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ON AREA AND SIDE LENGTHS OF TRIANGLES IN NORMED PLANES

BY

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Abstract. Let $\mathcal{M}^d$ be a d-dimensional normed space with norm $\|\cdot\|$ and let B be the unit ball in $\mathcal{M}^d$. Let us fix a Lebesgue measure V_B in $\mathcal{M}^d$ with $V_B(B) = 1$. This measure will play the role of the volume in $\mathcal{M}^d$. We consider an arbitrary simplex T in $\mathcal{M}^d$ with prescribed edge lengths. For the case $d = 2$, sharp upper and lower bounds of $V_B(T)$ are determined. For $d \geq 3$ it is noticed that the tight lower bound of $V_B(T)$ is zero.

1. Introduction and results. A subset K of $\mathbb{R}^d$ is said to be a *convex body* if it is convex, compact, and has non-empty interior. The convex and affine hull operations are abbreviated by conv and aff, respectively. Let o denote the origin in $\mathbb{R}^d$. It is known that the class of o-symmetric convex bodies in $\mathbb{R}^d$ is isomorphic to the class of all Euclidean norms in $\mathbb{R}^d$. In fact, if $\|\cdot\|$ is a norm in $\mathbb{R}^d$, then the unit ball $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$ with respect to $\|\cdot\|$ is an o-symmetric convex body. Vice versa, for every o-symmetric convex body B in $\mathbb{R}^d$ the functional

$$\|x\|_B := \min\{\alpha \geq 0 : x \in \alpha B\}$$

is a norm. In what follows, we consider an arbitrary normed space $\mathcal{M}^d$ viewed as the space $\mathbb{R}^d$ endowed with some norm $\|\cdot\|$; see [10], [7], and [6]. In the literature also the term *Minkowski space* is used for $\mathcal{M}^d$. The notation B is reserved for the *unit ball* with respect to $\|\cdot\|$. We fix a Lebesgue measure V_B in $\mathbb{R}^d$ such that $V_B(B) = 1$. Clearly, $V_B(\,\cdot\,) := V(\,\cdot\,)/V(B)$, where V is any Lebesgue measure in $\mathbb{R}^d$. Let us call $V_B(\,\cdot\,)$ the *normalized volume* in $\mathcal{M}^d$ (and for $d = 2$ the *normalized area*). Below, all metric notions referring to distance (e.g. length, diameter, perimeter etc.) will be considered with respect to the norm of $\mathcal{M}^d$.

In this paper we determine the complete system of inequalities for normalized area and side lengths of a triangle in a *normed plane* (i.e., a normed space of dimension two).

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Theorem 1. *For an arbitrary normed plane* $\mathcal{M}^2$ *and every triangle in* $\mathcal{M}^2$ *with side lengths* $0 < a_1 \le a_2 \le a_3$ *and area* A *we have*

$$\frac{1}{8}\, a_3(a_1 + a_2 - a_3) \le A \le \frac{a_1 a_2 a_3}{2(a_1 + a_2 + a_3)}. \tag{1}$$

Conversely, for all a_1, a_2, a_3, *and* A *satisfying* $0 < a_1 \le a_2 \le a_3$, $a_1+a_2 \ge a_3$ *and* (1), *there exists a normed plane* $\mathcal{M}^2$ *and a* (*possibly degenerate*) *triangle in* $\mathcal{M}^2$ *which has area* A *and side lengths* a_1, a_2, a_3. ∎

Theorem 1 extends the result from [12] providing tight bounds for the area of an equilateral triangle in a normed plane and characterizing the extremal cases. See also [4] and [10, Lemma 4.2.6], where such bounds are used in a short proof of a generalization of the *Blaschke–Lebesgue theorem* for normed planes. A theorem analogous to Theorem 1 and dealing with side lengths and circumradius of a triangle in a normed plane was obtained in [2]. See also [1] for results on inradii and side lengths of triangles in normed planes. For further results on the geometry of finite point sets in Euclidean and non-Euclidean spaces we refer to [8], [3], and [7].

The equality cases in (1) are described in the following

Proposition 2. *Let* $\mathcal{M}^2(B)$ *be a normed plane, let* p_1, p_2, p_3 *be pairwise distinct points in* $\mathcal{M}^2(B)$, *and* $T := \operatorname{conv}\{p_1, p_2, p_3\}$. *Set*

$$a_1 := \|p_2 - p_3\|, \qquad a_2 := \|p_3 - p_1\|, \qquad a_3 := \|p_1 - p_2\|, \tag{2}$$

$$A := V_B(T), \tag{3}$$

and assume that $a_1 \le a_2 \le a_3$. *Then*:

(i) *Equality on the right hand side of* (1) *is attained if and only if* p_1, p_2, p_3 *are not collinear and* $B = B_0$, *where*

$$B_0 := \operatorname{conv}\{\pm b_1, \pm b_2, \pm b_3\} \tag{4}$$

and

$$b_1 := (p_2 - p_3)/a_1, \quad b_2 := (p_3 - p_1)/a_2, \quad b_3 := (p_1 - p_2)/a_3 \tag{5}$$

(*see Figs.* 1 *and* 2).

