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A HIT-AND-MISS TOPOLOGY FOR 2^X, $C_n(X)$ AND $F_n(X)$

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Abstract. A hit-and-miss topology (τ_{HM}) is defined for the hyperspaces 2^X, $C_n(X)$ and $F_n(X)$ of a continuum X. We study the relationship between τ_{HM} and the Vietoris topology and we find conditions on X for which these topologies are equivalent.

1. Introduction. A *continuum* is a compact connected Hausdorff space and a *metric continuum* is a compact connected metric space; the spaces considered in this paper are continua unless we mention specifically that X is a metric continuum. Let X be a nondegenerate continuum, and $n \in \mathbb{N}$. Consider the following sets:

(1) $2^X = \{A \subseteq X : A \text{ is nonempty and compact}\}$.
(2) $C(X) = \{A \in 2^X : A \text{ is connected}\}$.
(3) $C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}$. If $n = 1$, then $C_1(X) = C(X)$.
(4) $F_n(X) = \{A \in 2^X : 1 \leq |A| \leq n\}$.

In continuum theory it is customary to endow these sets with the Vietoris topology τ_{V}; in this paper we define a hit-and-miss topology τ_{HM} on these spaces and we find conditions for which $\tau_{\mathrm{V}} = \tau_{\mathrm{HM}}$. The hit-and-miss topology considered in this paper differs from the ones already existing in the literature, because the family of sets to be missed is considerably smaller, for it consists of compact connected subsets of X (subcontinua of X); it follows that τ_{HM} is smaller than the Vietoris topology.

We divide this paper into two main parts. In the first part we study the topology τ_{HM} on 2^X and on $C_n(X)$. It is worth noticing that in view of 1.4 below one might think that τ_{HM} will coincide with the Vietoris topology if X is semi-locally connected; this, however, is not true as Example 2.3(2) shows. In the second part we study the topology τ_{HM} on $F_n(X)$. In the last

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section we consider the case when X is indecomposable, and we also consider a natural generalization of τ_{HM}.

Notation. If X is a topological space and $M \subseteq X$, then $\mathrm{int}_X(M)$ denotes the interior of M in X; $|M|$ denotes the cardinality of M, $\mathrm{Cl}(M)$ denotes the closure of M; and $\mathrm{Bd}(M)$ denotes the boundary of M.

If X is a space endowed with a metric d, then for $p \in X$ and $\varepsilon > 0$, $B_\varepsilon(p)$ denotes the set $\{x \in X : d(x,p) < \varepsilon\}$.

1.1. Definition. Let X be a continuum. For any $U \subseteq X$, we define the following subsets of 2^X:

$$\langle U\rangle = \{A \in 2^X : A \subseteq U\}, \qquad \langle U, X\rangle = \{A \in 2^X : A \cap U \neq \emptyset\}.$$

Then the collection

$$\mathfrak{S}_V = \{\langle U\rangle : U \text{ is open}\} \cup \{\langle U, X\rangle : U \text{ is open}\}$$

is a subbase of a topology in 2^X called the *Vietoris topology* (see [2]). Furthermore, let $U_1, \dots, U_k$ be subsets of X and define

$$\langle U_1, \dots, U_k\rangle = \Big\{A \in 2^X : A \subseteq \bigcup_{i=1}^{k} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i\Big\}.$$

It is known (see [2]) that the collection

$$\mathfrak{B} = \{\langle U_1, \dots, U_k\rangle : U_i \text{ is open for each } i \text{ and } k < \omega\}$$

is a base for τ_{V}.

The space $(2^X, \tau_{\mathrm{V}})$ is called the *hyperspace of compact subsets of* X, $(C(X), \tau_{\mathrm{V}})$ the *hyperspace of subcontinua of* X, $(C_n(X), \tau_{\mathrm{V}})$ the *n-fold hyperspace of* X, and $(F_n(X), \tau_{\mathrm{V}})$ the *n-fold symmetric product of* X.

1.2. Definition. As in 1.1, the collection

$$\mathfrak{S}_{\mathrm{HM}} = \{\langle U\rangle : X \setminus U \text{ is a continuum}\} \cup \{\langle U, X\rangle : U \text{ is open}\}$$

is a subbase of a topology in 2^X which we will denote by τ_{HM}.

1.3. Definition. Let $U_1, \dots, U_k$ be subsets of X and let V be a subset of X such that $X \setminus V$ has finitely many components. We define

$$(U_1, \dots, U_k; V) = \{A \in 2^X : A \subseteq V \text{ and } A \cap U_i \neq \emptyset \text{ for each } i\}.$$

1.4. Theorem. *The collection*

$$\mathfrak{B} = \{(U_1, \dots, U_k; V) : k < \omega,\ U_i \textit{ is open for each } i, \textit{ and } V \textit{ is an open set such that } X \setminus V \textit{ has finitely many components}\}$$

is a base for τ_{HM}.

Proof. Note that if an open set and $X \setminus V$ has finitely many components, then there exist open sets $V_1, \dots, V_k$ such that each $X \setminus V_i$ is connected and

$V = \bigcap_{i=1}^{k} V_i$. Hence,

$$\langle V\rangle = \Big\langle \bigcap_{i=1}^{k} V_i \Big\rangle = \bigcap_{i=1}^{k} \langle V_i\rangle.$$

So if $(U_1, \ldots, U_n; V)$ is an element of $\mathcal{B}$, then

$$(U_1, \ldots, U_n; V) = \Big(\bigcap_{i=1}^{n} \langle U_i, X\rangle\Big) \cap \bigcap_{i=i}^{k} \langle V_i\rangle.$$

Hence, every element of $\mathcal{B}$ can be written as a finite intersection of elements of $\mathfrak{S}_{\mathrm{HM}}$.

On the other hand, if V_1 and V_2 are open sets such that $X\setminus V_1$ and $X\setminus V_2$ are connected, then $X\setminus(V_1 \cap V_2)$ has finitely many components, for $X\setminus(V_1 \cap V_2) = (X\setminus V_1) \cup (X\setminus V_2)$. Hence,

$$\langle V_1\rangle \cap \langle V_2\rangle = (X; V_1 \cap V_2).$$

Now, for any two open sets U_1 and U_2 we have

$$\langle U_1, X\rangle \cap \langle U_2, X\rangle = (U_1, U_2; X).$$

Furthermore, if V is an open set and $X \setminus V$ has finitely many components, then

$$\langle U_1, X\rangle \cap \langle V\rangle = (U_1; V).$$

Hence, any finite intersection of elements of $\mathfrak{S}_{\mathrm{HM}}$ lies in $\mathcal{B}$. Finally, note that $\mathcal{B}$ is closed under finite intersections. ∎

2. The spaces $(2^X, \tau_{\mathrm{HM}})$ and $(C_n(X), \tau_{\mathrm{HM}})$. In this section we show that $\tau_{\mathrm{HM}} = \tau_{\mathrm{V}}$ on 2^X and on $C_n(X)$ if and only if the continuum X is locally connected. First, note that

$$\tau_{\mathrm{HM}} \subseteq \tau_{\mathrm{V}}, \tag{2.1}$$

for if $(U_1, \ldots, U_k; V)$ is a τ_{HM} basic open set, and $A \in (U_1, \ldots, U_k; V)$, then $A \in \langle U_1 \cap V, \ldots, U_k \cap V, V\rangle \subseteq (U_1, \ldots, U_k; V)$.

