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STATIONARY REFLECTION AND LEVEL BY LEVEL EQUIVALENCE

BY

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Abstract. We force and construct a model in which level by level equivalence between strong compactness and supercompactness holds, along with certain additional "inner model like" properties. In particular, in this model, the class of Mahlo cardinals reflecting stationary sets is the same as the class of weakly compact cardinals, and every regular Jónsson cardinal is weakly compact. On the other hand, we force and construct a model for the level by level equivalence between strong compactness and supercompactness in which on a stationary subset of the least supercompact cardinal κ, there are non-weakly compact Mahlo cardinals which reflect stationary sets. We also examine some extensions and limitations on what is possible in our theorems. Finally, we indicate how to ensure in our models that $\diamondsuit_\delta$ holds for every successor and Mahlo cardinal δ, and below the least supercompact cardinal κ, $\square_\delta$ holds on a stationary subset of κ. There are no restrictions in our main models on the structure of the class of supercompact cardinals.

1. Introduction and preliminaries. It is well-known that in canonical inner models, many large cardinals exhibit regularity properties that allow for precise characterizations. For instance, in L and higher inner models (see [7, Theorem 1.3, page 304] and [27]), the weakly compact cardinals are exactly the class of Mahlo cardinals admitting stationary reflection (1), and in higher inner models (see [18], [14, Theorem 20.23], and [22]), the regular Jónsson cardinals are precisely the Ramsey cardinals. These results are of course obtained by an analysis using standard inner model techniques.

The purpose of this paper is not only to construct via forcing a model for the level by level equivalence between strong compactness and supercompactness in which analogues of the above properties hold, but also to construct via forcing a model for the level by level equivalence between strong

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(1) In fact, in L and higher inner models, the weakly compact cardinals are exactly the class of inaccessible cardinals admitting stationary reflection. We will come back to this point at the end of the paper.

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compactness and supercompactness in which stationary reflection can occur on a stationary subset of the least supercompact cardinal κ composed of non-weakly compact Mahlo cardinals. Specifically, we prove the following as our main theorems.

THEOREM 1. *Let* $V \vDash$ *"ZFC +* $\mathcal{K} \neq \emptyset$ *is the class of supercompact cardinals". There is then a partial ordering* $\mathbb{P} \subseteq V$ *such that* $V^{\mathbb{P}} \vDash$ *"ZFC + GCH +* $\mathcal{K}$ *is the class of supercompact cardinals". In* $V^{\mathbb{P}}$, *level by level equivalence between strong compactness and supercompactness holds. Further*, *in* $V^{\mathbb{P}}$, *the Mahlo cardinals reflecting stationary sets are precisely the weakly compact cardinals. Finally*, *every regular Jónsson cardinal in* $V^{\mathbb{P}}$ *is weakly compact.*

THEOREM 2. *Let* $V \vDash$ *"ZFC +* $\mathcal{K} \neq \emptyset$ *is the class of supercompact cardinals +* κ *is the least supercompact cardinal". There is then a partial ordering* $\mathbb{P} \subseteq V$ *such that* $V^{\mathbb{P}} \vDash$ *"ZFC + GCH +* $\mathcal{K}$ *is the class of supercompact cardinals +* κ *is the least supercompact cardinal". In* $V^{\mathbb{P}}$, *level by level equivalence between strong compactness and supercompactness holds. Further*, *in* $V^{\mathbb{P}}$, *there is a stationary subset* $A \subseteq \kappa$ *composed of non-weakly compact Mahlo cardinals which reflect stationary sets.*

At the end of the paper, we will briefly discuss how to prove a generalized version of Theorem 1 in a universe containing relatively few large cardinals, and also mention why some restrictions are necessary. We will in addition indicate how to augment Theorems 1, 2, and the generalized version of Theorem 1 just mentioned to obtain models witnessing the same conclusions in which $\diamondsuit_\delta$ holds for every successor and Mahlo cardinal δ, and $\square_\delta$ holds below the least supercompact cardinal κ on a stationary set. Theorem 1 and its generalizations in which the models constructed satisfy L-like properties may be considered to follow the "outer model programme" as set forth by S. Friedman in [9].

Theorem 1 may be classified, in Woodin's phrase, as an "inner model theorem proven via forcing". This is because it satisfies properties analogous to those mentioned in the first paragraph above, along with a property one might expect in a "nice" inner model containing supercompact cardinals, namely GCH and the level by level equivalence between strong compactness and supercompactness. On the other hand, the model constructed for Theorem 2 does not have the properties one might necessarily expect in an inner model for a supercompact cardinal. Of course, it is presently unknown how to build any sort of inner model for supercompact cardinals along the lines of the inner models currently known ([2]). Thus, it is nec-

([2]) Woodin has announced he can construct an inner model of ZFC containing supercompact cardinals. His construction, however, yields a model without covering and indiscernibility; in particular, his model does not have sharps. This makes it quite differessary to use forcing to produce models such as those given in Theorems 1 and 2, and it is completely unclear what to expect in terms of combinatorial properties in a "reasonable" inner model containing supercompact cardinals.

Before presenting the proofs of our theorems, we briefly mention some preliminary information. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta], [\alpha, \beta), (\alpha, \beta]$, and (α, β) are as in the usual interval notation. The regular cardinal κ will be said to *reflect stationary sets* (or equivalently, to *admit stationary reflection*) if for every stationary $S \subseteq \kappa$, there is a limit ordinal $\delta < \kappa$ such that $S \cap \delta$ is a stationary subset of δ.

When forcing, $q \geq p$ will mean that *q is stronger than p*. If G is V-generic over $\mathbb{P}$, we will abuse notation somewhat and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V.