(ii) *If* $a_1 + a_2 > a_3$, *then* p_1, p_2, p_3 *are not collinear*, B_0 *is a convex hexagon, and equality on the left hand side of* (1) *is attained when* B *is the parallelogram with sides contained in the lines* $\pm \operatorname{aff}\{b_3, -b_1\}$ *and* $\pm \operatorname{aff}\{b_3, -b_2\}$, *which are spanned by sides of* B_0 (*see Figs.* 1 *and* 3).

(iii) *If* $a_1 + a_2 = a_3$, *then the left hand side of* (1) *is equal to zero and the equality* $A = 0$ *is attained if and only if* p_1, p_2, p_3 *are collinear.* ∎

Let $P_B(K)$ and $D_B(K)$ denote the *perimeter* and the *diameter*, respectively, of a planar convex set K with respect to the norm of $\mathcal{M}^2$. If K is de-

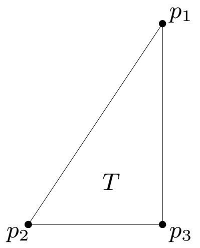

Fig. 1

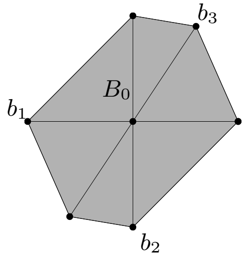

Fig. 2

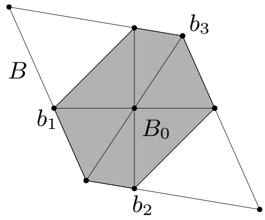

Fig. 3

generate to a segment, we define $P_B(K)$ as twice the length of the segment K. Using Theorem 1, we determine all possible triples $(P_B(T), D_B(T), V_B(T))$ such that $\mathcal{M}^2$ is an arbitrary normed plane and T is an arbitrary triangle in $\mathcal{M}^2$, which is possibly degenerate to a segment.

Corollary 3. *For an arbitrary normed plane $\mathcal{M}^2$ and every triangle in $\mathcal{M}^2$ with perimeter P, diameter D, and normalized area A we have*

$$2D \le P \le 3D, \qquad \frac{1}{8}\,D(P-2D) \le A \le \frac{(P-D)^2 D}{8P}. \tag{6}$$

Conversely, if $P > 0$, $D > 0$, and $A \ge 0$ satisfy (6), *then there exists a normed plane $\mathcal{M}^2$ and a (possibly degenerate) triangle in $\mathcal{M}^2$ with perimeter P, diameter D, and normalized area A.* ∎

Corollary 3 can also be represented in the form of inequalities for x and y given by

$$x := \frac{D_B(T)}{P_B(T)}, \qquad y := \frac{V_B(T)}{P_B(T)^2}.$$

Namely, we have

$$\tfrac{1}{3} \le x \le \tfrac{1}{2}, \qquad \tfrac{1}{8}x(1-2x) \le y \le \tfrac{1}{8}(1-x)^2 x.$$

The region consisting of all points (x, y) satisfying the above inequalities can obviously be used to describe all possible triples $(P_B(T), D_B(T), V_B(T))$. Analogous regions for systems of quantities associated to convex bodies in Euclidean space are called *Santaló diagrams*; see [9].

Now let us discuss possible extensions of the statement of Theorem 1 to higher dimensions. We consider $d+1$ pairwise distinct points $p_1, \ldots, p_{d+1}$ in $\mathcal{M}^d$ and define $T := \operatorname{conv}\{p_1, \ldots, p_{d+1}\}$. It turns out that for dimensions $d \ge 3$ the bounds of $V_B(T)$ in the case of prescribed distances within $\{p_1, \ldots, p_{d+1}\}$ have an essentially different form. The following proposition provides the lower bound. It partially extends the statement from [11] on the volume of an equilateral simplex in a finite-dimensional normed space.