2.1. Theorem. *Let X be a continuum. Then $(2^X, \tau_{\mathrm{HM}})$ is T_1.*

Proof. Let $A, B \in 2^X$ be such that $A \neq B$. Assume, without loss of generality, that there is $p \in B \setminus A$.

Let U be an open set such that $p \in U$ and $U \cap A = \emptyset$. Then $\mathcal{U} = (U; X)$ is an open set such that $B \in \mathcal{U}$ and $A \notin \mathcal{U}$.

Now, let $V = X \setminus \{p\}$. Then V is an open set whose complement is connected and $A \subseteq V$. Also $B \nsubseteq V$ since $p \in B$. Therefore, if $\mathcal{V} = (X; V)$, then $A \in \mathcal{V}$ and $B \notin \mathcal{V}$. This shows that 2^X is T_1. ∎

Since being T_1 is a hereditary property, we have the following corollary.

2.2. Corollary. *$(C_n(X), \tau_{\mathrm{HM}})$, and $(F_n(X), \tau_{\mathrm{HM}})$ are T_1 for all n.*

2.3. Examples.

(1) Let X be the $\sin(1/x)$ continuum, that is, X is the closure in the plane of the set $\{(x, \sin(1/x)) : 0 < x \leq 2/\pi\}$ (see Figure 1). For this continuum, $(C(X), \tau_{\mathrm{HM}})$ (and so $(2^X, \tau_{HM})$) is not T_2. To see this, let $\mathcal{U} = (U_1, \ldots, U_k; V)$ be a basic open set containing the subcontinuum $B = \{0\} \times [0,1]$ (see Figure 1) and let $\mathcal{O} = (O_1, \ldots, O_m; W)$ be a basic open set containing the limit bar $A = \{0\} \times [-1, 1]$. It is easy to see that there exists a subcontinuum C of $\{(s, \sin(1/s)) \in \mathbb{R}^2 : 0 < s \leq t\}$ such that C intersects each U_i and each O_i. This implies that $C \in \mathcal{U} \cap \mathcal{O}$. Hence $(C(X), \tau_{\mathrm{HM}})$ is not T_2.

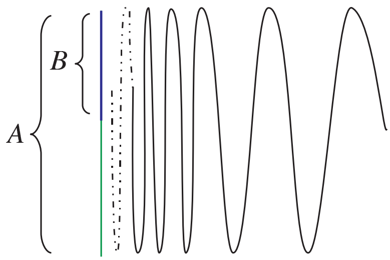

Fig. 1. The $\sin(1/x)$ continuum

(2) Let $L_0 = [0,1] \times \{0\}$, $D_0 = \{0\} \times [0,1]$, $D_1 = \{1\} \times [0,1]$, $C = [0, 1/4] \times \{0\}$, and for each $n \in \mathbb{N}$ let $L_n = [0,1] \times \{1/n\}$. The continuum $X = (\bigcup_{n \in \mathbb{N}} L_n) \cup L_0 \cup D_0 \cup D_1$ (see Figure 2) is semi-locally connected (see 2.5), yet $(C(X), \tau_{\mathrm{HM}})$ is not T_2. This can be shown in a similar way to Example (1). In fact, the subcontinua $A = D_0 \cup D_1 \cup L_2$ and $B = D_0 \cup D_1 \cup L_2 \cup C$ cannot be separated by disjoint open sets.

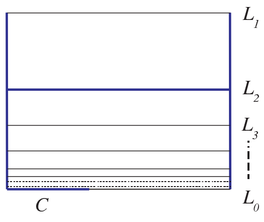

Fig. 2

For the remainder of the paper the symbol $\mathfrak{H}(X)$ will denote any of the hyperspaces defined in the previous section.

2.4. Proposition. *$(\mathfrak{H}(X), \tau_{\mathrm{HM}})$ is T_2 if and only if $(\mathfrak{H}(X), \tau_{\mathrm{HM}}) = (\mathfrak{H}(X), \tau_{\mathrm{V}})$.*

Proof. Since $(\mathfrak{H}(X), \tau_{\mathrm{HM}}) \subseteq (\mathfrak{H}(X), \tau_{\mathrm{V}})$, the identity map $i : (\mathfrak{H}(X), \tau_{\mathrm{V}}) \to (\mathfrak{H}(X), \tau_{\mathrm{HM}})$ is continuous. It is well known that $(\mathfrak{H}(X), \tau_{\mathrm{V}})$ is compact (see [2, Chapter 1]); and by hypothesis $(\mathfrak{H}(X), \tau_{\mathrm{HM}})$ is T_2, so i is a homeomorphism.

The sufficiency follows easily. ∎

Following the terminology of [5], we use the following definitions.

2.5. Definition. Given a continuum X and a subset A of X, we say that X is *semi-locally connected* (resp. *ω-semi-locally connected*) in A if for every open subset U of X such that $A \subset U$ there exists an open set V such that $A \subset V \subset U$ and $X \setminus V$ has finitely many (resp. at most countably many) components. If $A = \{p\}$, then we just say that X is *semi-locally connected* (resp. *ω-semi-locally connected*) at p. If X is *semi-locally connected* (resp. *ω-semi-locally connected*) at p for each $p \in X$, then we say that X is *semi-locally connected* (resp. *ω-semi-locally connected*).

Also, we extend this definition in the following way.

2.6. Definition. Given a continuum X and a hyperspace $\mathfrak{H}(X)$, we say that X is *semi-locally connected* (resp. *ω-semi-locally connected*) *in* $\mathfrak{H}(X)$ if for each $A \in \mathfrak{H}(X)$, X is semi-locally connected (resp. ω-semi-locally connected) in A.

2.7. Remark. Notice that if X is semi-locally connected (resp. ω-semi-locally connected) in 2^X, then X is semi-locally connected (resp. ω-semi-locally connected) in any other hyperspace $\mathfrak{H}(X)$.