Let κ be a regular cardinal. $\mathrm{Add}(\kappa, 1)$ is the standard partial ordering for adding a Cohen subset of κ. The partial ordering $\mathbb{P}$ is *κ-directed closed* if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$, there is an upper bound $p \in \mathbb{P}$. The ordering $\mathbb{P}$ is *κ-strategically closed* if in the two-person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$\prec\kappa$-strategically closed* if in the two-person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages (again choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. Note that if $\mathbb{P}$ is κ^+-directed closed, then $\mathbb{P}$ is κ-strategically closed. In addition, if $\mathbb{P}$ is κ-strategically closed and $f : \kappa \to V$ is a function in $V^{\mathbb{P}}$, then $f \in V$.

Suppose now that κ is a Mahlo cardinal. A partial ordering $\mathbb{P}(\kappa)$ whose use will be critical in the proof of Theorem 1 is the partial ordering for adding a nonreflecting stationary set of ordinals of a certain type to κ. Specifically, $\mathbb{P}(\kappa) = \{p :$ For some $\alpha < \kappa$, $p : \alpha \to \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, so that if $\beta < \sup(S_p)$ is inaccessible, then $S_p - (S_p \cap \beta)$ is composed of ordinals of cofinality at least $\beta\}$, ordered by $q \geq p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. It

ent from the usual kind of inner model, and means its structural properties are far more difficult to analyze.

is shown in [3] that forcing with $\mathbb{P}(\kappa)$ adds a nonreflecting stationary set of ordinals to κ and that $\mathbb{P}(\kappa)$ is $\prec\kappa$-strategically closed. This strategic closure property of $\mathbb{P}(\kappa)$, together with the fact that $|\mathbb{P}(\kappa)| = \kappa$, consequently imply that forcing with $\mathbb{P}(\kappa)$ over a model of GCH preserves GCH. It is further shown in [3] that for any inaccessible cardinal $\delta < \kappa$, there is a partial ordering $\mathbb{P}(\kappa/\delta)$ dense in $\mathbb{P}(\kappa)$ which is δ-directed closed.

We take this opportunity to recall briefly the combinatorial notions of diamond and square. If κ is a regular uncountable cardinal, $\diamondsuit_\kappa$ is the principle stating that there exists a sequence of sets $\langle S_\alpha : \alpha < \kappa\rangle$ such that $S_\alpha \subseteq \alpha$, with the additional property that for every $X \subseteq \kappa$, $\{\alpha < \kappa : X \cap \alpha = S_\alpha\}$ is a stationary subset of κ. If κ is an arbitrary uncountable cardinal, $\square_\kappa$ is the principle stating that there exists a sequence of sets $\langle C_\alpha : \alpha < \kappa^+$ and α is a limit ordinal$\rangle$ such that C_α is a closed, unbounded subset of α so that if $\mathrm{cof}(\alpha) < \kappa$, then C_α has order type below κ, with the additional property that for any limit point $\beta \in C_\alpha$, $C_\alpha \cap \beta = C_\beta$.

For κ a regular uncountable cardinal, it is possible to preserve $\diamondsuit_\kappa$ via certain forcing notions. For a general treatment of this topic, we refer readers to [25]. For our purposes, we will need the following two simple folklore facts.

FACT 1.1. *Suppose $V \vDash$ "κ is a regular uncountable cardinal for which $\diamondsuit_\kappa$ holds" and $\mathbb{P} \subseteq V$ is $\prec\kappa$-strategically closed. Then $V^{\mathbb{P}} \vDash$ "$\diamondsuit_\kappa$ holds".*

Proof. Suppose $V \vDash$ "$S = \langle S_\alpha : \alpha < \kappa\rangle$ is a diamond sequence for κ" and $\mathbb{P} \subseteq V$ is $\prec\kappa$-strategically closed. Assume $p \Vdash$ "$\dot{X} \subseteq \kappa$ and $\dot{C} \subseteq \kappa$ is club". Consider the game of length κ in which players I and II construct an increasing sequence of conditions. The game begins with player II choosing the trivial condition and player I choosing a condition extending p which decides the statements "$0 \in \dot{X}$" and "$0 \in \dot{C}$". At non-limit even stages $2\alpha > 0$, player II must choose a condition deciding the statements "$\alpha \in \dot{X}$" and "$\alpha \in \dot{C}$". By the $\prec\kappa$-strategic closure of $\mathbb{P}$, player II has a winning strategy for this game. We may thus assume that $\langle p_\alpha : \alpha < \kappa\rangle$ is an increasing sequence of conditions extending p such that p_α completely determines both $\dot{X} \cap \alpha$ and $\dot{C} \cap \alpha$, sets in V which we denote as X_α and C_α respectively.

Let $X' = \bigcup_{\alpha<\kappa} X_\alpha$ and $C' = \bigcup_{\alpha<\kappa} C_\alpha$. Both X' and C' are members of V, and C' is a club subset of κ. Hence, since S is a $\diamondsuit_\kappa$ sequence in V, let $\beta \in C'$ be such that $X' \cap \beta = S_\beta$. It is then the case that there is $\gamma > \beta$ with $p_\gamma \Vdash$ "$\beta \in \dot{C}$ and $\dot{X} \cap \beta = S_\beta$", which means that S remains in $V^{\mathbb{P}}$ a $\diamondsuit_\kappa$ sequence for κ. This completes the proof of Fact 1.1. ■

FACT 1.2. *Suppose $V \vDash$ "κ is a regular uncountable cardinal for which $\diamondsuit_\kappa$ holds" and $\mathbb{P} \in V$ is κ-c.c. and has cardinality κ. Then $V^{\mathbb{P}} \vDash$ "$\diamondsuit_\kappa$ holds".*

Sketch of proof. We give a proof sketch which was essentially told to us by Joel Hamkins. We quote liberally from his presentation. Suppose that