Proposition 4. *Let ϱ be a metric on $\{1, \ldots, d+1\}$ with $d \ge 3$. Then there exists a normed space $\mathcal{M}^d$ and points $p_1, \ldots, p_{d+1}$ in $\mathcal{M}^d$ satisfying* *the equalities*

$$\|p_i - p_j\| = \varrho(i,j) \qquad (i,j = 1,\dots,d+1)$$

and such that $V_B(T) = 0$, *where* $T := \operatorname{conv}\{p_1,\dots,p_{d+1}\}$. ■

The above proposition is a straightforward corollary of the result from [13], which states that for $d \ge 2$ every metric space with $d+2$ elements can be isometrically embedded into $\mathcal{M}^d$ for the case when $\|\cdot\|$ is the *maximum norm* (that is, $\|x\| = \max\{|x_1|,\dots,|x_d|\}$, where $x_1,\dots,x_d$ are the coordinates of x).

Regarding the upper bound of $V_B(T)$ for the case $d \ge 3$ we remark that this bound must have a much more complicated form than its two-dimensional analogue. In fact, assume that T is a simplex in $\mathcal{M}^d$, $d \ge 2$, with vertices $p_1,\dots,p_{d+1}$ and edge lengths $\varrho(i,j) := \|p_i - p_j\|$, $1 \le i < j \le d+1$. Then it can be shown that $V_B(T) \le V_{B_0}(T)$, where $B_0 := \operatorname{conv}\{(p_i - p_j)/a_{ij} : 1 \le i < j \le d+1\}$, and equality is attained if and only if $B = B_0$. However, it seems that there exists no simple analytical expression for $V(B_0)$ in terms of the edge lengths of T when $d \ge 3$. In fact, if $d = 2$ then B_0 is either a hexagon or a parallelogram. In contrast to the planar case, for $d \ge 3$ the polytope B_0 can have many more combinatorial types, which depend on the choice of the distances $\varrho(i,j)$.

2. The proofs. In the proofs below we usually determine a finite-dimensional normed space by its unit ball rather than by its norm. For that purpose we introduce the notation $\mathcal{M}^2(B)$ for a normed plane with unit ball B. Assume that $\mathcal{A}$ is a non-singular affine transformation in $\mathbb{R}^2$. Let us define the linear transformation $\mathcal{A}_0(x) := \mathcal{A}(x) - \mathcal{A}(o)$ with $x \in \mathbb{R}^2$. As a direct consequence of the Mazur–Ulam theorem (see [10, Theorem 3.1.2]) we have

$$\|\mathcal{A}(x) - \mathcal{A}(y)\|_{\mathcal{A}_0(B)} = \|x - y\|_B \tag{7}$$

for every $x, y \in \mathbb{R}^2$. Furthermore,

$$V_{\mathcal{A}_0(B)}(\mathcal{A}(K)) = V_B(K) \tag{8}$$

for every convex body K in $\mathbb{R}^2$. A hexagon H is said to be *affine regular* if it is an affine image of a hexagon which is regular in the Euclidean sense. It can be easily verified that an o-symmetric hexagon H is affine regular if and only if for the triple h_1, h_2, h_3 of the alternating vertices of H one has $h_1 + h_2 + h_3 = o$.

PROPOSITION 5. *Let* $0 < a_1 \le a_2 \le a_3$ *and* $a_3 \le a_1 + a_2$. *Let* p_1, p_2, p_3 *be non-collinear points in* $\mathbb{R}^2$, *and* b_1, b_2, b_3 *and* B_0 *be defined as in Proposition* 2. *Then*:

(i) *If* $\mathcal{A}$ *is a non-singular affine transformation such that*

$$(9)\qquad \mathcal{A}(p_1) = (0, a_2)^\top, \quad \mathcal{A}(p_2) = (-a_1, 0)^\top, \quad \mathcal{A}(p_3) = (0,0)^\top,$$

then

$$(10)\qquad \mathcal{A}_0(b_1) = (-1, 0)^\top, \quad \mathcal{A}_0(b_2) = (0, -1)^\top, \quad \mathcal{A}_0(b_3) = (a_1/a_3, a_2/a_3)^\top.$$

In particular, B *is an affine image of*

$$\operatorname{conv}\{\pm(1,0)^\top, \pm(0,1)^\top, \pm(a_1/a_3, a_2/a_3)^\top\}.$$

(ii) *The polygon* B_0 *is a parallelogram if and only if* $a_3 = a_1 + a_2$.
(iii) *If* $a_3 < a_1 + a_2$, *then* B_0 *is a convex hexagon.*
(iv) *The polygon* B_0 *is an affine regular hexagon if and only if* $a_1 = a_2 = a_3$.

