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AN INCONSISTENCY EQUATION INVOLVING MEANS

BY

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Abstract. We show that any quasi-arithmetic mean A_φ and any non-quasi-arithmetic mean M (reasonably regular) are inconsistent in the sense that the only solutions f of both equations

$$f(M(x,y)) = A_\varphi(f(x), f(y))$$

and

$$f(A_\varphi(x,y)) = M(f(x), f(y))$$

are the constant ones.

1. Background. The classical *Jensen functional equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$$

(see, e.g., M. Kuczma [6], J. Aczél [1], J. Aczél & J. Dhombres [2]) involves looking for mappings preserving the arithmetic mean. Recall that given an interval $I \subset \mathbb{R}$ every strictly monotonic and continuous function $\varphi\colon I \to \mathbb{R}$ generates the so-called *quasi-arithmetic mean* $A_\varphi\colon I \times I \to I$ by the formula

$$A_\varphi(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \qquad x, y \in I.$$

It is easily seen that functions f transforming a quasi-arithmetic mean A_φ of arguments x, y into a quasi-arithmetic mean A_ψ of $f(x)$, $f(y)$ have to be of the form $\psi^{-1} \circ g \circ \varphi$, where g is a Jensen function. Therefore, more interesting is the question of finding solutions of a *generalized Jensen functional equation*

$$f(M(x,y)) = N(f(x), f(y)), \tag{$*$}$$

where M, N stand for abstract means (see, e.g., Z. Daróczy & Zs. Páles [3]) and at least one of them is not quasi-arithmetic. In particular, the functional equations

$$f(L(x,y)) = L(f(x), f(y))$$

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and

$$f(L(x,y)) = \frac{f(x)+f(y)}{2}$$

with the *logarithmic mean*

$$L(x,y) = \begin{cases} (x-y)(\log x - \log y)^{-1} & \text{provided } x \neq y, \\ x & \text{otherwise.} \end{cases}$$

were studied by J. Matkowski [7] and P. Kahlig and J. Matkowski in [5], respectively. The main results of these papers state that if f is continuous at least at one point, then f is either constant or linear in the case of the first equation, whereas the latter equation admits no nonconstant solutions. It is worth emphasizing that non-constant solutions do exist only in the case where the means in question are of the same type (both non-quasi-arithmetic). The simultaneous occurrence of non-quasi-arithmetic (logarithmic) and quasi-arithmetic (actually arithmetic) means in the equation considered forces the unknown function to be constant.

The main aim of the present paper is to show that this phenomenon is not accidental. With no regularity assumptions whatsoever we shall show that equation $(*)$, where one of the means M, N is quasi-arithmetic and the other is not, admits no non-constant solutions. We think that this justifies the term *inconsistency equation* in the title of this article.

In the case where the non-quasi-arithmetic mean considered is the celebrated arithmetic-geometric Gauss mean, a result in that spirit was established in the paper of Z. Daróczy, Gy. Maksa and Zs. Páles [4, Corollary 2.2].

2. Preliminary results. For a two-place function M we will denote by M_a and ${}_aM$ the sections $M(a,\cdot)$ and $M(\cdot,a)$, respectively. The function M is called a *mean* if $\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}$ for all x,y from its domain; M is called a *strict mean* if the above inequalities are strict for $x \neq y$. Let us recall here a well-known fact, which we will use several times without explicit mention: any separately monotonic and separately continuous two-place function is jointly continuous.

Proposition 1. *Let $K \subset \mathbb{R}$ be a compact interval. Let $M\colon K \times K \to K$ be a mean with continuous and strictly increasing sections. For every $d \in K$ define*

$$S(d) = \{t \in K : d \in M_t(K)\},$$
$$T(d) = \{(u,v,s) \in K^3 : M(u,v) \in S(d),\ s \in S(M^{-1}_{M(u,v)}(d))\}.$$

Then:

(i) $S(d) = \{d\}$ *implies that* $d \in \partial K$;

(ii) *there exist continuous increasing functions* $\lambda, \mu\colon K \to K$ *such that*

$$S(d) = [\lambda(d), \mu(d)], \qquad d \in K;$$

(iii) *if* $d \in \operatorname{int} K$, *then* $\lambda(d) < d < \mu(d)$;
(iv) *for every* $d \in K$ *the function* $S(d) \ni x \mapsto M_x^{-1}(d)$ *is continuous*;
(v) *for every* $d \in K$ *the set* $T(d)$ *is connected.*