2.8. Proposition. *If a continuum X is locally connected, then it is semi-locally connected in 2^X.*

Proof. Let $A \in 2^X$, and let U be an open set in X such that $A \subseteq U$. Since X is locally connected and $\mathrm{Bd}(U)$ is compact, we can find a finite collection $\{W_1, \ldots, W_r\}$ of open connected sets such that

$$\mathrm{Bd}(U) \subseteq \bigcup_{i=1}^{r} W_i, \qquad \mathrm{Cl}(W_i) \cap A = \emptyset \qquad \text{for all } i = 1, \ldots, r.$$

Hence $K = (X \setminus U) \cup \bigcup_{i=1}^{r} \mathrm{Cl}(W_i)$ is closed and since every component of $X \setminus U$ intersects $\mathrm{Bd}(U)$ (Boundary Bumping Theorem, see 5.4 of [7]), K has a finite number of components. Therefore the set $V = X \setminus K$ is open, $A \subseteq V \subseteq U$, and $X \setminus V$ has finitely many components. ∎

2.9. Definition. Let X be a compactum. Define $\mathcal{T} : C(X) \to C(X)$ by

$$\mathcal{T}(A) = \{x \in X : W \cap A \neq \emptyset \text{ for each } W \in C(X) \text{ with } x \in \mathrm{int}_X(W)\}.$$

2.10. Proposition. *If a continuum X is semi-locally connected in $C(X)$, then it is locally connected.*

Proof. Let $K \in C(X)$ and $x \notin K$. By the normality of X, there exists an open set U in X such that $K \subseteq U$ and $x \notin \mathrm{Cl}(U)$. Since X is semi-locally connected in $C(X)$, there exists an open set V in X such that $K \subseteq V \subseteq U$ and $X \setminus V$ has finitely many components. Let A be the component of $X \setminus V$ that contains x. Then $x \in \mathrm{int}_X(A) \subseteq A \in C(X)$ and $A \cap K = \emptyset$. Therefore, $x \notin \mathcal{T}(K)$ for every $x \in X \setminus K$; hence, $\mathcal{T}(K) = K$. By (17.8) of [5], X is locally connected. ∎

2.11. Proposition. *If a continuum X is semi-locally connected in 2^X, then it is locally connected.*

2.12. Proposition. *If X is semi-locally connected in $\mathfrak{H}(X)$, then $(\mathfrak{H}(X), \tau_{\mathrm{HM}}) = (\mathfrak{H}(X), \tau_{\mathrm{V}})$.*

Proof. By (2.1), $\tau_{\mathrm{HM}} \subseteq \tau_{\mathrm{V}}$. So it suffices to show that for any basic open set $\mathcal{U}$ of τ_{V} there is a basic open set of τ_{HM} contained in $\mathcal{U}$.

Let $\mathcal{U} = \langle U_1, \ldots, U_n \rangle$ be a basic open set of τ_{V}, and let $A \in \mathcal{U}$. Then, by definition of $\mathcal{U}$, $A \subseteq U = \bigcup_{i=1}^n U_i$. Therefore, by hypothesis, there exists V such that $A \subseteq V \subseteq U$ and $X \setminus V$ has finitely many components.

Next we show that the set $\mathcal{V} = (U_1, \ldots, U_n; V)$ is contained in $\mathcal{U}$; indeed, if $B \in \mathcal{V}$, then $B \cap U_i \neq \emptyset$ for all i, and $B \subseteq V \subseteq U = \bigcup_{i=1}^n U_i$, which implies that $B \in \mathcal{U}$. Therefore $\tau_{\mathrm{V}} \subseteq \tau_{\mathrm{HM}}$. ∎

2.13. Proposition. *X is semi-locally connected in 2^X if and only if $(2^X, \tau_{HM}) = (2^X, \tau_{\mathrm{V}})$.*

Proof. If X is semi-locally connected in 2^X, then by 2.12 the topologies are the same.

Now assume $(2^X, \tau_{\mathrm{HM}}) = (2^X, \tau_{\mathrm{V}})$. Let $A \in 2^X$, and let $U \subseteq X$ be an open set such that $A \subseteq U$. We will construct an open set V such that $A \subseteq V \subseteq U$ and $X \setminus V$ has finitely many components. Let $\mathcal{U} = \langle U \rangle$. Hence $A \in \mathcal{U}$ and, by hypothesis, there exists $\mathcal{V} = (U_1, \ldots, U_n; V)$ such that $A \in \mathcal{V} \subseteq \mathcal{U}$. Then $A \subseteq V$ and $X \setminus V$ has finitely many components; to finish the proof we only need to show $V \subseteq U$. Let $x \in V$. Then $A \cup \{x\} \in 2^X$. Clearly, $A \cup \{x\} \subseteq V$, and since $A \cap U_i \neq \emptyset$ for all i, we have $(A \cup \{x\}) \cap U_i \neq \emptyset$ for all i. Therefore, $A \cup \{x\} \in \mathcal{V}$ and $A \cup \{x\} \in \mathcal{U} = \langle U \rangle$, implying $x \in U$. ∎

2.14. Lemma. *If $(C(X), \tau_{\mathrm{HM}}) = (C(X), \tau_{\mathrm{V}})$, then X is semi-locally connected.*

Proof. Let $p \in X$ and let U be an open set in X such that $p \in U$. If $\mathcal{U} = \langle U \rangle$, then $\{p\} \in \mathcal{U}$. Now, because $(C(X), \tau_{\mathrm{HM}}) = (C(X), \tau_{\mathrm{V}})$, there exists $\mathcal{V} = (U_1, \ldots, U_n; V)$ such that

$$\{p\} \in \mathcal{V} \subseteq \mathcal{U}. \tag{2.2}$$

Let W be an open set such that

$$p \in W \subseteq \mathrm{Cl}(W) \subseteq U \cap V \cap U_1 \cap \cdots \cap U_n. \tag{2.3}$$

Let C be a component of $X \setminus W$. We prove that

$$\text{if } C \cap (X \setminus U) \neq \emptyset, \text{ then } C \cap (X \setminus V) \neq \emptyset. \tag{2.4}$$

To see this, first note that $C \cap \mathrm{Cl}(W) \neq \emptyset$. Hence, from (2.3), $C \cap U_i \neq \emptyset$ for all i. Therefore, if $C \subseteq V$, then $C \in \mathcal{V}$. This, together with (2.2), implies $C \in \mathcal{U}$, that is, $C \subseteq U$. This shows that (2.4) holds.

Let $F = \mathrm{Cl}[(X \setminus V) \cup \bigcup\{C : C \text{ is a component of } X \setminus W \text{ and } C \cap (X \setminus U) \neq \emptyset\}]$. From (2.4) and the fact that $X \setminus V$ has finitely many components,

$$F \text{ has finitely many components.} \tag{2.5}$$

Let $O = X \setminus F$; by construction O is open and $p \in O$. From (2.5), the complement of O has a finite number of components. So, to show X is semi-locally connected at p it suffices to show $O \subseteq U$.

Let $x \in X \setminus U$. Then since $W \subseteq U$, $x \notin W$. Let C be the component of $X \setminus W$ that contains x. Since $x \notin U$, we have $C \cap (X \setminus U) \neq \emptyset$. Therefore, from the definition of F, $C \subseteq F$. Hence, $x \in F$, implying $x \notin O$. This shows that $O \subseteq U$, and the lemma is proved. ∎

2.15. Definition. Let X be a topological space, and let $p \in X$. Then X is *connected im kleinen* (*cik*) at p provided that p has a neighborhood base of connected neighborhoods (that is, connected sets that contain p in their interiors in X).