Proof. Part (i) can be verified by straightforward computations. The remaining parts are immediate consequences of (i). ∎

Let $H_1, \ldots, H_m$ be half-planes in $\mathbb{R}^2$ and let l_i be the boundary of H_i for $i = 1, \ldots, m$. If the polygonal region $P := H_1 \cap \cdots \cap H_m$ is uniquely determined by the lines $l_1, \ldots, l_m$, we denote it by $P(l_1, \ldots, l_m)$. For brevity we write $P(\pm l_1, \ldots, \pm l_m)$ rather than $P(l_1, \ldots, l_m, -l_1, \ldots, -l_m)$. In analytic expressions the elements of $\mathbb{R}^2$ are treated as column vectors. The following lemma is related to the result from [5] on parallelotopes of maximal volume contained in a given simplex.

LEMMA 6. *Let* l_1, l_2 *be distinct non-parallel lines in* $\mathbb{R}^2$ *and let* $p \in \mathbb{R}^2 \setminus (l_1 \cup l_2)$. *Let* l_3 *be a line containing* p, *intersecting both* l_1 *and* l_2 *and such that* P *is not the bisector of the side* $l_3 \cap P(l_1, l_2, l_3)$ *of* $P(l_1, l_2, l_3)$. *Then there exists a line* l_3' *obtained from* l_3 *by an arbitrarily small rotation around* p *such that the area of* $P(l_1, l_2, l_3')$ *is strictly smaller than the area of* $P(l_1, l_2, l_3)$.

Proof. The assertion of the lemma is invariant with respect to affine transformations. Hence we may assume that l_1, l_2 are coordinate axes and $p = (1,1)^\top$. Then, for an appropriate $t > 0$, the points $(0,0)^\top$, $(1+t, 0)^\top$, and $(0, 1+1/t)^\top$ are the vertices of $P(l_1, l_2, l_3)$. Thus the area of $P(l_1, l_2, l_3)$ is equal to $\frac{1}{2}(1+t)(1+1/t)$, and the statement of the lemma can be derived by computing the derivative of the above expression. ∎

LEMMA 7. *Let* H *be an o-symmetric hexagon in* $\mathbb{R}^2$ *and let* H_0 *be the hexagon whose vertices are the midpoints of the sides of* H. *Then* H_0 *is affine regular and* $V(H) = \frac{4}{3}V(H_0)$.

Proof. Let v_1, v_2, v_3 be consecutive vertices of H in the positive orientation. Then $\frac{1}{2}(v_1 + v_2)$, $\frac{1}{2}(v_3 - v_1)$, and $\frac{1}{2}(-v_2 - v_3)$ are alternating vertices of H_0, and their sum is equal to zero. Hence H_0 is affine regular. Let $\det(a, b)$ denote the determinant of a 2×2 matrix with columns a and b (in that order). We have

$$V(H) = 2(V(\mathrm{conv}\{o, v_1, v_2\}) + V(\mathrm{conv}\{o, v_2, v_3\}) + V(o, v_3, -v_1))$$
$$= \det(v_1, v_2) + \det(v_2, v_3) + \det(v_1, v_3)$$

and

$$V(H_0) = 6V(\mathrm{conv}\{o, \tfrac{1}{2}(v_1 + v_2), \tfrac{1}{2}(v_2 + v_3)\})$$
$$= \tfrac{3}{4}\det(v_1 + v_2, v_2 + v_3) = \tfrac{3}{4}(\det(v_1, v_2) + \det(v_2, v_3) + \det(v_1, v_3)).$$

Consequently, $V(H) = \frac{4}{3}V(H_0)$. ■

Now we are ready to prove the main theorem of the paper.

Proofs of Theorem 1 and Proposition 2. First we prove the main statement of Theorem 1 together with Proposition 2. Let $\mathcal{M}^2(B)$ be a normed plane and p_1, p_2, p_3 be pairwise distinct points in $\mathcal{M}^2(B)$ such that equalities (2) and (3) are satisfied and $a_1 \le a_2 \le a_3$. Let $T := \mathrm{conv}\{p_1, p_2, p_3\}$. Let us derive the right inequality of (1). For the proof we may assume that p_1, p_2, p_3 are not collinear. Then, by Proposition 5, B_0 is a parallelogram or a hexagon. We have $V(B) \ge V(B_0)$, where

$$\begin{aligned}(11)\quad V(B_0) &= 2(V(\mathrm{conv}\{o, b_1, -b_3\}) + V(\mathrm{conv}\{o, -b_3, b_2\}) \\ &\qquad + V(\mathrm{conv}\{o, b_2, -b_1\})) \\ &= 2\left(\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \frac{1}{a_3a_1}\right)V(T) = \frac{2(a_1 + a_2 + a_3)}{a_1a_2a_3}\,V(T).\end{aligned}$$

This yields the right inequality of (1) and Proposition 2(i). The proof of Proposition 2(iii) is straightforward.