Proof. (i) Observe first that we always have $d \in S(d)$. Assume that no other point belongs to $S(d)$ and suppose on the contrary that d lies in the interior of K. Choose any points $\xi_1 < d < \xi_2$ in K. Since $M_d(d) = d$ and the sections are strictly increasing, $M_d(\xi_1) < d < M_d(\xi_2)$. By the continuity of M, we may therefore find an open interval U whose points ξ satisfy $M(\xi, \xi_1) < d < M(\xi, \xi_2)$. Now, by the Darboux property of M_ξ, we have $d \in M_\xi(K)$, which implies that $\xi \in S(d)$. This yields $U \subset S(d)$; a contradiction.

(ii) The set $S(d)$ is non-empty (recall $d \in S(d)$) and closed. Indeed, if $(t_n)_{n\in\mathbb{N}}$ is a sequence of points of $S(d)$ which converges to t_0, we have

$$M(t_n, u_n) = d, \qquad n \in \mathbb{N},$$

for suitable $u_n \in K$, $n \in \mathbb{N}$. Then, choosing a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ which converges to some u_0, and passing to the limit, we obtain

$$M(t_0, u_0) = d.$$

Hence $d \in M_{t_0}(K)$, which means that $t_0 \in S(d)$.

To show that $S(d)$ is an interval suppose that $x \le y \le z$ and $x, z \in S(d)$. Then there are some $x', z' \in K$ such that

$$M(x, x') = d = M(z, z').$$

In view of the inequalities

$$M_y(x') = M(y, x') \ge M(x, x') = d,$$
$$M_y(z') = M(y, z') \le M(z, z') = d,$$

and the continuity of M_y, we conclude that the function M_y attains the value d, which means that $y \in S(d)$.

Consequently, for every $d \in K$ we may write $S(d) = [\lambda(d), \mu(d)]$. To prove monotonicity of λ and μ, assume that $d_1, d_2 \in K$, $d_1 < d_2$, and fix any $t \in K$ with $t < \lambda(d_1)$. If such a t does not exist then the inequality

$$\lambda(d_1) \le \lambda(d_2) \tag{2.1}$$

holds trivially; otherwise $t < \lambda(d_1) \le d_1$ and

$$M_t(d_1) = M(t, d_1) < d_1.$$

Since the section M_t does not attain the value d_1 (recall that $t \notin S(d_1)$), we must have $M_t(x) < d_1$ for all $x \in K$, whence $M_t(x) < d_2$ for all $x \in K$, which guarantees that $t \notin S(d_2)$. We have just shown that any number less than $\lambda(d_1)$ fails to fall into the interval $[\lambda(d_2), \mu(d_2)]$, which proves (2.1).

To show that μ is increasing assume again that $d_1, d_2 \in K$, $d_1 < d_2$, and fix any $t \in K$ with $t > \mu(d_2)$. If such a t does not exist then we trivially have

$$\mu(d_1) \le \mu(d_2). \tag{2.2}$$

In the other case $d_2 \le \mu(d_2) < t$ and

$$M_t(d_2) = M(t, d_2) > d_2,$$

hence $M_t(x) > d_1$ for all $x \in K$, and $t \notin S(d_1)$. Since any number greater than $\mu(d_2)$ fails to fall into the interval $[\lambda(d_1), \mu(d_1)]$, inequality (2.2) holds true.

Now we are going to show that μ is a left-continuous function. Suppose the contrary: there is an increasing sequence $(d_n)_{n\in\mathbb{N}}$ of numbers from K which converges to $d_0 \in K$ and

$$S(d_n) \subset [\lambda(d_n), \gamma], \qquad n \in \mathbb{N}, \tag{2.3}$$

for some $\gamma \in K$ with $\gamma < \mu(d_0)$. Let us distinguish two possible cases.

If $\lambda(d_0) = \mu(d_0) = d_0$ then, in the light of assertion (i), d_0 has to be the right end-point of the interval K. Since $M(d_0, d_0) = d_0$, there exist $\xi \in (\gamma, d_0)$ and $n \in \mathbb{N}$ such that $M(\xi, d_0) = d_n$. This means that $\xi \in S(d_n)$ and contradicts (2.3).