It is obvious that if X is locally connected at p, then X is *cik* at p. However, the converse is false even for continua (see Figure 22 of [2]). Nevertheless, it is well known that if a topological space is *cik* at every point, then it is locally connected at every point.

2.16. Proposition. *If* $(C(X), \tau_{\mathrm{HM}}) = (C(X), \tau_{\mathrm{V}})$, *then* X *is locally connected.*

Proof. From 2.14 we know that X is semi-locally connected.

Suppose X is not locally connected. Then there is $p \in X$ such that X is not *cik* at p. Consider the following set of natural numbers:

$$M = \{n \in \mathbb{N} : p \text{ has a local base } \mathcal{B} \text{ such that } \forall V \in \mathcal{B},\ X \setminus V \text{ has } n \text{ components}\}.$$

We consider two cases.

Case 1: $M \neq \emptyset$. Let $m = \min M$. Because X is not *cik* at p and by the definition of M there exists an open set $V \in \mathcal{B}$ with the following properties:

(2.6) $X \setminus V$ has m components;

(2.7) if W is open and $W \subseteq V$, then $X \setminus W$ has at least m components;

(2.8) V has infinitely many components.

Denote by $D_1, \ldots, D_m$ the components of $X \setminus V$, let $\{C_\alpha : \alpha \in \Lambda\}$ be the components of V, and denote by C_p the component containing p.

By the Boundary Bumping Theorem, for each α, $\mathrm{Cl}(C_\alpha)$ intersects at least one D_i. Furthermore,

$$\text{if } \mathrm{Cl}(C_\alpha) \cap D_i \neq \emptyset, \text{ then } \mathrm{Cl}(C_\alpha) \cap D_j = \emptyset \text{ for all } j \neq i. \tag{2.9}$$

Otherwise, if there is an α such that $\mathrm{Cl}(C_\alpha)$ intersects at least two components of $X \setminus V$, then $W = V \setminus \mathrm{Cl}(C_\alpha)$ would be an open set such that $W \subseteq V$ and $X \setminus W$ has at most $m-1$ components, contradicting (2.7).

Case 2: $M = \emptyset$. Let m_0 be a fixed natural number. There exists an open set V such that

$$\text{for every open set } W \subseteq V,\ X \setminus W \text{ has more than } m_0 \text{ components.}$$

Furthermore, we can choose V that satisfies (2.9) as follows. Assume $X \setminus V$ has n components with $n > m_0$. If (2.9) is not satisfied, then there exists C_k such that $\mathrm{Cl}(C_k)$ intersects at least two components D_i and D_j of $X \setminus V$. Then $W' = V \setminus \mathrm{Cl}(C_k)$ is an open set such that $W' \subseteq V$ and $X \setminus W'$ has at most $n-1$ components. We repeat this process until we obtain the required set.

For any of the two cases, the argument continues as follows:

Since X is not *cik* at p, there exists a subset $\{C_k : k \in \mathbb{N}\}$ of $\{C_\alpha : \alpha \in \Lambda\}$ such that

$$\text{for all } \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ such that } B_\varepsilon(p) \cap C_k \neq \emptyset \text{ for all } k \geq N.$$

Since the set of components of $X \setminus V$ is finite, there is one that intersects infinitely many $\mathrm{Cl}(C_k)$. Assume, without loss of generality, that D_1 intersects $\mathrm{Cl}(C_k)$ for all k. From (2.9) we see that

$$\mathrm{Cl}(C_k) \cap D_i = \emptyset \quad \text{for all } i \neq 1 \text{ and all } k. \tag{2.10}$$

We may assume that $\mathrm{Cl}(C_k)$ converges to C with $p \in C$. Then $C \setminus D_1 \neq \emptyset$.

Let $A = D_1$ and $B = D_1 \cup \mathrm{Cl}(C)$. Then $A, B \in C(X)$ and $A \neq B$. We will show that any two open sets in τ_{HM} containing A and B, respectively, intersect.

Let $\mathcal{U} = (U_1, \ldots, U_u; T)$ and $\mathcal{O} = (O_1, \ldots, O_s; Z)$ be basic open sets such that $A \in \mathcal{U}$ and $B \in \mathcal{O}$.

Since $A \subseteq T$ and $\mathrm{Cl}(C_k) \cap A \neq \emptyset$ for all k, we have $C_k \cap T \neq \emptyset$ for all k.

Claim. *Let*

$$\varXi = \{k \in \mathbb{N} : C_k \cap (X \setminus T) \neq \emptyset\}, \qquad \varUpsilon = \{k \in \mathbb{N} : C_k \cap (X \setminus Z) \neq \emptyset\}.$$

Then $|\varXi|$ and $|\varUpsilon|$ are finite.

Proof. Let $k, s \in \Xi$, $k \neq s$, and let E_k and E_s be components of $X \setminus T$ such that $C_k \cap E_k \neq \emptyset$ and $C_s \cap E_s \neq \emptyset$. We will show, by contradiction, that $E_k \cap E_s = \emptyset$. So, suppose $E_k \cap E_s \neq \emptyset$; then $E_k = E_s$.

Since $X \setminus T \subseteq V \cup D_2 \cup \cdots \cup D_n$, we obtain $E_k \subseteq V \cup D_2 \cup \cdots \cup D_n$. Since E_k is connected, if $E_k \cap D_i \neq \emptyset$, then $E_k \subseteq V \cup D_i$. This implies that $C_k \cap D_i \neq \emptyset$, contrary to (2.10).

Thus, $E_k \subseteq V$, which implies that C_k and C_s are the same component of V, so $k = s$, which is a contradiction. Therefore $E_k \cap E_s = \emptyset$, and thus for every $k \in \Xi$ there exists a component E_k of $X \setminus T$ such that $C_k \cap E_k \neq \emptyset$. Since $X \setminus T$ has finitely many components, Ξ is finite.

With a similar argument we prove that Υ is finite. This proves the Claim.

Hence, there is $N_1 \in \mathbb{N}$ such that

$$C_k \subseteq T \quad \text{for all } k \geq N_1, \tag{2.11}$$

and there exists $N_2 \in \mathbb{N}$ such that

$$C_k \subseteq Z \quad \text{for all } k \geq N_2. \tag{2.12}$$

Also, there exists $N_3 \in \mathbb{N}$ such that

$$\text{if } C \cap O_j \neq \emptyset, \text{ then } C_k \cap O_j \neq \emptyset \text{ for all } k \geq N_3. \tag{2.13}$$

Now, let $t \in \mathbb{N}$, with $t \geq \max\{N_1, N_2, N_3\}$, and let $E = D_1 \cup C_t$. Since $A = D_1$, E intersects all the U_i's, and from (2.11), $E \subseteq T$. Hence,

$$E \in \mathcal{U}.$$

Since $B = D_1 \cup C$, and from (2.13), we see that E intersects all the O_i's, and from (2.12), $E \subseteq Z$. Hence,

$$E \in \mathcal{O}.$$

Therefore $\mathcal{U} \cap \mathcal{O} \neq \emptyset$, implying that $(C(X), \tau_{\mathrm{HM}})$ is not T_2, a contradiction to the hypothesis since $(C(X), \tau_{\mathrm{V}})$ is T_2. Thus X must be locally connected. ■

The following theorem follows immediately from 2.14 and 2.16.