$\langle A_\alpha : \alpha < \kappa \rangle$ is a $\diamondsuit_\kappa$ sequence, G is V-generic over $\mathbb{P}$, and $\mathbb{P}$ has cardinality κ and satisfies κ-c.c. Let $B_\alpha = i_G(A_\alpha)$ provided that A_α codes a $\mathbb{P}$-name, and $B_\alpha = \emptyset$ otherwise. If C is any subset of κ in $V[G]$, then let $\dot{C}$ be a name for C. Since $|\mathbb{P}| = \kappa$ and $\mathbb{P}$ satisfies κ-c.c., we may assume that $\dot{C}$ is hereditarily of cardinality at most κ. Therefore, we may let $C^* \subseteq \kappa$, $C^* \in V$, code $\dot{C}$. In V, C^* is anticipated on a stationary set, i.e., $S = \{\alpha < \kappa : C^* \cap \alpha = A_\alpha\}$ is stationary. Further, S is a $\mathbb{P}$-name. By [16, Exercise H2, page 247], since $\mathbb{P}$ is κ-c.c., S remains stationary in $V[G]$. And, on a club, $i_G(C^* \cap \alpha) = C \cap \alpha$. Thus, C is anticipated on a stationary subset by $\langle B_\alpha : \alpha < \kappa \rangle$. This completes the proof sketch of Fact 1.2. ∎

The notion of level by level equivalence between strong compactness and supercompactness was introduced by Shelah and the author in [6]. In that paper, the following theorem was proven.

Theorem 3. *Let $V \vDash$ "ZFC + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + GCH + $\mathcal{K}$ is the class of supercompact cardinals + For every pair of regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact".*

We will say that any model witnessing the conclusions of Theorem 3 is a model for *the level by level equivalence between strong compactness and supercompactness*. Note that the exception in Theorem 3 is provided by a theorem of Menas [21], who showed that if κ is a measurable limit of cardinals δ which are λ strongly compact, then κ is λ strongly compact but need not be λ supercompact. Observe also that Theorem 3 is a strengthening of the result of Kimchi and Magidor [15], who showed it is consistent for the classes of strongly compact and supercompact cardinals to coincide precisely, except at measurable limit points.

A result which will be key in the proof of Theorem 1 is an amalgamation of Hamkins' Gap Forcing Theorem of [10, 11] together with [10, 11, Corollary 16]. We therefore state the theorem we will be using now, along with some associated terminology, quoting freely from [10, 11]. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed". In Hamkins' terminology of [10, 11], $\mathbb{P}$ admits a gap at δ. In his terminology, $\mathbb{P}$ is *mild* with respect to a cardinal κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size below κ has a "nice" name τ in V of size below κ, i.e., there is a set y in V, $|y| < \kappa$, such that any ordinal forced by a condition in $\mathbb{P}$ to be in τ is an element of y. Also, as in the terminology of [10, 11] (and elsewhere), an embedding $j : \overline{V} \to \overline{M}$ is *amenable* to $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific theorem we will be using is then the following.

THEOREM 4 (Hamkins). *Suppose that $V[G]$ is a forcing extension obtained by forcing with a partial ordering $\mathbb{P}$ that admits a gap at some $\delta < \kappa$ and $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^{\delta} \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and and κ is λ strongly compact in $V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V.*

Finally, we mention that we are assuming familiarity with standard large cardinal notions. Interested readers may consult [13] or [14] for further details. We note only that the cardinal κ is *supercompact up to the cardinal* λ if κ is δ supercompact for every $\delta < \lambda$.

2. The proofs of Theorems 1 and 2

Proof of Theorem 1. Let $V \models$ "ZFC + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals". Without loss of generality, by first doing a preliminary forcing if necessary, we may also assume that V is as in Theorem 3, i.e., that GCH and level by level equivalence between strong compactness and supercompactness hold in V. This allows us to define the partial ordering $\mathbb{P}$ used in the proof of Theorem 1 as the (possibly proper class) reverse Easton iteration which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then is trivial forcing, except at cardinals which are in V both non-Ramsey and Mahlo. At such a cardinal κ, we force with the partial ordering $\mathbb{P}(\kappa)$. Standard arguments (see [13]) then show that for $\mathbb{Q}$ any initial segment (proper or improper) of $\mathbb{P}$, $V^{\mathbb{Q}} \models$ "ZFC + GCH" and V and $V^{\mathbb{Q}}$ have the same cardinals and cofinalities.

LEMMA 2.1. *If $V \models$ "$\kappa < \lambda$ are such that κ is λ supercompact and λ is regular", then $V^{\mathbb{P}} \models$ "κ is λ supercompact".*

Proof. Suppose that κ and λ are as in the hypotheses of Lemma 2.1. Fix $j : V \to M$ an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$, and write $\mathbb{P} = \mathbb{P}_{\lambda+1} * \dot{\mathbb{P}}^{\lambda+1}$. Since $\Vdash_{\mathbb{P}_{\lambda+1}}$ "$\dot{\mathbb{P}}^{\lambda+1}$ is γ-strategically closed for γ the least inaccessible cardinal above λ", it suffices to show that $V^{\mathbb{P}_{\lambda+1}} \models$ "κ is λ supercompact".

To do this, we use a variant of the argument given in the proof of [3, Lemma 3.1]. Write $\mathbb{P}^{\lambda+1}$ as $\mathbb{P}^0 * \dot{\mathbb{P}}^1$, where $\mathbb{P}^0$ is $\mathbb{P}^{\lambda+1}$ defined through stage κ, i.e., $\mathbb{P}^0 = \mathbb{P}_\kappa$, and $\dot{\mathbb{P}}^1$ is a term for the rest of $\mathbb{P}^{\lambda+1}$, i.e., the portion acting on the non-Ramsey Mahlo cardinals in the half-open interval $(\kappa, \lambda]$. If it is not the case that $V^{\mathbb{P}_{\lambda+1}} \models$ "κ is λ supercompact", then let $p = \langle p_0, \dot{p}_1 \rangle \in \mathbb{P}^0 * \dot{\mathbb{P}}^1$ be such that $p \Vdash$ "κ is not λ supercompact". By our remarks in Section 1, we assume without loss of generality that each nontrivial coordinate of p_1 is a term for a condition in the appropriate $\mathbb{P}(\delta/\kappa)$.