If $\lambda(d_0) < \mu(d_0)$ we may choose

$$\xi \in (\max\{\gamma, \lambda(d_0)\}, \mu(d_0)).$$

Since $\xi \in S(d_0)$ and $\xi \notin S(d_n)$ for $n \in \mathbb{N}$, the section M_ξ attains the value d_0, whereas it fails to attain any of the values d_n ($n \in \mathbb{N}$). This forces the function M_ξ to be constantly greater than or equal to d_0. Because M_ξ is increasing and $d_0 \in M_\xi(K)$, it follows that $M_\xi(\inf K) = d_0$. We have thus shown that ${}_{\inf K}M(\xi) = d_0$ for every ξ from a non-degenerate interval, which contradicts the fact that the section ${}_{\inf K}M$ is strictly increasing.

For the right continuity of the function μ suppose that there exists a decreasing sequence $(d_n)_{n\in\mathbb{N}}$ of numbers from K convergent to $d_0 \in K$ such that

$$[\lambda(d_n), \gamma] \subset S(d_n), \qquad n \in \mathbb{N},$$

for some $\gamma \in K$ with $\gamma > \mu(d_0)$. Since $S(d_n) = \{d_n\}$ is possible exclusively when $d_n \in \partial K$ (see (i)), without loss of generality we may assume that $\lambda(d_1) < \gamma$ and hence $\lambda(d_n) < \gamma$ for all $n \in \mathbb{N}$ (recall that $(\lambda(d_n))_{n\in\mathbb{N}}$ is decreasing). Choose

$$\xi \in (\max\{\lambda(d_1), \mu(d_0)\}, \gamma).$$

Then obviously $\xi \in S(d_n)$ for all $n \in \mathbb{N}$ and the section M_ξ attains all of the values $\{d_n : n \in \mathbb{N}\}$. The compactness of K implies that M_ξ attains the value d_0 as well. This, however, contradicts the fact that $\xi \notin S(d_0)$.

The continuity of the function λ may be obtained in an analogous way.

(iii) Assume that $d \in \operatorname{int} K$. In order to prove that $\lambda(d) < d$ it is enough to show that there exists $u \in S(d)$ such that $u < d$.

If for every $v \in K$ with $v > d$ the section ${}_vM$ were everywhere strictly greater than d, then we would have ${}_vM(\inf K) = M(\inf K, v) > d$. Letting $v \to d^+$ we obtain $M(\inf K, d) \geq d$, which leads to a contradiction:

$$d = M(d,d) > M(\inf K, d) \geq d.$$

We have thus proved that there exist $v \in K$ with $v > d$ and $w \in K$ such that $M(w,v) \leq d$. On the other hand, $M(d,v) > d$; hence we can find $u \in K \cap [w,d)$ satisfying $M(u,v) = d$. This means exactly that $u \in S(d)$.

The inequality $d < \mu(d)$ may be proved analogously.

(iv) Fix a sequence $(x_n)_{n\in\mathbb{N}}$ of elements from $S(d)$ which converges to $x_0 \in S(d)$. Let $y_0 \in K$ be the point for which $M(x_0, y_0) = d$. For every $n \in \mathbb{N}$ there exists (exactly one) $y_n \in I$ such that $M(x_n, y_n) = d$. Our aim is to prove that $y_n \to y_0$ as $n \to \infty$.

Choose a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ which converges to some $y \in K$. Letting $k \to \infty$ in the equation

$$M(x_{n_k}, y_{n_k}) = d$$

we obtain $M(x_0, y) = d$, which yields $y = y_0$.

(v) Fix $(u', v', s'), (u'', v'', s'') \in T(d)$. If $(u', v') = (u'', v'')$, then

$$S(M^{-1}_{M(u',v')}(d)) = S(M^{-1}_{M(u'',v'')}(d))$$

and since the latter set is an interval, the points (u', v', s') and (u'', v'', s'') may be connected in $T(d)$ simply by a vertical segment. From now on, we assume that $(u', v') \neq (u'', v'')$.

Assume, without loss of generality, that $M(u', v') \leq M(u'', v'')$; then either $u' \leq u''$ or $v' \leq v''$. Suppose that the first inequality holds true (the other case may be treated analogously). Now, we shall consider two possibilities.