Next we obtain Proposition 2(ii) and the left inequality of (1) for the case $a_3 < a_1 + a_2$. From now on we assume that $a_3 < a_1 + a_2$. Then p_1, p_2, p_3 are obviously not collinear, and so T is not degenerate to a segment. By Proposition 5(iii), B_0 is a hexagon. We denote by $\mathcal{B}'$ the class of all o-symmetric convex bodies B' satisfying

$$a_1 = \|p_2 - p_3\|_{B'}, \qquad a_2 = \|p_3 - p_1\|_{B'}, \qquad a_3 = \|p_1 - p_2\|_{B'}.$$

The class $\mathcal{B}'$ can obviously be described as the class of o-symmetric bodies B' for which B_0 is inscribed in B', that is, every vertex of B_0 is a boundary point of B'. Clearly, $B \in \mathcal{B}'$. In what follows we distinguish several cases depending on the properties of B and provide some $B' \in \mathcal{B}'$ satisfying $V(B') \ge V(B)$ or even $V(B') > V(B)$. The last two inequalities are obviously equivalent to the inequalities $V_{B'}(T) \le V_B(T)$ and $V_{B'}(T) < V_B(T)$, respectively. In such a way we single out a narrow subclass of bodies B' from $\mathcal{B}'$ which minimize $V_{B'}(T)$. The proof of the following claim is straightforward.

Claim 1. *For $i = 1, 2, 3$ let l_i be a supporting line of B at b_i. Then $B' := P(\pm l_1, \pm l_2, \pm l_3) \in \mathcal{B}'$ and $V(B') \geq V(B)$.*

In view of the above claim, we see that if $B \neq B' := P(\pm l_1, \pm l_2, \pm l_3)$ with l_1, l_2, l_3 as in the statement of the claim, then $V(B') > V(B)$, and so $\min_{B' \in \mathcal{B}'} V_B(T)$ is not attained at B. Hence we proceed with our considerations for the case $B = P(\pm l_1, \pm l_2, \pm l_3)$. Clearly, in this case B is a parallelogram or a hexagon.

The following two claims are concerned with the situation when every side of B contains precisely one vertex of B_0.

Claim 2. *Assume that every side of B contains precisely one vertex of B_0 and that there exists a side I of B and a vertex v of B_0 such that $v \in I$ and v does not bisect I. Then there exists $B' \in \mathcal{B}'$ such that $V(B') > V(B)$.*

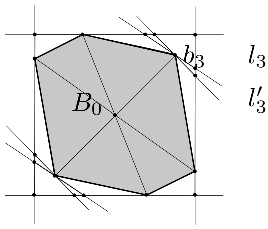

Fig. 4

Without loss of generality we assume that $v = b_3$. Then, by Lemma 6, there exists a line l'_3 obtained from l_3 by a slight rotation around the point b_3 such that the area of the triangle $P(l'_3, -l_1, -l_2)$ is strictly smaller than the area of the triangle $P(l_3, -l_1, -l_2)$. Then we can set $B' := P(\pm l_1, \pm l_2, \pm l'_3)$; see Fig. 4.

Claim 3. *Assume that every side of B contains precisely one vertex of B_0 and every side of B is bisected by a vertex of B_0. Then $a := a_1 = a_2 = a_3$ and $V_B(T) = \frac{1}{8}a^2$.*

By Lemma 7, B_0 is an affine regular hexagon and $V(B) = \frac{4}{3}V(B_0)$. Furthermore, by Proposition 5(iv) we get $a := a_1 = a_2 = a_3$. Applying the previous observations and (11) we obtain

$$V_B(T) = \frac{V(T)}{V(B)} = \frac{3V(T)}{4V(B_0)} = \frac{1}{8}\, a^2,$$

which shows the assertion of the claim.

The following two claims are concerned with the case when some side of B contains two vertices of B_0.

Claim 4. *Assume that B is a hexagon and some side of B contains two vertices of B_0. Then there exists $B' \in \mathcal{B}'$ such that $V(B') > V(B)$.*

Without loss of generality we may assume that l_1 contains b_1 and $-b_3$. Since B is a hexagon, we can set $B' := P(\pm l_1, \pm l_2)$, obtaining the assertion of the claim.