2.17. Theorem. *If $(C_n(X), \tau_{\mathrm{HM}}) = (C_n(X), \tau_{\mathrm{V}})$, then X is semi-locally connected and locally connected.*

We summarize the previous results in the following theorem.

2.18. Theorem. *Let X be a continuum and $n \in \mathbb{N}$. The following are equivalent.*

(a) *X is locally connected.*
(b) *X is semi-locally connected in $C_n(X)$.*
(c) *X is semi-locally connected in 2^X.*
(d) *$(C_n(X), \tau_{\mathrm{HM}}) = (C_n(X), \tau_{\mathrm{V}})$.*
(e) *$(2^X, \tau_{\mathrm{HM}}) = (2^X, \tau_{\mathrm{V}})$.*

(f) $(C_n(X), \tau_{\mathrm{HM}})$ *is* T_2.
(g) $(2^X, \tau_{\mathrm{HM}})$ *is* T_2.

Proof. (b)⇒(d) and (c)⇒(e) are given by 2.12, (a)⇒(b) and (a)⇒(c) are given by 2.8, (c)⇒(a) is given by 2.10, (e)⇒(c) is given by 2.13, (d)⇒(a) is given by 2.16, and (d)⇔(f) and (e)⇔(g) are given by 2.4. ∎

2.19. Remark. Let (X, τ) be a Hausdorff compact space. In [8], Z. M. Rakowski defined τ^* as the topology generated by the closed subbase $\mathcal{B} = \{A \subset X : A \text{ is a subcontinuum of } X\}$ and proved that if (X, τ) is a hereditary unicoherent continuum, then $\tau = \tau^*$ is equivalent to (X, τ) being locally connected. So conditions (a)–(g) in 2.18 are equivalent to (h) $\tau = \tau^*$.

3. The space $(F_n(X), \tau_{\mathrm{HM}})$. In this section we focus on aposyndetic continua.

3.1. Definition. A continuum X is said to be *aposyndetic at* p if for every $q \in X$, $q \neq p$, there exists a subcontinuum M of X such that $p \in \mathrm{int}_X(M)$ and $q \notin M$. If X is aposyndetic at every point, then X is said to be *aposyndetic*.

A continuum X is *finitely aposyndetic* if for every $p \in X$ and every nonempty finite subset F of X such that $p \notin F$ there exists a subcontinuum M of X such that $p \in \mathrm{int}_X(M)$ and $M \cap F = \emptyset$.

A continuum X is *n-aposyndetic* if for every $p \in X$ and every nonempty subset F of X such that $p \notin F$ and $|F| \leq n$ there exists a subcontinuum M of X such that $p \in \mathrm{int}_X(M)$ and $M \cap F = \emptyset$.

Burton Jones proved that every semi-locally connected continuum is aposyndetic (see Theorem 3 of [3]).

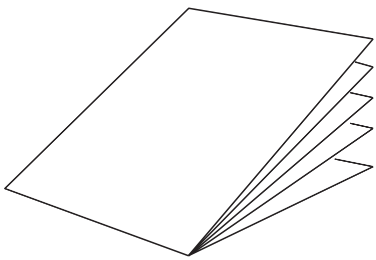

Fig. 3. Book continuum

Some results from the previous section do not hold true for the hyperspace $F_n(X)$. In particular, 2.10 is false when $C(X)$ is replaced with $F_n(X)$. To see this, let X be the Cartesian product of the unit interval and the harmonic fan (the cone over the set $\{0\} \cup \{1/n : n \in \mathbb{N}\}$) (see Figure 3). Clearly

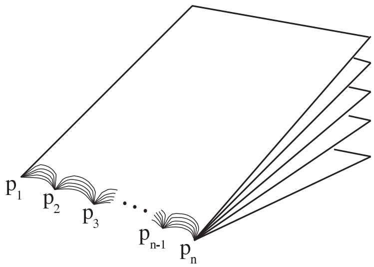

Fig. 4. Book with n binding points

X is not locally connected. Given any $A \in F_n(X)$ and an open set U of X such that $A \subseteq U$, we can find an $\varepsilon > 0$ such that $V = \bigcup_{x \in A} B_\varepsilon(x) \subseteq U$ and $X \setminus V$ has finitely many components. On the other hand, let Y be a "book" continuum whose binding consists of a fixed number of points $\{p_1, \ldots, p_n\}$ (see Figure 4). Then for $A = \{p_1, \ldots, p_n\} \in F_n(X)$ and any open set U containing A there is no open set V such that $A \subseteq V \subseteq U$ and $X \setminus V$ has finitely many components. Note that X is finitely aposyndetic while Y is not.

3.2. Theorem. *Let X be a continuum. The following are equivalent.*

(a) *X is n-aposyndetic.*
(b) *X is semi-locally connected in $F_n(X)$.*
(c) $(F_{n+1}(X), \tau_{\mathrm{HM}}) = (F_{n+1}(X), \tau_{\mathrm{V}})$.
(d) $(F_{n+1}(X), \tau_{\mathrm{HM}})$ *is T_2.*

Proof. (a)⇒(c). To prove this implication we show that $(F_{n+1}(X), \tau_{\mathrm{HM}})$ is T_2; then 2.4 implies (c). Let $A, B \in F_{n+1}(X)$ be two different points. We divide the proof into three cases.

Case 1: $|A| = |B| = n+1$. Let $A = \{a_1, \ldots, a_{n+1}\}$ and $B = \{b_1, \ldots, b_{n+1}\}$. Since $A \neq B$ and they have the same number of elements there exist $a \in A \setminus B$ and $b \in B \setminus A$, say $a = a_1$ and $b = b_1$.

Since A and B are finite sets, we can find for each i open sets U_i and V_i with the following properties:

$$U_i \cap (A \cup B) = \{a_i\} \quad \text{and} \quad V_i \cap (A \cup B) = \{b_i\}; \tag{3.1}$$

$$\text{if } i \neq j, \text{ then } U_i \cap U_j = \emptyset \text{ and } V_i \cap V_j = \emptyset; \tag{3.2}$$

$$\text{if } a_i \neq b_j \text{ then } U_i \cap V_j = \emptyset. \tag{3.3}$$

Let $\mathcal{U} = (U_1, \ldots, U_{n+1}; X \setminus \{b_1\})$ and $\mathcal{V} = (V_1, \ldots, V_{n+1}; X \setminus \{a_1\})$. By construction $\mathcal{U} \cap \mathcal{V} = \emptyset$, $A \in \mathcal{U}$ and $B \in \mathcal{V}$.