First, assume $v' > v''$. Since ${}_{v'}M(u') \leq M(u'', v'')$ and ${}_{v'}M(u'') > M(u'', v'')$, there exists $t_0 \in [u', u'']$ such that

$$M(t_0, v') = M(u'', v''). \tag{2.4}$$

Put

$$\gamma(t) = (t, v') \quad \text{for } t \in [u', t_0].$$

One sees immediately that $M \circ \gamma$ is increasing in $[u', t_0]$ and

$$M(\gamma(t_0)) = M(u'', v''), \quad M(\gamma(u')) = M(u', v').$$

For an arbitrary $t \in (t_0, u'']$ (if any exists) consider the section $M_t|_{[v'',v']}$. Since

$$M_t(v'') \leq M(u'', v'') \quad \text{and} \quad M_t(v') = M(t, v') > M(t_0, v') = M(u'', v''),$$

there exists $\xi_t \in [v'', v']$ such that

$$M_t(\xi_t) = M(u'', v''), \quad \text{i.e.} \quad \xi_t = M_t^{-1}(M(u'', v'')).$$

Define

$$\gamma(t) = \xi_t \qquad \text{for } t \in (t_0, u''].$$

We have thus defined a function $\gamma \colon [u', u''] \to K^2$. The inclusion $(t_0, u''] \subset S(M(u'', v''))$ jointly with (iv) gives the continuity of γ in $(t_0, u'']$. Its continuity in $[u', t_0)$ is obvious, whereas the continuity at t_0 may be checked directly. Indeed, equation (2.4) implies that $t_0 \in S(M(u'', v''))$ and

$$M_{t_0}^{-1}(M(u'', v'')) = v'.$$

Hence, once again making use of (iii), we infer that

$$\lim_{t \to t_0+} \gamma(t) = (t_0, v').$$

Obviously,

$$\lim_{t \to t_0-} \gamma(t) = (t_0, v').$$

The function $M \circ \gamma \colon [u', u''] \to K$ is continuous. Further, since it is increasing in $[u', t_0]$ (as mentioned before) and constantly equal to $M(u'', v'')$ in $[t_0, u'']$ (which follows directly from the definition), we infer that it increases on $[u', u'']$.

If we recall that $M(u', v') \leq M(u'', v'')$ and we are working under the assumption $v' > v''$, we see that necessarily $u' < u''$. Consider any increasing homeomorphism $\varphi \colon [0, 1] \to [u', u'']$ and define a function $\widetilde{\gamma} \colon [0, 1] \to K^2$ as

$$\widetilde{\gamma} = \gamma \circ \varphi.$$

This is a continuous curve in K^2 such that

$$\widetilde{\gamma}(0) = \gamma(u') = (u', v'), \tag{2.5}$$

$$\widetilde{\gamma}(1) = \gamma(u'') = (u'', v''). \tag{2.6}$$

Moreover, since $M \circ \gamma$ is increasing, the function

$$M \circ \widetilde{\gamma} = (M \circ \gamma) \circ \varphi$$

is increasing as well.

Now, assume that $v' \leq v''$. This case is much simpler. Indeed, define $\gamma \colon [v', v'' + u'' - u'] \to K^2$ as follows:

$$\gamma(t) = \begin{cases} (u', t) & \text{if } t \in [v', v''], \\ (t - v'' + u', v'') & \text{if } t \in (v'', v'' + u'' - u']. \end{cases}$$

It is immediately seen that γ is continuous and $M \circ \gamma$ is increasing in $J := [v', v'' + u'' - u']$. Since $(u', v') \neq (u'', v'')$, as assumed at the very beginning of the proof, the interval J is non-degenerate and we may consider any increasing homeomorphism $\varphi\colon [0,1] \to J$. It remains to define $\widetilde{\gamma}\colon [0,1] \to K^2$ as $\widetilde{\gamma} = \gamma \circ \varphi$ to obtain a continuous curve $\widetilde{\gamma}$ satisfying conditions (2.5) and (2.6) and such that the function $M \circ \widetilde{\gamma}$ is increasing.

Thus, in any case we have

$$M(\widetilde{\gamma}(0)) = M(u', v') \in S(d), \qquad M(\widetilde{\gamma}(1)) = M(u'', v'') \in S(d).$$