CLAIM 5. *Assume that B is a parallelogram and some side of B contains two vertices of B_0. Then there exists $B' \in \mathcal{B}'$ such that $V(B') \geq V(B)$ and every side of B' lies in the affine hull of a side of B_0.*

Without loss of generality we assume that $B = P(\pm l_1, \pm l_2)$ and l_1 contains b_1 and $-b_3$. If some vertex of B_0 is also a vertex of B, there is nothing to prove and we may just set $B' := B$. Let us consider the opposite case. Let $\pm v_1, \pm v_2$ be vertices of B such that v_1 is the intersection point of l_1 and $-l_2$, and v_2 is the intersection point of l_1 and l_2. Without loss of generality we may also assume that $\|v_2 - b_2\| \leq \|-v_1 - b_2\|$ (that is, b_2 is closer to the vertex v_2 of B than to $-v_1$ in the non-strict sense). Let v_1' be the intersection point of l_1 and $\operatorname{aff}\{b_3, -b_2\}$. It is easy to see that $\operatorname{conv}\{b_2, v_2, -b_3\}$ and $\operatorname{conv}\{b_2, -v_1', -v_1\}$ are homothetic triangles, and that the former triangle is smaller than the latter one. Thus, the assertion of the claim follows by setting $B' := \operatorname{conv}\{\pm v_1', \pm v_2\}$; see Fig. 5.

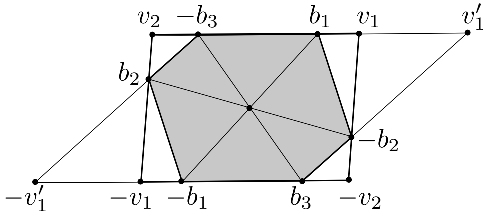

Fig. 5

There are precisely three parallelograms B' satisfying the condition of Claim 5. It turns out that we can determine which of them has the largest area.

CLAIM 6. *Let m_1, m_2, m_3 be the lines given by*

$$m_1 := \operatorname{aff}\{b_1, -b_3\}, \quad m_2 := \operatorname{aff}\{b_2, -b_1\}, \quad m_3 := \operatorname{aff}\{b_3, -b_2\}.$$

Then the area of $P(\pm m_1, \pm m_3)$ is not less than the area of $P(\pm m_1, \pm m_2)$ and $P(\pm m_2, \pm m_3)$.

By Proposition 5(i) there exists a non-singular affine transformation $\mathcal{A}$ such that equalities (9) and (10) are satisfied. In view of (7) and (8) we may assume, without loss of generality, that $\mathcal{A}$ is the identity. We have $B_0 = P(\pm m_1, \pm m_2, \pm m_3)$. The triangles $\operatorname{conv}\{m_1 \cap m_3, b_1, -b_2\}$ and $\operatorname{conv}\{-(m_1 \cap m_2), -b_2, b_3\}$ have equal angles at their common vertex $-b_2$. The side $\operatorname{conv}\{-b_2, -(m_1 \cap m_2)\}$ of $\operatorname{conv}\{-(m_1 \cap m_2), -b_2, b_3\}$ is not longer than the side $\operatorname{conv}\{b_1, -b_2\} = \operatorname{conv}\{(-1,0)^\top, (0,1)^\top\}$ of $\operatorname{conv}\{m_1 \cap m_3, b_1, -b_2\}$, since $-(m_1 \cap m_2)$ is a point lying on the segment

$\operatorname{conv}\{(0,1)^\top,(1,2)^\top\}$; see also Fig. 6. Furthermore, the side $\operatorname{conv}\{-b_2,b_3\}$ of $\operatorname{conv}\{-(m_1\cap m_2),-b_2,b_3\}$ is not longer than the side $\operatorname{conv}\{-b_2,m_1\cap m_3\}$ of $\operatorname{conv}\{m_1\cap m_3,b_1,-b_2\}$, since

$$b_3\in\operatorname{conv}\{-b_1,-b_2,(1,1)^\top\}$$

and

$$\operatorname{conv}\{(-1,1)^\top,b_1,-b_2\}\subseteq\operatorname{conv}\{m_1\cap m_3,b_1,-b_2\}.$$

The latter implies that the area of $\operatorname{conv}\{-(m_1\cap m_2),-b_2,b_3\}$ is not larger than the area of $\operatorname{conv}\{m_1\cap m_3,b_1,-b_2\}$. Analogously, we can also show that the area of $\operatorname{conv}\{-b_1,b_3,m_2\cap m_3\}$ is not larger than the area of $\operatorname{conv}\{-b_1,b_2,-(m_1\cap m_3)\}$. Consequently, the areas of the parallelograms $P(\pm m_1,\pm m_2)$ and $P(\pm m_2,\pm m_3)$ are not larger than the area of $P(\pm m_1,\pm m_3)$, which is the assertion of Claim 6.