Case 2: $|A| = n+1$ and $|B| \leq n$. Let $A = \{a_1, \ldots, a_{n+1}\}$ and $B = \{b_1, \ldots, b_s\}$. Assume that $a_1 \notin B$.

Since X is n-aposyndetic and $|B| \leq n$ there exists a subcontinuum M of X such that $a_1 \in \mathrm{int}_X(M)$ and $M \cap B = \emptyset$. As in Case 1, we construct open sets U_i and V_i satisfying (3.1)–(3.3), and we can choose U_1 such that $U_1 \subseteq \mathrm{int}_X(M)$.

Let $\mathcal{U} = (U_1, \ldots, U_{n+1}; X)$ and $\mathcal{V} = (V_1, \ldots, V_s; X \setminus M)$. Then $A \in \mathcal{U}$, $B \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Case 3: $|A| \le n$ and $|B| \le n$. Assume, without loss of generality, that $|A| \ge |B|$. Then we proceed as in Case 1 or 2 depending on whether $|A| = |B|$ or $|A| > |B|$.

(c)⇒(b). Let $A \in F_n(X)$ and let U be an open set in X such that $A \subseteq U$. Let $\mathcal{U} = \langle U \rangle$. Since $A \in F_n(X) \subseteq F_{n+1}(X)$, there exists $\mathcal{V} = (U_1, \ldots, U_s; V)$ such that $A \in \mathcal{V} \subseteq \mathcal{U}$ and $X \setminus V$ has finitely many components. We show that $V \subseteq U$. Indeed, let $x \in V$. Then $A \cup \{x\} \in F_{n+1}$ since $|A| \le n$. Now, because $A \in \mathcal{V}$, $(A \cup \{x\}) \cap U_i \ne \emptyset$ and $A \cup \{x\} \subseteq V$. Hence $A \cup \{x\} \in \mathcal{V} \subseteq \mathcal{U}$. Therefore $A \cup \{x\} \subseteq U$, implying $x \in U$. This shows that $V \subseteq U$.

(b)⇒(a). Let $p \in X$ and let F be a nonempty subset of X such that $p \notin F$ and $|F| \le n$.

There exist two open sets O and U such that $p \in O$, $F \subseteq U$ and $\mathrm{Cl}(O) \cap \mathrm{Cl}(U) = \emptyset$.

Let $V \subseteq U$ be an open set such that $F \subseteq V$ and $X \setminus V$ has finitely many components. Then the component M of $X \setminus V$ that contains p is a subcontinuum of X such that $p \in \mathrm{int}_X(M)$ and $M \cap F = \emptyset$. Thus X is n-aposyndetic.

(c)⇔(d). These implications follow from 2.4. ■

4. Observations. In this section we analyze the case when X is an indecomposable continuum. We also define a new topology τ_ω which generalizes τ_{HM} and we prove, for this new topology, similar results to the ones given in the previous section.

Indecomposable case

4.1. Definition. A continuum is called *decomposable* if it can be written as the union of two nonempty nondegenerate proper subcontinua. A continuum is called *indecomposable* if it is not decomposable (for more on indecomposable continua see [7]).

4.2. Theorem. *Let X be a continuum. The following are equivalent*:

(a) *X is indecomposable.*
(b) *$X \in \mathrm{Cl}(\mathcal{U})$ for all nonempty open sets $\mathcal{U}$ of $(2^X, \tau_{\mathrm{HM}})$ (or of $C_n(X)$ for all n).*
(c) *$\mathcal{U} \cap \mathcal{V} \ne \emptyset$ for all nonempty open sets $\mathcal{U}, \mathcal{V}$ of $(2^X, \tau_{\mathrm{HM}})$ (or of $C_n(X)$ for all n).*

Proof. (a)⇒(b). Let $\mathcal{U} = (U_1, \ldots, U_u; W)$ be a basic open set. Assume $X \notin \mathcal{U}$, otherwise we are done.

Let $\mathcal{V} = (V_1, \ldots, V_r; O)$ be a basic open set such that $X \in \mathcal{V}$. We will show that $\mathcal{V}$ intersects $\mathcal{U}$, thus showing that $X \in \mathrm{Cl}(\mathcal{U})$.

By the definition of $\mathcal{U}$, $X \setminus W$ has finitely many components $\{C_1, \ldots, C_k\}$. Since $X \notin \mathcal{U}$, each C_i is a proper subcontinuum of X. Hence, because X is indecomposable, each C_i is contained in a composant K_i (see [7]).

Now, let R be a composant different from each K_i; such an R exists (and is dense in X) because X is indecomposable. By the density of R, we can find a proper subcontinuum A of X such that

(1) $A \subseteq R$,
(2) A intersects each U_i and each V_j.

Notice, from our choice of R, that $R \subseteq W$. Therefore $A \subseteq W$; this and (2) imply $A \in \mathcal{U}$. Also $A \in \mathcal{V}$ from (2) and since $A \subseteq X \subseteq O$. This proves (a)⇒(b).

(b)⇒(a). Suppose X is decomposable, hence $X = S_1 \cup S_2$, where S_1 and S_2 are proper subcontinua of X.

Let $U = X \setminus S_1$, and let $\mathcal{U} = (X; U)$. Note that $\mathcal{U} = \langle U \rangle$. We will construct an open set $\mathcal{V}$ containing X such that $\mathcal{V} \cap \mathcal{U} = \emptyset$, thus showing that $X \notin \mathrm{Cl}(\mathcal{U})$.

Since $X \setminus S_2$ is open and $U \subseteq S_2$, there is a nonempty open set $V \subseteq X \setminus S_2$ such that

$$V \cap U = \emptyset. \tag{4.1}$$

Let $\mathcal{V} = (V; X)$. By construction $X \in \mathcal{V}$. Now, $\mathcal{V} \cap \mathcal{U} = (V; U)$, hence by definition $A \in (V; U)$ if and only if $A \cap V \neq \emptyset$ and $A \subseteq U$. Then from (4.1) we have $(V; U) = \emptyset$.

(a)⇒(c). Let $\mathcal{U} = (U_1, \ldots, U_n; W)$ and $\mathcal{V} = (V_1, \ldots, V_k; O)$ be two basic open sets. We can assume, from (a)⇒(b), that $X \notin \mathcal{U}$ and $X \notin \mathcal{V}$.

By definition, $X \setminus W = \{C_1, \ldots, C_m\}$ and $X \setminus O = \{D_1, \ldots, D_r\}$ where C_i and D_j are continua for all i and j.