As $M \circ \widetilde{\gamma}$ is increasing and $S(d)$ is an interval, we have

$$M(\widetilde{\gamma}(t)) \in S(d) \quad \text{for all } t \in [0,1], \tag{2.7}$$

whence we can define

$$[0,1] \ni t \mapsto \theta(t) = M^{-1}_{M(\widetilde{\gamma}(t))}(d).$$

By (iii), the function θ is continuous. Write

$$S(\theta(t)) = [\widetilde{\lambda}(t), \widetilde{\mu}(t)],$$

where $\widetilde{\lambda} := \lambda \circ \theta$ and $\widetilde{\mu} := \mu \circ \theta$ with λ and μ as in assertion (ii). By the same assertion, $\widetilde{\lambda}$ and $\widetilde{\mu}$ are both continuous, and obviously $\widetilde{\lambda} \leq \widetilde{\mu}$. By (2.5) and (2.6), we have $s' \in S(\theta(0))$ and $s'' \in S(\theta(1))$; in other words,

$$\widetilde{\lambda}(0) \leq s' \leq \widetilde{\mu}(0), \qquad \widetilde{\lambda}(1) \leq s'' \leq \widetilde{\mu}(1).$$

This allows one to define a continuous function $\psi\colon [0,1] \to K$ such that $\psi(0) = s'$, $\psi(1) = s''$ and $\widetilde{\lambda} \leq \psi \leq \widetilde{\mu}$. Then

$$\psi(t) \in S(M^{-1}_{M(\widetilde{\gamma}(t))}(d)) \quad \text{for all } t \in [0,1].$$

This, jointly with condition (2.7), guarantees that the function

$$[0,1] \ni t \mapsto (\widetilde{\gamma}(t), \psi(t))$$

has values in the set $T(d)$. Moreover, it is a continuous function whose values at 0 and 1 equal (u', v', s') and (u'', v'', s''), respectively. ∎

PROPOSITION 2. *Let $I \subset \mathbb{R}$ be an interval and let $M\colon I \times I \to I$ be a mean. If for every compact interval $K \subset I$ the restriction $M|_{K \times K}$ is a quasi-arithmetic mean, then M is quasi-arithmetic.*

Proof. Choose a sequence $(K_n)_{n \in \mathbb{N}}$ of compact subintervals of I such that $K_n \subset K_{n+1}$ for $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} K_n = I$. For every $n \in \mathbb{N}$ let $\varphi_n\colon K_n \to \mathbb{R}$ stand for a continuous and strictly increasing generator of the restriction $M|_{K_n \times K_n}$. We define functions $\psi_n\colon K_n \to \mathbb{R}$, for $n \in \mathbb{N}$, inductively as follows.

Let $\psi_1 := \varphi_1$. Fix $n \geq 2$ and assume we have already defined continuous and strictly increasing generators $\psi_1\colon K_1 \to \mathbb{R}, \ldots, \psi_{n-1}\colon K_{n-1} \to \mathbb{R}$ of the restrictions $M|_{K_1\times K_1},\dots,M|_{K_{n-1}\times K_{n-1}}$, respectively, in such a way that

$$\psi_j|_{K_{j-1}} = \psi_{j-1} \quad \text{for all } j,\ 2\le j\le n-1.$$

The function φ_n generates the mean M in the interval K_{n-1}, as well as in K_n. Since both ψ_{n-1} and $\varphi_n|_{K_{n-1}}$ are continuous and strictly increasing generators of $M|_{K_{n-1}}$, there exist $a,b\in\mathbb{R}$ with $a>0$ such that

$$\psi_{n-1} = a\varphi_n|_{K_{n-1}} + b.$$

On the other hand, the function $\psi_n\colon K_n\to\mathbb{R}$ defined by

$$\psi_n = a\varphi_n + b$$

is a continuous and strictly increasing generator of $M|_{K_n}$, as is φ_n. This finishes the inductive process of constructing the sequence $(\psi_n)_{n\in\mathbb{N}}$ of continuous and strictly increasing mappings generating the sequence of restrictions $(M|_{K_n})_{n\in\mathbb{N}}$. Furthermore, we have

$$\psi_{n+1}|_{K_n} = \psi_n \quad \text{for all } n\in\mathbb{N}. \tag{2.8}$$

It remains to define a mapping $\varphi\colon I\to\mathbb{R}$ by putting

$$\varphi(x) = \psi_n(x) \quad \text{for } x\in I,$$

where $n\in\mathbb{N}$ is such that $x\in K_n$. It is immediate, in view of (2.8), that the definition is correct. It is also straightforward that φ is a continuous and strictly increasing mapping which generates the mean M in the whole interval I. ∎

Proposition 3. *Let $I\subset\mathbb{R}$ be an interval and let $M: I\times I\to I$ be a continuous and strict mean. Assume that a set $E\subset I$ has the property that $M(c,d)\in E$ whenever $c,d\in E$. If $a,b\in E$, then E is dense in $[a,b]$.*