Let G_0 be V-generic over $\mathbb{P}^0$ such that $p_0 \in G_0$. Since $\mathbb{P}^0$ may be defined so as to have cardinality κ, by the Lévy–Solovay results [20], we know that the set $A = \{\delta \in (\kappa, \lambda] : \delta \text{ is a non-Ramsey Mahlo cardinal}\}$ is the same in both V and $V[G_0]$. Consequently, working in $V[G_0]$ and once again using our remarks from Section 1, let $\mathbb{P}^3$ be the reverse Easton iteration of partial orderings which, for every $\delta \in A$, add nonreflecting stationary sets of ordinals using $\mathbb{P}(\delta/\kappa)$.

Note now that if G_1 is $V[G_0]$-generic over $\mathbb{P}^3$ and $p_1 \in G_1$, then G_1 must also generate a $V[G_0]$-generic filter G_1^* over $\mathbb{P}^1$. To see this, it clearly suffices to show that G_1 meets all dense open subsets of $\mathbb{P}^1$ above p_1. If D is such a set, then let $D_1 = \{q \in \mathbb{P}^3 : q \text{ extends some element of } D\}$. The set D_1 is clearly open. If $q \in \mathbb{P}^3$, then $q \in \mathbb{P}^1$, so by density, there is $q' \geq q$, $q' \in D$. By using our remarks from Section 1 if necessary to find a term which is forced to extend each term denoting a nontrivial coordinate of q' to a term for an element of the appropriate $\mathbb{P}(\delta/\kappa)$, we obtain $q'' \geq q' \geq q$, $q'' \in D_1$. Thus, G_1 meets D_1 and hence meets D, so G_1 generates a $V[G_0]$-generic filter G_1^* over $\mathbb{P}^1$.

By the definition of $\mathbb{P}$ and the closure properties of M, $j(\mathbb{P}^0 * \dot{\mathbb{P}}^1) = \mathbb{P}^0 * \dot{\mathbb{P}}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, where $\dot{\mathbb{Q}}$ is a term for the portion of $j(\mathbb{P}^0 * \dot{\mathbb{P}}^1)$ acting on ordinals in the open interval $(\lambda, j(\kappa))$, and $\dot{\mathbb{R}}$ is a term for $j(\dot{\mathbb{P}}^1)$, i.e., the portion of $j(\mathbb{P}^0 * \dot{\mathbb{P}}^1)$ acting on ordinals in the interval $(j(\kappa), j(\lambda)]$. If G_1 is $V[G_0]$-generic over $\mathbb{P}^3$ and $p_1 \in G_1$, then by the preceding paragraph, G_1 generates a $V[G_0]$-generic filter G_1^* over $\mathbb{P}^1$. Since $\mathbb{P}^1$ is λ^+-c.c. in $V[G_0]$, $M[G_0][G_1^*]$ remains λ-closed with respect to $V[G_0][G_1^*]$. Consequently, by GCH in $V[G_0][G_1^*]$, the usual diagonalization argument (as given, e.g., in the construction of the generic object G_1 in [5, Lemma 2.4]) may be used to build in $V[G_0][G_1^*]$ an $M[G_0][G_1^*]$-generic object G_2 over $\mathbb{Q}$. (This argument uses the $\prec\lambda^+$-strategic closure of $\mathbb{Q}$ in both $M[G_0][G_1^*]$ and $V[G_0][G_1^*]$, together with the fact that by GCH, there are only $2^\lambda = \lambda^+$ many dense open subsets of $\mathbb{Q}$ present in $M[G_0][G_1^*]$, to meet all of the required sets.) We may then lift j in $V[G_0][G_1^*]$ to $j : V[G_0] \to M[G_0][G_1^*][G_2]$.

Since $G_1 \subseteq G_1^*$ and G_1 is $V[G_0]$-generic over a partial ordering ($\mathbb{P}^3$) that is κ-directed closed in $V[G_0]$, $j''G_1$ generates in $V[G_0][G_1^*][G_2]$ a compatible set of conditions of cardinality $\lambda < j(\kappa)$ in a partial ordering ($j(\mathbb{P}^3)$) that is $j(\kappa)$-directed closed in $M[G_0][G_1^*][G_2]$. Therefore, by the fact $M[G_0][G_1^*][G_2]$ is λ-closed with respect to $V[G_0][G_1^*][G_2] = V[G_0][G_1^*]$, we can let r be a master condition for $j''G_1$ and once again use the usual diagonalization argument in $V[G_0][G_1^*]$ to build G_3 to be an $M[G_0][G_1^*][G_2]$-generic object over $j(\mathbb{P}^3)$ containing r. By elementarity, it will be the case that G_3 generates an $M[G_0][G_1^*][G_2]$-generic object G_3^* over $\mathbb{R} = j(\mathbb{P}^1)$ containing r. As usual, we will then be able to see that in $V[G_0][G_1^*]$, j lifts to $j : V[G_0][G_1^*] \to M[G_0][G_1^*][G_2][G_3^*]$, so κ is λ supercompact in $V[G_0][G_1^*]$. This, however, contradicts that $p = \langle p_0, p_1\rangle \in G_0 * G_1^*$ and $p \Vdash$ "κ is not λ supercompact". This contradiction completes the proof of Lemma 2.1. ∎

LEMMA 2.2. *$V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds".*

Proof. The proof of Lemma 2.2 follows closely the proofs of [2, Lemma 3.2] and [1, Lemma 1.3]. Suppose $V^{\mathbb{P}} \models$ "$\kappa < \lambda$ are regular cardinals such that κ is λ strongly compact and κ is not a measurable limit of cardinals δ which are λ supercompact". By Lemma 2.1, any cardinal δ such that δ is λ supercompact in V remains λ supercompact in $V^{\mathbb{P}}$. This means that $V \models$ "$\kappa < \lambda$ are regular cardinals such that κ is not a measurable limit of cardinals δ which are λ supercompact".

