}_0 \subset \mathcal{P}(A)$ such that $\mathcal{F} \subset \sigma(\mathcal{F}_0)$".

Of course, $*(X) \Rightarrow *(Y)$ for any two sets X and Y such that $|X| \geq |Y|$, so only cardinality is really important here. We now recall some important facts.

It is known that $\mathcal{P}(\omega_1) \otimes_\sigma \mathcal{P}(\omega_1) = \mathcal{P}(\omega_1 \times \omega_1)$, and that MA implies $\mathcal{P}(\mathfrak{c}) \otimes_\sigma \mathcal{P}(\mathfrak{c}) = \mathcal{P}(\mathfrak{c} \times \mathfrak{c})$. Both results can be found in [5] and [7]. From this, one can easily prove $*(\omega_1)$, and MA $\Rightarrow *(\mathfrak{c})$ ([2, 3, 6]). The following theorem gives us a condition for larger sets.

THEOREM 1.2. *Suppose that $2^{\aleph_1} \leq \mathfrak{c}$ and $*(\mathfrak{c})$. Then $*(X)$ for any X.*

Proof. Let $\mathcal{F}$ be a family of subsets of X such that $|\mathcal{F}| \leq \aleph_1$. We can assume that $|X| \geq \mathfrak{c}$ and $\mathcal{F} \neq \emptyset$. For any $x \in X$, let $A_x = \bigcap_{A\in\mathcal{F}} A^{\chi_A(x)}$, where $Z^1 = Z$ and $Z^0 = X \setminus Z$ for $Z \subseteq X$. There are no more than $2^{|\mathcal{F}|} \leq 2^{\aleph_1} \leq \mathfrak{c}$ such sets.

Note that if $A_x \neq A_y$, then $A_x \cap A_y = \emptyset$, because for at least one set in $\mathcal{F}$, this set is in one of the families that we intersect to obtain A_x and A_y, and the complement of this set is in the other one. One can also easily see that $x \in A_x$ for every $x \in X$, and if $x \in F \in \mathcal{F}$ then $A_x \subseteq F$. So the family $\mathcal{A} = \{A_x : x \in X\}$ is in fact a partition of X such that every $F \in \mathcal{F}$ is equal to $\bigcup_{x\in F} A_x$.

Let $\mathcal{U}$ be the family $\{\bigcup \mathcal{A}' : \mathcal{A}' \subseteq \mathcal{A}\}$. It is evident that $\mathcal{U}$ is a σ-field, and we know that $\mathcal{F} \subseteq \mathcal{U}$. Hence $\sigma(\mathcal{F}) \subseteq \mathcal{U}$. Let S be a selector from $\mathcal{A}$. For $A, B \in \mathcal{U}$, if $A \neq B$, then $A \cap S \neq B \cap S$ because A and B are different unions of sets from the partition $\mathcal{A}$ of X.

We define $h : \mathcal{U} \to \mathcal{P}(S)$ by $h(U) = U \cap S$ for $U \in \mathcal{U}$. This function is an isomorphism between $\mathcal{U}$ and $\mathcal{P}(S)$. Then $h[\mathcal{F}]$ is a family of cardinality $\aleph_1$ on S of cardinality $\leq \mathfrak{c}$. Now, from $*(\mathfrak{c})$ we know that there exists a countable family $\mathcal{G} \subseteq \mathcal{P}(S)$ such that $h[\mathcal{F}] \subseteq \sigma(\mathcal{G})$. Hence, $\mathcal{F} = h^{-1}[h[\mathcal{F}]] \subseteq \sigma(h^{-1}[\mathcal{G}])$, and $h^{-1}[\mathcal{G}]$ is a countable family. ∎

Note that one cannot weaken the set-theoretic conditions in this theorem. It is obvious that the conclusion is not true when $*(\mathfrak{c})$ is false. The following example will show us the importance of the assumption $2^{\aleph_1} \leq \mathfrak{c}$.

PROPOSITION 1.3. *Suppose that $2^{\aleph_1} > \mathfrak{c}$ and $|Z| \geq 2^{\aleph_1}$. Then $*(Z)$ is false.*

Proof. Define $C_1 = 2^{\omega_1}$. Let $\mathcal{H} = \{H_\alpha : \alpha < \omega_1\}$, where $H_\alpha = \{x \in C_1 : x(\alpha) = 1\}$ for $\alpha < \omega_1$. Note that $\mathcal{H}$ separates points in C_1. Every σ-field on C_1 that contains $\mathcal{H}$ has to separate points in C_1. If $\mathcal{A}$ is a countably generated σ-field on C_1, then it is atomic and of cardinality $\leq \mathfrak{c}$ (see [2, pp. 8, 14]), so the atoms of $\mathcal{A}$ form a partition of C_1 into $\leq \mathfrak{c}$ pieces. As $|C_1| = 2^{\aleph_1}$ and $2^{\aleph_1} > \mathfrak{c}$, there exists an atom of $\mathcal{A}$ which is not a singleton. This means that $\mathcal{A}$ does not separate points of C_1, so it does not contain $\mathcal{H}$.