Proof. Suppose the contrary: there exists an open interval $(\alpha,\beta)\subset[a,b]$ such that $E\cap(\alpha,\beta)=\emptyset$. We may assume that (α,β) is maximal in the sense that its endpoints belong to the closure of E. Thus, there exist sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of elements from E such that $a_n\le\alpha$, $b_n\ge\beta$, $n\in\mathbb{N}$, which converge to α and β, respectively. Plainly, $M(a_n,b_n)\to M(\alpha,\beta)\in(\alpha,\beta)$. Consequently, $E\ni M(a_n,b_n)\in(\alpha,\beta)$ for sufficiently large $n\in\mathbb{N}$; a contradiction. ∎

3. Main results. Now, we are in a position to establish the results exhibiting the inconsistency of quasi-arithmetic and non-quasi-arithmetic means.

Theorem 1. *Let $I,J\subset\mathbb{R}$ be intervals, let $A_\varphi\colon J\times J\to J$ be a quasi-arithmetic mean and $M\colon I\times I\to I$ be a mean with continuous and strictly increasing sections. If there exists a non-constant solution $f\colon I\to J$ of the* *functional equation*

$$f(M(x,y)) = A_\varphi(f(x), f(y)), \qquad x, y \in I, \tag{3.1}$$

then M is quasi-arithmetic.

Proof. If $(K_n)_{n\in\mathbb{N}}$ is any increasing sequence of compact subintervals of I such that $\bigcup_{n\in\mathbb{N}} K_n = I$, then $f|_{K_n}$ is non-constant for sufficiently large $n \in \mathbb{N}$. Therefore, in the light of Proposition 2, it is enough to prove our assertion for I being compact.

Equation (3.1) may be rewritten in the form

$$(\varphi \circ f)(M(x,y)) = \frac{(\varphi\circ f)(x) + (\varphi \circ f)(y)}{2}, \qquad x, y \in I,$$

which states that the non-constant function $g := \varphi \circ f$ satisfies

$$g(M(x,y)) = \frac{g(x)+g(y)}{2}, \qquad x, y \in I. \tag{3.2}$$

Fix $u, v, s, t \in I$ and note that

$$\begin{aligned} g(M(M(u,v), M(s,t))) &= \frac{g(M(u,v)) + g(M(s,t))}{2} \\ &= \frac{g(u)+g(v)+g(s)+g(t)}{4}. \end{aligned}$$

By symmetry of the right-hand side with respect to v and s, we infer that

$$g(M(M(u,v), M(s,t))) = g(M(M(u,s), M(v,t))). \tag{3.3}$$

In what follows, we show that g is *philandering* (i.e. non-constant on any non-degenerate interval). If g were constantly equal to some b on $(c,d) \subset I$, where $c < d$, then we could choose a maximal interval (C,D) containing (c,d) and such that $g|_{(C,D)} = b$. We shall prove that $C = \inf I$ and $D = \sup I$.

Suppose that $\inf I < C$. We first prove that

$$\bigvee_{u\in(C,D)} \ \bigvee_{y\in S(u)\cap(C,D)} (M_y^{-1}(u) < C). \tag{3.4}$$

Suppose the contrary: $u \geq M_y(C)$ for any $u \in (C,D)$ and $y \in S(u) \cap (C,D)$, which is equivalent to the inequality

$$u \geq M(\sup[S(u)\cap(C,D)], C). \tag{3.5}$$

Plainly,

$$\sup[S(u)\cap(C,D)] = \min\{\mu(u), D\}. \tag{3.6}$$

By Proposition 1(ii), the function μ is continuous, hence the function given by (3.6) is continuous as well. Since $\inf I < C$ we may let $u \to C+$ in (3.5) to obtain

$$\begin{aligned} C &\geq \lim_{u \to C+} M(\sup[S(u) \cap (C, D)], C) = \lim_{u \to C+} M(\min\{\mu(u), D\}, C) \\ &= M(\min\{\lim_{u \to C+} \mu(u), D\}, C) = M(\min\{\mu(C), D\}, C) > C; \end{aligned}$$

a contradiction (in the last inequality we apply Proposition 1(iii) and the fact that $C \in \operatorname{int} I$). Thus, (3.4) has been proved.