Note that it is possible to write $\mathbb{P} = \mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| = \omega$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is ω-strategically closed". Further, by the definition of $\mathbb{P}$, it is easily seen that $\mathbb{P}$ is mild with respect to κ. Therefore, by Theorem 4, $V \models$ "κ is λ strongly compact". Hence, by level by level equivalence between strong compactness and supercompactness in V, $V \models$ "κ is λ supercompact", so another application of Lemma 2.1 shows that $V^{\mathbb{P}} \models$ "κ is λ supercompact". This completes the proof of Lemma 2.2. ∎

LEMMA 2.3. *$V^{\mathbb{P}} \models$ "$\mathcal{K}$ is the class of supercompact cardinals".*

Proof. By Lemma 2.1, if κ is λ supercompact in V for $\lambda > \kappa$ regular, then κ is λ supercompact in $V^{\mathbb{P}}$. Further, by the factorization of $\mathbb{P}$ as $\mathbb{Q} * \dot{\mathbb{R}}$ given in Lemma 2.2 and an application of Theorem 4, any cardinal κ which is λ supercompact in $V^{\mathbb{P}}$ had to have been λ supercompact in V. Thus, $\mathcal{K}$ is precisely the class of supercompact cardinals in $V^{\mathbb{P}}$. This completes the proof of Lemma 2.3. ∎

LEMMA 2.4. *$V^{\mathbb{P}} \models$ "If κ is a Ramsey cardinal in V, then κ is weakly compact".*

Proof. Suppose $V \models$ "κ is a Ramsey cardinal". Write $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}$. By the definition of $\mathbb{P}$, $\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}$ is κ-strategically closed". Thus, to prove Lemma 2.4, it suffices to show that $V^{\mathbb{P}_\kappa} \models$ "κ is weakly compact".

To do this, we adapt an argument from Theorem 1.4 of Hamkins' paper [12], quoting liberally from his presentation. By the definition of $\mathbb{P}$, it is clearly the case that κ remains inaccessible in $V^{\mathbb{P}_\kappa}$. It therefore is enough to show that κ has the tree property in $V^{\mathbb{P}_\kappa}$. Suppose as a consequence that $\dot{T}$ is a name for a κ-tree in $V^{\mathbb{P}_\kappa}$. In V, let N be a transitive elementary substructure of $H(\kappa^+)$ of size κ containing $\mathbb{P}_\kappa$ and $\dot{T}$ which is closed under $<\kappa$ sequences. Since κ, being Ramsey in V, is also weakly compact in V, there is an elementary embedding $j : N \to M$ having critical point κ. As in [12, Theorem 1.4], we may also assume that $|M| = \kappa$ and $V \models$ "$M^{<\kappa} \subseteq M$".

Write $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{R}}$. For any V-generic object G over $\mathbb{P}_\kappa$, the fact $\mathbb{P}_\kappa$ is κ-c.c. allows us to infer that $V[G] \models$ "$M[G]^{<\kappa} \subseteq M[G]$". Further, regardless if $\mathbb{R}$ acts nontrivially on κ, it is the case that $M[G] \models$ "$\mathbb{R}$ is $\prec\kappa$-strategically closed". Therefore, since by the fact $V[G] \models$ "$|M[G]| = \kappa$", there are only κ many dense open subsets of $\mathbb{R}$ present in $M[G]$, and since $V[G] \models$ "$M[G]^{<\kappa} \subseteq M[G]$", we may use the diagonalization argument mentioned in the proof of Lemma 2.1 to meet the κ many dense open subsets of $\mathbb{R}$ and construct in $V[G]$ an $M[G]$-generic object H for $\mathbb{R}$. Then j lifts in $V[G]$ to an elementary embedding $j : N[G] \to M[G][H]$.

Because $\dot{T} \in N$, $T \in N[G]$. Since T is a κ-tree in both $V[G]$ and $N[G]$, by elementarity, $j(T)$ is a $j(\kappa)$-tree in $M[G][H]$. Any element on the κth level of $j(T)$ gives a branch of length κ through T. This means that κ has the tree property in $V[G]$, as desired. This completes the proof of Lemma 2.4. ∎

Lemma 2.5. *In $V^{\mathbb{P}}$, the Mahlo cardinals reflecting stationary sets are precisely the weakly compact cardinals.*

Proof. If κ is weakly compact, then clearly, κ is both Mahlo and reflects stationary sets. For the reverse direction, suppose $V^{\mathbb{P}} \models$ "κ is Mahlo and reflects stationary sets". Let δ be a V-Mahlo non-Ramsey cardinal, and write $\mathbb{P} = \mathbb{P}_{\delta+1} * \dot{\mathbb{Q}}$. Since $\Vdash_{\mathbb{P}_{\delta+1}}$ "δ contains a nonreflecting stationary set of ordinals and $\dot{\mathbb{Q}}$ is δ-strategically closed", $V^{\mathbb{P}} \models$ "δ contains a nonreflecting stationary set of ordinals". Since any cardinal Mahlo in $V^{\mathbb{P}}$ had to have been Mahlo in V, κ had to have been a Ramsey cardinal in V. By Lemma 2.4, $V^{\mathbb{P}} \models$ "κ is weakly compact". This completes the proof of Lemma 2.5. ∎

Lemma 2.6. *Every regular Jónsson cardinal in $V^{\mathbb{P}}$ is weakly compact.*

Proof. Suppose $V^{\mathbb{P}} \models$ "κ is a regular Jónsson cardinal". Since $V^{\mathbb{P}} \models$ GCH, by [23, Chapters 3 and 4], κ must also be a Mahlo cardinal. By a result of Tryba [26] (also independently due to Woodin—see [14, Proposition 8.17, page 96]), κ must reflect stationary sets at some limit ordinal $\lambda < \kappa$. Hence, by Lemma 2.5, κ is weakly compact in $V^{\mathbb{P}}$. This completes the proof of Lemma 2.6. ∎

Since V is of course an inner model of $V^{\mathbb{P}}$, Lemma 2.5 and its proof and Lemma 2.6 easily imply that any regular Jónsson cardinal in $V^{\mathbb{P}}$ is Ramsey in an inner model (namely V). We note that by [19, Theorem 3], if the class of non-Ramsey Mahlo cardinals is actually a set, then V is definable within $V^{\mathbb{P}}$ using a certain set parameter. This tells us that this inner model is in a certain sense definable within $V^{\mathbb{P}}$, thereby enhancing the analogy with the canonical inner models mentioned in the first paragraph of this paper.