Let $A \in \mathcal{U}$ and $B \in \mathcal{V}$ be two subcontinua of X. As in (a)⇒(b), take A and B in a composant R different from the ones containing the C_is and D_js. Let $a \in A$ and $b \in B$. Since $a, b \in R$, there exists a proper subcontinuum E of X such that $a, b \in E$. It follows that $A \cup B \cup E \subseteq R$ is a proper subcontinuum of X that intersects each U_i and each V_j. Hence $A \cup B \cup E \in \mathcal{U} \cap \mathcal{V}$.

(c)⇒(b). This implication follows easily. ■

We finish this section with some generalizations.

The τ_ω topology. In 2^X, consider the topology generated by the following subbase:

$$\mathfrak{S}_{\mathrm{HM}} = \{\langle U\rangle : U \text{ is open and } X\backslash U \text{ has at most countably many components}\} \cup \{\langle U, X\rangle : U \text{ is open}\}.$$

We will denote this topology by τ_ω. It is easy to see, using similar arguments to those in 1.4, that the collection

$$\mathfrak{B} = \{(U_1, \ldots, U_k; V) : k < \omega,\ U_i \text{ is open for each } i, \text{ and } V \text{ is an open set such that } X \setminus V \text{ has at most countably many components}\}$$

is a base for τ_ω.

4.3. Remark. From the definition of τ_ω, we see that $\tau_{\mathrm{HM}} \subseteq \tau_\omega \subseteq \tau_{\mathrm{V}}$. It follows that 2.1, 2.2, and 2.4 hold with τ_{HM} replaced by τ_ω. However, for any of the spaces in 2.3 the space $(C(X), \tau_\omega)$ is T_2 and hence $(C(X), \tau_\omega) = (C(X), \tau_{\mathrm{V}})$, so that 2.14 and 2.16 do not hold for τ_ω. Also note that 4.2 holds with τ_{HM} replaced by τ_ω.

4.4. Remark. Using the same arguments and replacing τ_{HM} with τ_ω and "semi-locally connected" with "ω-semi-locally connected" we find that 2.12 and 2.13 can be generalized as well. Notice that the spaces in 2.3 are ω-semi-locally connected in 2^X but none of them is locally connected, therefore 2.10 does not hold when "semi-locally connected" is replaced with "ω-semi-locally connected".

We recall the definition of Suslinian space.

4.5. Definition. A compact space X is called *Suslinian* provided that every collection of mutually disjoint nondegenerate subcontinua of X is countable.

4.6. Remark. Mimicking the proof of 2.14 we get the following generalization of 2.14.

4.7. Proposition. *If $(\mathfrak{H}(X), \tau_\omega) = (\mathfrak{H}(X), \tau_{\mathrm{V}})$, then X is ω-semi-locally connected.*

The above observations can be summarized as follows.

4.8. Theorem. *Let X be a continuum and $n \in \mathbb{N}$. Consider the following statements:*

(a) *X is Suslinian.*
(b) *X is ω-semi-locally connected in $C_n(X)$.*
(c) *X is ω-semi-locally connected in 2^X.*
(d) $(C_n(X), \tau_\omega) = (C_n(X), \tau_{\mathrm{V}})$.
(e) $(2^X, \tau_\omega) = (2^X, \tau_{\mathrm{V}})$.
(f) *$(C_n(X), \tau_\omega)$ is T_2.*

(g) $(2^X, \tau_\omega)$ *is* T_2.
(h) X *is* ω*-semi-locally connected.*
(i) *For all* $x \in X$ *and for every open set* U *with* $x \in U$ *there exists a closed neighborhood* N *of* x *such that* $N \subseteq U$ *and* N *has at most countably many components.*

Then

(1) (b)$\Rightarrow$(d), (d)$\Rightarrow$(h), (e)$\Rightarrow$(h), (c)$\Leftrightarrow$(e), (i)$\Rightarrow$(b), (i)$\Rightarrow$(c), (d)$\Leftrightarrow$(f), *and* (e)$\Leftrightarrow$(g).
(2) (a)$\not\Rightarrow$(b), (a)$\not\Rightarrow$(c), (a)$\not\Rightarrow$(h). *Thus*, *from* (1), (a)$\not\Rightarrow$(d),(e),(g),(f).
(3) *If* X *is a hereditarily unicoherent metric continuum*, *then* (c)$\Rightarrow$(a) *and* (d)$\Rightarrow$(a).

Proof. (1) (d)$\Leftrightarrow$(f) and (e)$\Leftrightarrow$(g) follow from 4.3; (b)$\Rightarrow$(d) and (c)$\Leftrightarrow$(e) follow from 4.4; (d)$\Rightarrow$(h) and (e)$\Rightarrow$(h) follow from 4.7.

The proofs of (i)$\Rightarrow$(b) and (i)$\Rightarrow$(c) are similar to the proof of 2.8: just replace "X is locally connected" with (i), "open connected sets" with "closed neighborhoods with at most countably many components", and "finite number of components" with "at most countably many components".

(2) The dendroid constructed in §3, p. 135, of [4] is a Suslinian continuum which does not satisfy (b), (c), nor (h) since the complement of every arbitrarily small neighborhood of r has uncountably many components.

(3) To prove (3), we first need to prove some lemmas.

4.9. LEMMA. *Let* X *be a hereditarily unicoherent metric continuum. If* X *is not Suslinian*, *then there exists an open set* U *in* X *with uncountably many components.*

Proof. Since X is not Suslinian, there exists an uncountable family $\mathcal{C} = \{C_\lambda : \lambda \in \Lambda\}$ of pairwise disjoint nondegenerate subcontinua of X. Since Λ is uncountable and C_λ nondegenerate for all λ, there exists $\varepsilon > 0$ such that the set $\mathcal{B} = \{C_\lambda \in \mathcal{C} : \operatorname{diam}(C_\lambda) \geq \varepsilon\}$ is uncountable.

The open cover $\{B_{\varepsilon/4}(x) : x \in X\}$ of X has a finite subcover $\{B_{\varepsilon/4}(x_i) : i \leq m\}$. Hence, there exists $j \in \{1, \ldots, m\}$ such that $B_{\varepsilon/4}(x_j)$ intersects uncountably many C_λ's from $\mathcal{B}$.

If $B_{\varepsilon/4}(x_j)$ has uncountably many components, then we are done. So, suppose otherwise.

Let $\mathcal{D} = \{C_\lambda \in \mathcal{B} : C_\lambda \cap B_{\varepsilon/4}(x_j) \neq \emptyset\}$. Since $B_{\varepsilon/4}(x_j)$ does not have uncountably many components, there exists a subcontinuum C of $\operatorname{Cl}(B_{\varepsilon/4}(x_j))$ that intersects uncountably many members of $\mathcal{D}$; for simplicity, suppose it intersects all of them. Since $\operatorname{diam}(C_\lambda) \geq \varepsilon$, there exists $k \neq j$ such that $B_{\varepsilon/4}(x_k)$ intersects uncountably many elements of $\mathcal{D}$; denote by $\mathcal{D}'$ this set. Let $U = B_{\varepsilon/4}(x_k) \setminus \operatorname{Cl}(B_{\varepsilon/3}(x_j))$. We will show that U has uncountably many components.