Choose any u and y as in (3.4) and put

$$K = S(u) \cap (C, D).$$

Observe that K is an interval. By Proposition 1(iv), the function

$$K \ni x \mapsto \psi(x) := M_x^{-1}(u)$$

is continuous, so $\psi(K)$ is an interval. Since $\psi(u) = u$, and hence

$$\psi(K) \cap (C, D) \neq \emptyset,$$

the set $\psi(K) \cup (C, D)$ is an interval. Condition (3.4) guarantees that this interval also strictly contains (C, D). Now, we show that g is constantly b on this interval. Since $g|_{(C,D)} = b$, only arguments from $\psi(K)$ need to be considered. Fix any $z \in \psi(K)$, $z = M_x^{-1}(u)$ for some $x \in K$. We have

$$\begin{aligned} b &= g(u) = g(M(x, M_x^{-1}(u))) = g(M(x, z)) \\ &= \frac{g(x) + g(z)}{2} = \frac{b + g(z)}{2}, \end{aligned}$$

which implies that $g(z) = b$. This contradicts the maximality of (C, D).

The proof of $D = \sup I$ is analogous. The equality $g|_{\operatorname{int} I} = b$ together with (3.2) easily implies that g is constant, $g = b$.

Summarizing, any non-constant solution g of equation (3.2) is necessarily philandering.

Setting, for brevity, $u * v = M(u, v)$ for $u, v \in I$ put

$$\begin{aligned} H_1(u, v, s, t) &= (u * v) * (s * t), \qquad u, v, s, t \in I, \\ H_2(u, v, s, t) &= (u * s) * (v * t), \qquad u, v, s, t \in I. \end{aligned}$$

Fix $d \in I$. A straightforward calculation shows that

$$H_1^{-1}(\{d\}) = \{(u, v, s, t) \in I^4 : (u, v, s) \in T(d),\, t = (M_s^{-1} \circ M_{u*v}^{-1})(d)\}.$$

This means that $H_1^{-1}(\{d\})$ is the graph of the function

$$T(d) \ni (u, v, s) \mapsto (M_s^{-1} \circ M_{u*v}^{-1})(d),$$

which, in view of Proposition 1(iv), (v), is continuous and defined on a connected domain. Hence, $H_1^{-1}(\{d\})$ is a connected subset of I^4.

If the continuous function H_2 attained at least two different values on the connected set $H_1^{-1}(\{d\})$, then it would attain there all values from some non-degenerate interval. Equality (3.3), which states precisely that

$$g(H_1(u,v,s,t)) = g(H_2(u,v,s,t)), \qquad (u,v,s,t) \in I^4,$$

would then imply that g is constant on that interval, which contradicts the fact that g is philandering. Consequently, there exists a constant $v(d)$ such that $H_1^{-1}(\{d\}) \subset H_2^{-1}(\{v(d)\})$. Similarly, there exists a constant $w(d)$ such that $H_2^{-1}(\{v(d)\}) \subset H_1^{-1}(\{w(d)\})$. Thus,

$$H_1^{-1}(\{d\}) \subset H_2^{-1}(\{v(d)\}) \subset H_1^{-1}(\{w(d)\}),$$

whence obviously $w(d) = d$ and

$$H_1^{-1}(\{d\}) = H_2^{-1}(\{v(d)\}).$$

However, this means that H_2 is a function of H_1, that is, $H_2 = \gamma \circ H_1$ with some $\gamma \colon I \to I$.

For every $d \in I$ one has

$$\begin{aligned} d &= (d*d)*(d*d) = H_2(d,d,d,d) = \gamma(H_1(d,d,d,d)) = \gamma((d*d)*(d*d)) \\ &= \gamma(d), \end{aligned}$$

i.e. γ is the identity function. This implies that $H_1 = H_2$. In other words, the operation $* = M$ satisfies the bisymmetry equation. Since (3.2) gives also the equality

$$g(u*v) = g(v*u), \qquad u,v \in I,$$

a reasoning similar to the one above leads to the symmetry of $*$. This, jointly with $M(u,u) \equiv u$ and the injectivity of the sections M_a, ${}_aM$ for $a \in I$, shows that M satisfies all the assumptions of the celebrated theorem of Aczél (see, e.g., J. Aczél & J. Dhombres [2, p. 287]) which asserts that M is then quasi-arithmetic. ∎