Lemmas 2.1–2.6 complete the proof of Theorem 1. ∎

Proof of Theorem 2. Let $V \models$ "ZFC + $\mathcal{K}$ is the class of supercompact cardinals + κ is the least supercompact cardinal". Without loss of generality, as in the proof of Theorem 1, we assume in addition that $V \models$ "GCH + Level by level equivalence between strong compactness and supercompactness holds".

We now define a certain stationary subset $A \subseteq \kappa$. By [5, Lemma 2.1] and the succeeding remarks, κ is a limit of strong cardinals, since κ is (at least) 2^κ supercompact and strong. Let μ be a normal measure over κ having Mitchell rank 1 ([3]). In the ultrapower $M = V^\kappa/\mu$, the statement "κ is a measurable cardinal having trivial Mitchell rank which is a limit of strong cardinals" is true, since κ is the critical point of the elementary embedding generated by μ. Therefore, by reflection, $A = \{\delta < \kappa : \delta$ is a measurable limit of strong cardinals having trivial Mitchell rank$\} \in \mu$, which automatically implies that A is a stationary subset of κ.

Given the set A, we are now ready to present the partial ordering $\mathbb{P}$ used in the proof of Theorem 2. Let $B = \{\delta < \kappa : \delta$ is a strong cardinal which is not a limit of strong cardinals$\}$. Easily, $A \cap B = \emptyset$. Define $\mathbb{P}$ as the reverse Easton iteration having length κ which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then does trivial forcing except when $\delta \in A \cup B$. If $\delta \in B$, then we force with $\mathrm{Add}(\delta, 1)$. If $\delta \in A$, then we force with the partial ordering $\mathbb{Q}_\delta$ of [17, page 69] adding a δ-Souslin tree (via homogeneous trees of successor height less than δ, ordered by end-extension).

Because $\mathbb{P}$ is κ-c.c., an application of [16, Exercise H2, page 247] tells us that A remains stationary after forcing with $\mathbb{P}$. In addition, the arguments of [17, pages 68–71] tell us that for $\delta \in A$, $V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta} = V^{\mathbb{P}_{\delta+1}} \models$ "δ is a non-weakly compact Mahlo cardinal which reflects stationary sets". Since by [17, page 70], for $\delta \in A$, it is the case that $\mathbb{Q}_\delta$ is $\prec\delta$-strategically closed, we may now infer that $V^{\mathbb{P}} \models$ "A is a stationary subset of κ composed of non-weakly compact Mahlo cardinals which reflect stationary sets".

The following is the natural analogue of Lemma 2.1.

Lemma 2.7. *If $V \models$ "$\delta < \lambda$ are such that δ is λ supercompact and λ is regular", then $V^{\mathbb{P}} \models$ "δ is λ supercompact".*

Proof. If $\delta > \kappa$, then Lemma 2.7 easily follows by the results of [20]. We consequently assume for the remainder of the proof of Lemma 2.7 that $\delta \le \kappa$. It is clear that $\delta \notin A$, since by GCH and the fact $\lambda > \delta$, δ is at least 2^δ supercompact and hence has nontrivial Mitchell rank. In addition, as we just mentioned, [5, Lemma 2.1] and the succeeding remarks show that if δ is (at least) 2^δ supercompact and strong, then δ is a limit of strong cardinals.

([3]) Relevant facts and definitions concerning the Mitchell ordering of normal measures and supercompact cardinals may be found in [13].

From this, it immediately follows that $\delta \notin B$, so δ must be a trivial stage of forcing.

Let $\gamma = \sup(\{\alpha < \delta : \alpha$ is a nontrivial stage of forcing$\})$, and write $\mathbb{P} = \mathbb{P}_\gamma * \dot{\mathbb{Q}}$. If $\delta = \kappa$, then $\dot{\mathbb{Q}}$ is a term for trivial forcing. If $\delta < \kappa$, then since δ is a trivial stage of forcing, β, the first member on which $\dot{\mathbb{Q}}$ is forced to act nontrivially, must be above δ. Further, it is the case that $\lambda < \beta$. This is since otherwise, $V \models$ "δ is α supercompact for every $\alpha < \beta$ and β is strong". Thus, as mentioned in the proof of [5, Lemma 2.4], δ must be supercompact, which contradicts that $V \models$ "$\delta < \kappa$ and κ is the least supercompact cardinal". Consequently, regardless if $\delta = \kappa$ or $\delta < \kappa$, to show that $V^{\mathbb{P}} = V^{\mathbb{P}_\gamma * \dot{\mathbb{Q}}} \models$ "δ is λ supercompact", it suffices to show that $V^{\mathbb{P}_\gamma} \models$ "δ is λ supercompact".