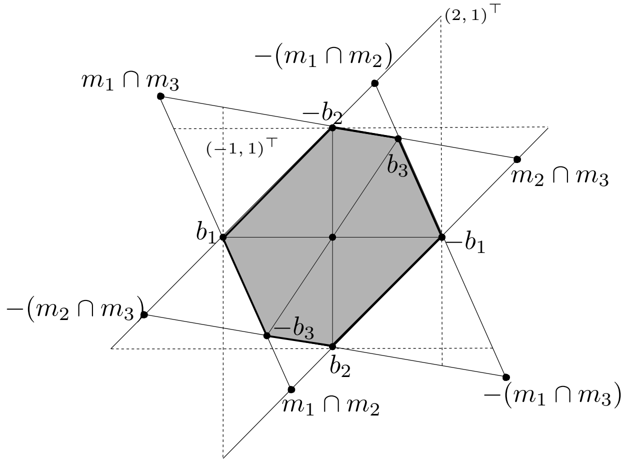

Fig. 6

Let us evaluate $V_B(T)$ for the case $B=P(\pm m_1,\pm m_3)$. We set $\alpha_1:=a_1/a_3$ and $\alpha_2:=a_2/a_3$ so that $b_3=(\alpha_1,\alpha_2)^\top$. The line m_1 can be determined by the parametric equation $(1-t_1)(-b_3)+t_1b_1$ with the parameter $t_1\in\mathbb{R}$, and the line m_3 by the parametric equation $(1-t_2)b_3+t_2(-b_2)$ with $t_2\in\mathbb{R}$. The parameters t_1,t_2 correspond to the point $m_1\cap m_3$ if

$$(1-t_1)(-b_3)+t_1b_1=(1-t_2)b_3+t_2(-b_2).$$

Rearranging the terms we arrive at the equation

$$t_1(b_1+b_3)+t_2(b_2+b_3)=2b_3$$

with unknowns t_1 and t_2. This equation can be reformulated in the matrix form

$$\begin{bmatrix}\alpha_1-1 & \alpha_1\\ \alpha_2 & \alpha_2-1\end{bmatrix}\begin{bmatrix}t_1\\ t_2\end{bmatrix}=2\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}.$$

By Cramer's rule, $t_i = \Delta_i/\Delta$ for $i = 1, 2$, where

$$\begin{aligned} \Delta &:= (\alpha_1 - 1)(\alpha_2 - 1) - \alpha_1\alpha_2 = 1 - \alpha_1 - \alpha_2, \\ \Delta_1 &:= -2\alpha_1, \qquad \Delta_2 := -2\alpha_2. \end{aligned}$$

Thus,

$$t_1 = \frac{2\alpha_1}{\alpha_1 + \alpha_2 - 1}, \qquad t_2 = \frac{2\alpha_2}{\alpha_1 + \alpha_2 - 1}.$$

Consequently,

$$\begin{aligned} m_1 \cap m_3 &= (1 - t_2)b_3 + t_2(-b_2) = (\alpha_1 + \alpha_2 - 1)^{-1}((\alpha_1 - \alpha_2 - 1)b_3 + 2\alpha_2(-b_2)) \\ &= (\alpha_1 + \alpha_2 - 1)^{-1} \begin{bmatrix} (\alpha_1 - \alpha_2 - 1)\alpha_1 \\ (1 + \alpha_1 - \alpha_2)\alpha_2 \end{bmatrix}, \end{aligned}$$

and so

$$\begin{aligned} V(P(\pm m_1, \pm m_3)) &= 2(\alpha_1 + \alpha_2 - 1)^{-1} \begin{vmatrix} \alpha_1 & (\alpha_1 - \alpha_2 - 1)\alpha_1 \\ \alpha_2 & (1 + \alpha_1 - \alpha_2)\alpha_2 \end{vmatrix} \\ &= 2(\alpha_1 + \alpha_2 - 1)^{-1}\alpha_1\alpha_2 \begin{vmatrix} 1 & \alpha_1 - \alpha_2 - 1 \\ 1 & 1 + \alpha_1 - \alpha_2 \end{vmatrix} = \frac{4\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - 1}. \end{aligned}$$

Thus, for $B = P(\pm m_1, \pm m_3)$ we get

$$\begin{aligned} V_B(T) &= V(T)/V(B) = \frac{1}{2}\, a_1 a_2 \cdot \frac{1}{4}\, (\alpha_1 + \alpha_2 - 1)\, \frac{1}{\alpha_1\alpha_2} = \frac{1}{8}\, a_3^2(\alpha_1 + \alpha_2 - 1) \\ &= \frac{1}{8}\, a_3(a_1 + a_2 - a_3). \end{aligned}$$