Since $|Z| \geq 2^{\aleph_1}$, there is a function $f : C_1 \xrightarrow{\text{1-1}} Z$. If $F : \mathcal{P}(Z) \to \mathcal{P}(C_1)$ is defined by $F(B) = f^{-1}[B]$, then F is a homomorphism. There is no countable family $\mathcal{G} \subset \mathcal{P}(Z)$ such that $\{f[H] : H \in \mathcal{H}\} \subseteq \sigma(\mathcal{G})$: if there were, then $\sigma(F[\mathcal{G}])$ would contain $\mathcal{H}$. ∎

2. Minimal generators of $\mathcal{P}(\kappa)$. In this section we give an application of property $*$.

Definition 2.1. Let $\mathcal{A} = \sigma(\mathcal{H})$ be a σ-field on X. Then $\mathcal{H}$ is a *minimal generator* of $\mathcal{A}$ if $\sigma(\mathcal{G}) \neq \mathcal{A}$ for any $\mathcal{G} \subsetneq \mathcal{H}$.

First examples of σ-fields without minimal generators were given in [1]. We recall a simple lemma from [8].

Lemma 2.2. *Let $\mathcal{A}$ be a σ-field on X that satisfies the following conditions*:

(i) *$\mathcal{A}$ is not countably generated.*
(ii) *For any family $\mathcal{F} \subseteq \mathcal{A}$ of cardinality ω_1, there exists a countably generated σ-field $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\mathcal{F} \subseteq \mathcal{A}_0$.*

Then $\mathcal{A}$ does not have a minimal generator.

We now focus on σ-fields $\mathcal{P}(\kappa)$ (κ a cardinal). MA implies that $\mathcal{P}(\kappa)$ has a minimal generator for every $\kappa < \mathfrak{c}$, and CH implies that $\mathcal{P}(\omega_1)$ does not have a minimal generator. These facts can be found in [8] along with the following two problems.

> [PP17] *Is it provable in ZFC that there exists a cardinal $\kappa > \omega_1$ such that $\mathcal{P}(\kappa)$ does not have a minimal generator?*
>
> [PP18] *Is it provable in ZFC that there exists a cardinal $\kappa > \omega_1$ such that $\mathcal{P}(\kappa)$ has a minimal generator?*

The lemma cannot be used to give a positive answer to PP17, because it is consistent that $\mathcal{P}(\kappa)$ satisfies (ii) for no $\kappa > \omega_1$. This follows easily from Proposition 1.3 if we assume CH $+ \aleph_2 = 2^{\mathfrak{c}}$. Thus, we still do not know the exact answer to PP17. However, we know that the answer to PP18 is negative.

Theorem 2.3. *Suppose that $2^{\aleph_1} \leq \mathfrak{c}$, $*(\mathfrak{c})$ holds and $\mathfrak{c} = \aleph_2$. Then $\mathcal{P}(\kappa)$ does not have a minimal generator for $\kappa > \omega_1$.*

Proof. Let $\kappa > \omega_1$ be a fixed cardinal. We will show that $\mathcal{P}(\kappa)$ satisfies the hypotheses of the lemma.

First, we know that $\kappa \geq \aleph_2 = \mathfrak{c}$. Thus $|\mathcal{P}(\kappa)| > \mathfrak{c}$, and hence $\mathcal{P}(\kappa)$ is not countably generated. Second, if $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ has cardinality $\aleph_1$, then Theorem 1.2 implies that $\mathcal{F}$ is contained in a countably generated sub-σ-field of $\mathcal{P}(\kappa)$. ∎

COROLLARY 2.4. *It is consistent that for $\kappa > \omega_1$, $\mathcal{P}(\kappa)$ does not have a minimal generator.*

Namely, it is known that MA + $\aleph_2 = \mathfrak{c}$ is relatively consistent with ZFC and that it implies the assumption of Theorem 2.3 ([9] or [7], and [5]).

3. Generalizations. First, we generalize property $*$.

DEFINITION 3.1. Let A be any set, and let κ be a cardinal. Then $*_\kappa(A)$ stands for the statement: "For every $\mathcal{F} \subseteq \mathcal{P}(A)$ of cardinality κ there exists a countably generated σ-field $\mathcal{B} \subseteq \mathcal{P}(A)$ such that $\mathcal{F} \subseteq \mathcal{B}$".

Note that $*(\cdot)$ is equivalent $*_{\omega_1}(\cdot)$. We can now generalize Theorem 1.2.