To do this, we first observe that if $\gamma < \delta$, then $|\mathbb{P}_\gamma| < \delta$, so by the results of [20], $V^{\mathbb{P}_\gamma} \models$ "δ is λ supercompact". We hence assume without loss of generality that $\gamma = \delta$. Let then $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\lambda)$ such that $M \models$ "δ is not λ supercompact". Since $\mathrm{cp}(j) = \delta$, $\gamma = \delta$, $\lambda > \delta$, and GCH holds in V, it follows that $M \models$ "δ has nontrivial Mitchell rank and is a limit of strong cardinals". Also, $M \models$ "No cardinal $\eta \in (\delta, \lambda]$ is strong", because if not, then by closure, $M \models$ "δ is α supercompact for all $\alpha < \eta$, where η is strong". As we have already observed, this means that $M \models$ "δ is supercompact", a contradiction to the fact that $M \models$ "δ is not λ supercompact". It hence immediately follows that $j(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{Q}}'$, where the first ordinal on which $\dot{\mathbb{Q}}'$ is forced to act nontrivially is above λ. Once again, the usual diagonalization argument (to which we referred in the proof of Lemma 2.1) then applies and shows that j lifts in $V^{\mathbb{P}_\delta} = V^{\mathbb{P}_\gamma}$ to $j : V^{\mathbb{P}_\delta} \to M^{j(\mathbb{P}_\delta)}$, i.e., $V^{\mathbb{P}_\gamma} \models$ "δ is λ supercompact". This completes the proof of Lemma 2.7. ∎

By writing $\mathbb{P} = \mathbb{Q} * \dot{\mathbb{R}}$, where $\mathbb{Q}$ is nontrivial, $|\mathbb{Q}| = \omega$, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is ω-strategically closed", the same proof as presented in Lemma 2.3 shows that $V^{\mathbb{P}} \models$ "$\mathcal{K}$ is the class of supercompact cardinals". From this, it immediately follows that $V^{\mathbb{P}} \models$ "κ is the least supercompact cardinal". By using the factorization of $\mathbb{P}$ just given and replacing a reference to Lemma 2.1 with a reference to Lemma 2.7, the same proof as found in Lemma 2.2 applies and shows that $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds". Since standard arguments once again show that $V^{\mathbb{P}} \models \mathrm{GCH}$, this completes the proof of Theorem 2. ∎

3. Some additional comments and concluding remarks. As we have already mentioned, in L and higher inner models, the weakly compact cardinals are precisely the class of inaccessible cardinals admitting stationary reflection. One may wonder whether this phenomenon is also possible in the context of the level by level equivalence between strong compactness and supercompactness. The methods previously discussed in fact allow us to establish the following theorem, which is a generalized version of Theorem 1.

THEOREM 5. *Let $V \vDash$ "ZFC + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + GCH + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal". In $V^{\mathbb{P}}$, level by level equivalence between strong compactness and supercompactness holds. Further, in $V^{\mathbb{P}}$, the inaccessible cardinals reflecting stationary sets are precisely the weakly compact cardinals. Finally, every regular Jónsson cardinal in $V^{\mathbb{P}}$ is weakly compact.*

Sketch of proof. Suppose $V \vDash$ "ZFC + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal". Without loss of generality, as in the proofs of Theorems 1 and 2, we assume in addition that $V \vDash$ "GCH + Level by level equivalence between strong compactness and supercompactness holds". For δ an inaccessible cardinal, redefine $\mathbb{P}(\delta)$ to be the partial ordering for adding a nonreflecting stationary set of ordinals of cofinality ω to δ. ($\mathbb{P}(\delta)$ is composed of characteristic functions of subsets of δ consisting of ordinals of cofinality ω which are nonstationary at their supremum nor have any initial segments which are stationary, ordered by end-extension—a more precise definition may be found in [5, Section 1].) The partial ordering $\mathbb{P}$ used in the proof of Theorem 5 is the reverse Easton iteration of length κ which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then is trivial forcing, except at cardinals which are in V both non-Ramsey and inaccessible. At such a cardinal δ, we force with the partial ordering $\mathbb{P}(\delta)$.

If $V \vDash$ "$\delta < \lambda$ are such that δ is λ supercompact and λ is regular" and $j : V \to M$ is an elementary embedding witnessing the λ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\lambda)$, then since λ must be below the least inaccessible above δ, we have $j(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{Q}}$, where the first ordinal at which $\dot{\mathbb{Q}}$ is forced to act nontrivially is well above λ. The usual diagonalization argument therefore once again applies and allows us to show that $V^{\mathbb{P}} \vDash$ "δ is λ supercompact". With $\mathcal{K}$ having κ as its only member, the arguments of Lemmas 2.2–2.6 suitably modified then imply that $V^{\mathbb{P}}$ is as desired. This completes the proof sketch of Theorem 5. ∎

Of course, the large cardinal structure of both our ground model and generic extension in Theorem 5 is severely limited. One may wonder if this is indeed necessary. The following theorem, told to us by James Cummings, shows that some restrictions are required.

THEOREM 6 (Folklore). *Suppose κ is a regular limit of cardinals δ which are κ strongly compact. Suppose in addition that the regular cardinals below κ are nonstationary* (*e.g., if κ is the least regular limit of cardinals δ which are κ strongly compact*). *Then κ admits stationary reflection.*

Proof. Take $S \subseteq \kappa$ as being stationary. Define $f : S \to \kappa$ by $f(\delta) = \mathrm{cof}(\delta)$. Since $f(\delta) \le \delta$, by Fodor's theorem, either f is the identity on a stationary subset of S, or $f(\delta) = \alpha$ for some fixed cardinal α and all δ in a stationary subset of S. If the former holds, then the regular cardinals must be a stationary subset of κ, contradictory to our hypotheses. Thus, fix $T \subseteq S$ stationary and α such that $f(\delta) = \alpha$ for all $\delta \in T$. Since κ is a limit of cardinals δ which are κ strongly compact, let $\kappa_0 \in (\alpha, \kappa)$ be such that κ_0 is κ strongly compact. Since $\alpha < \kappa_0 < \kappa$, κ_0 is κ strongly compact, and κ is regular, it follows that κ admits stationary reflection for stationary subsets composed of ordinals of cofinality α. Thus, there is some $\delta < \kappa$ for which $T \cap \delta$, and hence $S \cap \delta$, is stationary. This completes the proof of Theorem 6. ∎

As mentioned in Section 1, it is possible to augment the results of Theorems 1, 2, and 5 so as to obtain $\diamondsuit_\delta$ for every successor and Mahlo cardinal δ (4) and $\square_\delta$ for every δ in a stationary subset of the least supercompact cardinal. To do this, by [1, Theorem 1], we assume without loss of generality that our ground model V not only satisfies GCH and the level by level equivalence between strong compactness and supercompactness, but also is a model for $\diamondsuit_\delta$ for every regular uncountable cardinal δ and $\square_\delta$ for every δ in a certain stationary subset of the least supercompact cardinal κ. We then force with the partial ordering $\mathbb{P}$ used in the proofs of either Theorem 1, 2, or 5 of this paper. By our work above, the resulting model consequently witnesses the conclusions of any of these theorems. We therefore have to show that we may additionally infer the remaining desired properties.