Summarizing all the claims above, we see that $\min_{B' \in \mathcal{B}'} V_B(T)$ is attained for $B = P(\pm m_1, \pm m_3)$ (assuming that (9) and (10) hold true with $\mathcal{A}$ equal to the identity) or for B satisfying the assumptions of Claim 3. Since for $a := a_1 = a_2 = a_3$ one has $\frac{1}{8} a_3(a_1 + a_2 - a_3) = \frac{1}{8} a^2$, we see that $\min_{B' \in \mathcal{B}'} V_B(T)$ is attained for $B = P(\pm m_1, \pm m_3)$. This proves the lower bound in (1) and Proposition 2(ii).

It remains to prove the converse statement in Theorem 1. Assume that $0 < a_1 \le a_2 \le a_3$, $a_1 + a_2 \ge a_3$ and (1) is satisfied. Above we have shown the existence of normed planes $\mathcal{M}^2(B')$ and $\mathcal{M}^2(B'')$ such that for some points $p_1', p_2', p_3' \in \mathcal{M}^2(B')$ and $p_1'', p_2'', p_3'' \in \mathcal{M}^2(B)$ one has

$$\begin{aligned} a_1 &= \|p_2' - p_3'\|_{B'}, \qquad a_2 = \|p_3' - p_1'\|_{B'}, \qquad a_3 = \|p_1' - p_2'\|_{B'}, \\ a_1 &= \|p_2'' - p_3''\|_{B''}, \qquad a_2 = \|p_3'' - p_1''\|_{B''}, \qquad a_3 = \|p_1'' - p_2''\|_{B''}, \end{aligned}$$

and

$$V_{B'}(T') = \frac{1}{8}\, a_3(a_1 + a_2 - a_3), \qquad V_{B''}(T'') = \frac{a_1 a_2 a_3}{2(a_1 + a_2 + a_3)}$$

for $T' := \operatorname{conv}\{p_1', p_2', p_3'\}$ and $T'' := \operatorname{conv}\{p_1'', p_2'', p_3''\}$. In view of (7) and (8), without loss of generality we may assume that $p_i := p_i' = p_i''$ for $i = 1, 2, 3$. Hence we also have $T := T' = T''$. For $0 \le t \le 1$ let $\mathcal{M}^2(B_t)$ be the normed plane with the norm satisfying

$$\|x\|_{B_t} = (1-t)\|x\|_{B'} + t\|x\|_{B''}$$

for every $x \in \mathbb{R}^2$. For every $0 \le t \le 1$ we obviously have

$$a_1 = \|p_2 - p_3\|_{B_t}, \qquad a_2 = \|p_3 - p_1\|_{B_t}, \qquad a_3 = \|p_1 - p_2\|_{B_t}.$$

Furthermore, B_t is equal to B' and B'' for $t = 0$ and $t = 1$, respectively. In view of continuity of $V(B_t)$ in the parameter t, for some $0 \le t \le 1$ the equality $V_{B_t}(T) = A$ is satisfied. This proves the converse statement in Theorem 1. ∎

Proof of Corollary 3. Let us prove the main statement. Assume that T is a triangle in $\mathcal{M}^2$, possibly degenerate to a segment, and set $P := P_B(T)$, $D := D_B(T)$, and $A := V_B(T)$. Then the inequality $2D \le P$ is an obvious consequence of the triangle inequality. The proof of $P \le 3D$ is straightforward. The inequality $\frac{1}{8}D(P-2D) \le A$ is a direct reformulation of the left hand side of (1). Now let us obtain the inequality $A \le \frac{1}{8}(P-D)^2 D$. We may assume that T is not degenerate to a segment. Let a_1, a_2, a_3 be the side lengths of T with $a_1 \le a_2 \le a_3$. Then $a_3 = D$. We have

$$A \overset{(1)}{\le} \frac{a_1 a_2 D}{2P} = \frac{D}{8P}\left((a_1+a_2)^2 - (a_1-a_2)^2\right) \le \frac{D}{8P}(a_1+a_2)^2 = \frac{(P-D)^2 D}{8P}.$$

This finishes the proof of necessity. For the proof of sufficiency we consider an arbitrary triple P, D, A with $P > 0$, $D > 0$, $A \ge 0$ that satisfies (6). Then, putting $a_3 := D$ and $a_1 := a_2 := \frac{1}{2}(P-D)$, we readily get from (6) the inequalities $a_1 \le a_2 \le a_3$ and $a_3 \le a_1 + a_2$. The converse statement of the corollary follows from the converse statement of Theorem 1. ∎

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