Indeed, otherwise there exists a subcontinuum E of $\mathrm{Cl}(U)$ that intersects all but countably many elements of $\mathcal{D}'$. Then there exist two elements C_λ and C_γ of $\mathcal{D}'$ such that C and E intersect both C_λ and C_γ. Because X is a dendroid, $C \cup E \cup C_\lambda \cup C_\gamma$ is a dendroid. Since C_λ and C_γ are disjoint, and C and E are disjoint, it follows that $C \cup E \cup C_\lambda \cup C_\gamma$ contains a simple closed curve, which contradicts $C \cup E \cup C_\lambda \cup C_\gamma$ being a dendroid. ∎

4.10. Lemma. *Let X be a continuum and let U be an open subset of X. If U has uncountably many components, then there is $p \in U$ such that for each open set V with $p \in V \subset U$, V intersects uncountably many components of U.*

Proof. Assume, to the contrary, that for each $p \in U$ there is an open set V_p with $p \in V_p \subset U$ and V_p intersects countably many components. The collection $\mathcal{V} = \{V_p : p \in U\}$ is an open cover of U. Since U is Lindelöf, there is a sequence $\{p_i\}_{i\in\mathbb{N}}$ such that $U = \bigcup_{i\in\mathbb{N}} V_{p_i}$. Since, for each $i \in \mathbb{N}$, V_{p_i} intersects countably many components, U has countably many components, which is a contradiction. ∎

We now prove (3) of 4.8.

Proof. Let X be a hereditarily unicoherent metric continuum.

(c)⇒(a). Suppose X is not Suslinian. Then by 4.9 there is an open set U with uncountably many components. Let $p \in U$ be the point from 4.10 and let $A = X \setminus U$.

Let W be an open set such that $A \subseteq W$ and $p \notin \mathrm{Cl}(W)$. By construction, if V is an open set such that $A \subseteq V \subseteq W$, then $X \setminus V$ has uncountably many components. Hence, X is not ω-semi-locally connected in 2^X.

(d)⇒(a). We prove this implication by contradiction: we will construct two subcontinua A and B of X and we will show that any two open sets containing A and B intersect, thus showing that $(C(X), \tau_\omega)$ is not T_2, contrary to hypothesis.

So, suppose that X is not Suslinian. From 4.9 there is an open set U with uncountably many components. Let $p \in U$ be the point from 4.10. From 4.7, X is ω-semi-locally connected, so we can find an open neighborhood V of p such that V has uncountably many components $\{C_\lambda : \lambda \in \Lambda\}$, and $X \setminus V$ has at most countably many components $\{D_i : i \in \mathbb{N}\}$. Thus, there is D_j such that $D_j \cap \mathrm{Cl}(C_\lambda) \neq \emptyset$ for uncountably many C_λ's. Without loss of generality assume $j = 1$.

Now, let C_λ and C_γ be two different components such that $\mathrm{Cl}(C_\lambda) \cap D_1 \neq \emptyset$ and $\mathrm{Cl}(C_\gamma) \cap D_1 \neq \emptyset$. If $\mathrm{Cl}(C_\lambda) \cap D_k \neq \emptyset$ for $k \neq 1$, then $\mathrm{Cl}(C_\gamma) \cap D_k = \emptyset$; otherwise the continuum $\mathrm{Cl}(C_\lambda) \cup \mathrm{Cl}(C_\gamma) \cup D_1 \cup D_k$ would contain a simple closed curve, contrary to the fact that X is a hereditarily unicoherent continuum.

The previous argument shows that there is an uncountable collection ξ of C_λ's such that $\mathrm{Cl}(C_\lambda) \cap D_1 \neq \emptyset$, and $\mathrm{Cl}(C_\lambda) \cap D_i = \emptyset$ for all $i \neq 1$. Using a similar argument to the one in 4.10, we can find a point q in V such that every open set W with $q \in W \subseteq V$ intersects uncountably many elements of $\mathcal{E}$.

Let C_q be the component of V that contains q. Since q is in the closure of $\mathcal{E}$ and $C(X)$ is compact, it follows that $\mathrm{Cl}(C_q) \cap D_1 \neq \emptyset$. Let $A = D_1$ and $B = \mathrm{Cl}(C_q) \cup D_1$, and let $\mathcal{U} = (U_1, \ldots, U_n; T)$ and $\mathcal{O} = (O_1, \ldots, O_s; Z)$ be two basic open sets such that $A \in \mathcal{U}$ and $B \in \mathcal{O}$. A modification of the Claim in the proof of 2.16, for the countable case, shows that $\mathcal{U}$ and $\mathcal{O}$ intersect, thus implying that $(C(X), \tau_\omega)$ is not T_2, which is a contradiction. Therefore X must be Suslinian. ∎

4.11. Question. Which implications in 4.8 can be reversed, and which cannot?

The τ_ω topology in $F_n(X)$. For the spaces $(F_n(X), \tau_\omega)$ we have the following theorem.

4.12. Theorem. *Let X be a continuum. Then*

(a) *X is countably aposyndetic* (1)

implies the following:

(b) *X is ω-semi-locally connected in $F_n(X)$ for all $n \in \mathbb{N}$.*
(c) *$(F_n(X), \tau_\omega) = (F_n(X), \tau_{\mathrm{V}})$ for all $n \geq 2$.*
(d) *$(F_n(X), \tau_\omega)$ is T_2 for all $n \geq 2$.*

Furthermore, (b), (c), *and* (d) *are equivalent.*

Proof. (a)⇒(c). Assume X is countably aposyndetic. Then, in particular, X is n-aposyndetic for every $n \in \mathbb{N}$. Therefore, from (a)⇒(c) of 3.2, $(F_n(X), \tau_{\mathrm{HM}}) = (F_n(X), \tau_{\mathrm{V}})$ for all $n \geq 2$; this and the fact that $\tau_{\mathrm{HM}} \subseteq \tau_\omega \subseteq \tau_{\mathrm{V}}$ imply (c).

(b)⇒(c). This implication follows from the appropriate generalization of 2.12 (see 4.4).

(c)⇒(b). The proof of (c)⇒(b) is the same as the proof of (c)⇒(b) of 3.2 with "has finitely many components" replaced by "has at most countably many components".

(c)⇔(d). These implications follow from the corresponding generalization of 2.4 (see 4.3). ∎

4.13. Example. Let X be the harmonic fan, i.e. the cone over $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. Observe that (b), (c), and (d) (in the previous theorem) are

(1) A continuum X is *countably aposyndetic* if for every $p \in X$ and every nonempty countable closed subset F of X such that $p \notin F$ there exists a subcontinuum M of X such that $p \in \mathrm{int}_X(M)$ and $M \cap F = \emptyset$.

satisfied for X. However, X is not aposyndetic, so (a) is not satisfied. This shows that neither of (b), (c), nor (d) imply (a).

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