THEOREM 3.2. *Suppose that $*_\kappa(\mathfrak{c})$ holds and $2^\kappa \leq \mathfrak{c}$. Then $*_\kappa(Z)$ is true for every set Z.*

The proof is similar to the proof of Theorem 1.2. We can also generalize Proposition 1.3 with ease.

PROPOSITION 3.3. *Suppose that $2^\kappa > \mathfrak{c}$. If $|Z| \geq 2^\kappa$, then $*_\kappa(Z)$ is false.*

The next theorem shows a similar relation between $\mathcal{P}(\kappa) \otimes_\sigma \mathcal{P}(\mathfrak{c})$ and $\mathcal{P}(\kappa) \otimes_\sigma \mathcal{P}(J)$, where J is any set.

THEOREM 3.4. *Suppose that $2^\kappa \leq \mathfrak{c}$. Let J be any set. If $\mathcal{P}(\kappa) \otimes_\sigma \mathcal{P}(\mathfrak{c}) = \mathcal{P}(\kappa \times \mathfrak{c})$, then $\mathcal{P}(\kappa) \otimes_\sigma \mathcal{P}(J) = \mathcal{P}(\kappa \times J)$.*

Proof. Let $Z \in \mathcal{P}(\kappa \times J)$. We will show that Z is in $\sigma(\mathcal{R}(\mathcal{P}(\kappa), \mathcal{P}(J)))$.

Note that $Z = \bigcup_{\alpha<\kappa} \{\alpha\} \times Z_\alpha$, where $Z_\alpha \in \mathcal{P}(J)$ for all $\alpha < \kappa$. Let us take a closer look at $\mathcal{F} = \{Z_\alpha : \alpha < \kappa\}$. As in the proof of Theorem 1.2 we can obtain $\mathcal{A}$, $\mathcal{U}$ and S with the same properties and $|\mathcal{A}| = |S| \leq 2^{|\mathcal{F}|} \leq 2^\kappa \leq \mathfrak{c}$.

We can see that $\mathcal{U}_1 = \{\bigcup_{\alpha<\kappa} \{\alpha\} \times U_\alpha : \forall_{\alpha<\kappa}\ U_\alpha \in \mathcal{U}\}$ is a σ-field and $\kappa \times S$ is a selector from $\mathcal{A}_1 = \{\{\alpha\} \times A : \alpha < \kappa \wedge A \in \mathcal{A}\}$. Every set in $\mathcal{U}_1$ is a union of some sets from $\mathcal{A}_1$. The family $\mathcal{U}$ has the property that $Z_\alpha \in \mathcal{U}$ for all $\alpha < \kappa$. Thus, Z is in $\mathcal{U}_1$.

Without loss of generality we may assume that $S \subseteq \mathfrak{c}$. We define an isomorphism $h : \mathcal{U}_1 \to \mathcal{P}(\kappa \times S)$ by setting $h(U) = U \cap (\kappa \times S)$ for $U \in \mathcal{U}_1$.

Note that if $\mathcal{P}(\kappa) \otimes_\sigma \mathcal{P}(\mathfrak{c}) = \mathcal{P}(\kappa \times \mathfrak{c})$ then $\mathcal{P}(\kappa) \otimes_\sigma \mathcal{P}(S) = \mathcal{P}(\kappa \times S)$ because $S \subseteq \mathfrak{c}$. Since $h(Z) \in \mathcal{P}(\kappa \times S)$, we deduce that $h(Z) \in \mathcal{P}(\kappa) \otimes_\sigma \mathcal{P}(S)$. Hence $Z = h^{-1}[h(Z)] \in h^{-1}[\sigma(\mathcal{R}(\mathcal{P}(\kappa), \mathcal{P}(S)))]$. One can easily see that h^{-1} maps rectangles to rectangles, and since h is an isomorphism,

$$Z \in \sigma(h^{-1}[\mathcal{R}(\mathcal{P}(\kappa), \mathcal{P}(S))]) \subseteq \sigma(\mathcal{R}(\mathcal{P}(\kappa), \mathcal{P}(J))). \ \blacksquare$$

If we assume MA, then Theorem 3.2 is a corollary of Theorem 12 in [4] (Theorem FHJ).

It is not clear to us, however, if $*_\kappa(\mathfrak{c}) + 2^\kappa \leq \mathfrak{c}$ is equivalent to the conditions in that theorem, all of which are satisfied only if $2^\kappa \leq \mathfrak{c}$. This problem is known as the CE problem on D. H. Fremlin's problem list. Note that the conclusion in Theorem 3.4 is exactly condition (iii) in Theorem FHJ so we can add the following condition there: $2^\kappa \leq \mathfrak{c}$ and $\mathcal{P}(\kappa) \otimes_\sigma \mathcal{P}(\mathfrak{c}) = \mathcal{P}(\kappa \times \mathfrak{c})$.

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