By our previous work, κ remains the least supercompact cardinal in $V^{\mathbb{P}}$. To see that $\square_\delta$ holds on a stationary subset of κ, we note that since forcing with (any version of) $\mathbb{P}$ preserves cardinals and cofinalities, it easily follows that each instance of $\square_\delta$ remains an instance of $\square_\delta$ in $V^{\mathbb{P}}$. Because $\mathbb{P}$ may be written as $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$, where $\mathbb{P}_\kappa$ satisfies κ-c.c. and $\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}$ is κ-strategically

(4) Shelah has recently shown in [24] that GCH implies $\diamondsuit_\delta$ holds for every successor cardinal greater than or equal to $\aleph_2$. (It is of course impossible for GCH to imply that $\diamondsuit_{\aleph_1}$ holds, since by the results of [8], there is a model containing no Souslin trees in which $2^{\aleph_0} = \aleph_1$.) It is unknown, however (see [24, Question 0.5]), if there is an analogous ZFC theorem (with or without GCH) when δ is inaccessible. Thus, a supplemental forcing is necessary to obtain $\diamondsuit_{\aleph_1}$, and the present state of knowledge seems to require that further additional forcing be done in order to obtain $\diamondsuit_\delta$ at every Mahlo cardinal δ.

closed" ([5]), another application of [16, Exercise H2, page 247] tells us that any stationary subset of κ in V remains stationary in $V^{\mathbb{P}}$. This means that in $V^{\mathbb{P}}$, $\square_\delta$ holds on a stationary subset of the least supercompact cardinal. Then, since for any Mahlo cardinal δ, we may write $\mathbb{P}$ as $\mathbb{P}_\delta * \dot{\mathbb{Q}}$, where $|\mathbb{P}_\delta| \le \delta$ and $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{Q}}$ is (at least) $\prec\delta$-strategically closed", an application of Facts 1.1 and 1.2 yields that $\diamondsuit_\delta$ is preserved in $V^{\mathbb{P}}$. This means we are able to prove the following theorems.

THEOREM 7. *Let $V \models$ "ZFC + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + GCH + $\mathcal{K}$ is the class of supercompact cardinals". In $V^{\mathbb{P}}$, level by level equivalence between strong compactness and supercompactness holds, as does $\diamondsuit_\delta$ for every successor and Mahlo cardinal δ and $\square_\delta$ for every δ in a stationary subset of the least supercompact cardinal. Further, in $V^{\mathbb{P}}$, the Mahlo cardinals reflecting stationary sets are precisely the weakly compact cardinals. Finally, every regular Jónsson cardinal in $V^{\mathbb{P}}$ is weakly compact.*

THEOREM 8. *Let $V \models$ "ZFC + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + κ is the least supercompact cardinal". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + GCH + $\mathcal{K}$ is the class of supercompact cardinals + κ is the least supercompact cardinal". In $V^{\mathbb{P}}$, level by level equivalence between strong compactness and supercompactness holds, as does $\diamondsuit_\delta$ for every successor and Mahlo cardinal δ and $\square_\delta$ for every δ in a stationary subset of κ. Further, in $V^{\mathbb{P}}$, there is a stationary subset of κ composed of non-weakly compact Mahlo cardinals which reflect stationary sets.*

THEOREM 9. *Let $V \models$ "ZFC + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + GCH + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal". In $V^{\mathbb{P}}$, level by level equivalence between strong compactness and supercompactness holds, as does $\diamondsuit_\delta$ for every successor and Mahlo cardinal δ and $\square_\delta$ for every δ in a stationary subset of κ. Further, in $V^{\mathbb{P}}$, the inaccessible cardinals reflecting stationary sets are precisely the weakly compact cardinals. Finally, every regular Jónsson cardinal in $V^{\mathbb{P}}$ is weakly compact.*

If desired, it is possible to augment the models of Theorems 7–9 still further, so that they satisfy additional instances of $\square$. For example, it is shown in [4] that there are weak forms of $\square$ which may hold above every supercompact cardinal in a universe in which the level by level equivalence between strong compactness and supercompactness is also true. Our methods demonstrate that these additional instances of this weak version of $\square$ (we

([5]) In what follows, depending upon the exact definition of $\mathbb{P}$, $\dot{\mathbb{Q}}$ may be a term for trivial forcing.

refer readers of this paper to [4] for the exact statement) may be assumed to be present in the models of Theorems 7–9.

Of course, our theorems leave many questions open. We conclude our paper by posing a few of them. For instance, in any of our models, are the regular Jónsson cardinals precisely the Ramsey cardinals, as is the case in higher inner models? Are the regular Jónsson cardinals precisely the weakly compact cardinals? Is there even a reasonably uniform characterization of the regular Jónsson cardinals? Or, counter-intuitively, are the non-weakly compact Mahlo cardinals admitting stationary reflection in Theorems 2 and 8 also Jónsson cardinals? In light of the gap between the assumptions of Theorems 5 and 6, is it possible to prove a generalization of Theorem 5 for a universe with a richer large cardinal structure? Finally, is it possible to extend Theorems 2 and 8 so that there exist non-weakly compact Mahlo cardinals which reflect stationary sets above some supercompact cardinal(s)? By [17, pages 69–70], the partial ordering $\mathbb{Q}_\delta$ used in the proofs of Theorems 2 and 8 isn't even ω_1-directed closed, so a positive answer to this question would require the introduction of a highly directed closed partial ordering which forces the existence of the desired kind of Mahlo cardinal.

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