THEOREM 2. *Let $I, J \subset \mathbb{R}$ be intervals, let $A_\varphi \colon I \times I \to I$ be a quasi-arithmetic mean and $M \colon J \times J \to J$ be a mean with continuous and strictly increasing sections. If $f \colon I \to J$ is a non-constant solution of the functional equation*

$$f(A_\varphi(x,y)) = M(f(x), f(y)), \qquad x,y \in I, \tag{3.7}$$

then M is quasi-arithmetic on the interval $(\inf f(I), \sup f(I))$. *In particular, if M is not quasi-arithmetic, then there are no surjective solutions of* (3.7), *and if for every non-degenerate subinterval P of J the restriction $M|_{P\times P}$ is non-quasi-arithmetic, then the only solutions of* (3.7) *are the constant ones.*

Proof. Let $f \colon I \to J$ be a non-constant solution to (3.7) with $\alpha := \inf f(I)$ and $\beta := \sup f(I)$. Then $\alpha < \beta$ and

$$f \circ \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) = M(f(x), f(y)), \qquad x,y \in I,$$

which implies that $g := f \circ \varphi^{-1} : \varphi(I) \to J$ solves the functional equation

$$g\left(\frac{u+v}{2}\right) = M(g(u), g(v)), \qquad u, v \in \varphi(I). \tag{3.8}$$

Moreover, since φ^{-1} maps $\varphi(I)$ onto I, we get $\inf g(\varphi(I)) = \alpha < \beta = \sup g(\varphi(I))$. Fix $u, v, s, t \in \varphi(I)$ and note that (3.8) yields

$$g\left(\frac{u+v+s+t}{4}\right) = M(M(g(u), g(v)), M(g(s), g(t))).$$

By symmetry of the left-hand side with respect to v and s, we infer that

$$M(M(g(u), g(v)), M(g(s), g(t))) = M(M(g(u), g(s)), M(g(v), g(t))).$$

Therefore, M satisfies the bisymmetry equation

$$M(M(a, b)), M(c, d)) = M(M(a, c)), M(b, d)) \tag{3.9}$$

for $a, b, c, d \in g(\varphi(I))$.

Let $[\gamma, \delta] \subset (\alpha, \beta)$ be any compact interval. Then there exist $u_0, v_0 \in \varphi(I)$ such that $g(u_0) < \gamma \leq \delta < g(v_0)$. Equation (3.8), jointly with Proposition 3 applied for $E := g(\varphi(I))$, implies that $g(\varphi(I))$ is dense in $[g(u_0), g(v_0)]$. By the arbitrariness of $[\gamma, \delta]$, the set $g(\varphi(I))$ is dense in (α, β). Thus, by virtue of equation (3.9) and the continuity of M, we infer that M satisfies the bisymmetry equation on (α, β). Likewise, (3.7) gives the symmetry of M as well as the identity $M(u, u) \equiv u$ (which also results from the fact that M is a mean). Thus, it remains to apply Aczél's theorem quoted above to conclude the proof. ■

4. Concluding remarks. The basic result of P. Kahlig and J. Matkowski [5] states that given a non-empty open interval $I \subset (0, \infty)$, any solution $f : I \to \mathbb{R}$ of the equation

$$f(L(x, y)) = \frac{f(x) + f(y)}{2}, \qquad x, y, \in I$$

continuous at least at one point is necessarily constant. This becomes a special case of our Theorem 1 (for $M = L|_{I \times I}$ and φ being the identity function). Moreover, we need no regularity assumption on f; in particular, the requirement of continuity at a point turns out to be unnecessary.

Three of the four possible situations concerning equation $(*)$, namely: both M and N are quasi-arithmetic, N is quasi-arithmetic but M is not, M is quasi-arithmetic but N is not, have thus been examined in detail. The case where both M and N are non-quasi-arithmetic is the only missing one.

The main result of J. Matkowski's paper [7, Theorem 2] states that if $M = N = L$, $I = (0, \infty)$ and $f : (0, \infty) \to (0, \infty)$ is a solution of the functional equation $f(L(x, y)) = L(f(x), f(y))$, $x, y \in I$, admitting a point of continuity, then f is either constant or linear. This shows that the remaining situation is qualitatively dissimilar to the previous ones even in the case where $M = N$ and, at present, the behavior of the corresponding solutions (even satisfying some regularity conditions) is hardly predictable. Therefore, we have left that case untouched for the time being.

A slight asymmetry between the assertions of Theorems 1 and 2 results from the fact that equation (3.7) says nothing about the values of the (potentially) non-quasi-arithmetic mean M off the rectangle $[\inf f(I), \sup f(I)]^2$. Therefore, the inconsistency phenomenon occurs if for every subrectangle R of the domain of M the mean $M|_{R\times R}$ fails to be quasi-arithmetic.

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