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ABSTRACT PARABOLIC PROBLEMS WITH NON-LIPSCHITZ CRITICAL NONLINEARITIES

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Abstract. The Cauchy problem for a semilinear abstract parabolic equation is considered in a fractional power scale associated with a sectorial operator appearing in the linear main part of the equation. Existence of local solutions is proved for non-Lipschitz nonlinearities satisfying a certain critical growth condition.

1. Introduction. In this article we consider a semilinear abstract parabolic problem with a non-Lipschitz nonlinearity and prove the existence of local solutions in some large phase space of initial data when the nonlinear term satisfies a certain critical growth condition.

Suppose that X is a Banach space and $-A : \mathrm{dom}(A) \subset X \to X$ is a linear operator which generates a C^0 analytic semigroup $\{e^{-At}\} \subset L(X)$. Denote by $\mathcal{FPS}$ the fractional power scale $\{X^\sigma : \sigma \geq 0\}$ generated by (X, A) (see [1]) and suppose that

$$\text{(1.1)} \qquad \begin{array}{l} F \text{ is a continuous map from } X^\alpha \text{ into } X^\beta \text{ for} \\ \text{certain } X^\alpha, X^\beta \in \mathcal{FPS} \text{ with } 1 > \alpha - \beta \geq 0. \end{array}$$

With the above set-up consider the Cauchy problem

$$\text{(1.2)} \qquad \dot{u}(t) + Au(t) = F(u(t)), \qquad t > 0,$$

$$\text{(1.3)} \qquad u(0) = u_0$$

and recall the following result that goes back to [12, Theorem 1] (see also [13, Theorem 6.2.1]).

Proposition 1.1. *If A is a sectorial operator in a Banach space X with compact resolvent,* (1.1) *holds and $u_0 \in X^\alpha$, then there exists a local X^α mild solution $u(t)$ of* (1.2)–(1.3)*; that is, there exists $\tau > 0$ and a function $u \in C([0,\tau], X^\alpha)$ satisfying for $t \in [0,\tau]$ the Cauchy integral formula*

$$\text{(1.4)} \qquad u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} F(u(s))\, ds.$$

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If furthermore $F : X^\alpha \to X^\beta$ *takes bounded sets to bounded sets, then* $u(t)$ *has a continuation (denoted the same and called a maximally defined* X^α*-solution) onto a maximal interval of existence* $[0, \tau_{u_0})$ *such that*

$$\textit{either} \quad \tau_{u_0} = \infty \quad \textit{or} \quad \limsup_{t \to \tau_{u_0}^-} \|u(t)\|_{X^\alpha} = \infty.$$

A nontrivial task arises when (1.2) is to be solved with initial data in a certain larger space X^ζ with $\zeta < \alpha$ and F is not continuous on bounded sets from X^ζ into X^δ for any $\delta \geq 0$ such that $1 > \zeta - \delta \geq 0$; in fact, the map F may not be well defined on X^ζ with values in X^δ for any $\delta \geq 0$ such that $1 > \zeta - \delta \geq 0$. This is the situation that will be investigated in the main body of this paper.

Our concern will be to prove local solvability of (1.2)–(1.3) in a possibly large fractional power space from the scale $\mathcal{FPS}$ assuming that the nonlinear term F satisfies (1.1) and the growth condition

$$\exists_{\rho>1}\, \exists_{c>0} \quad \|F(v)\|_{X^\beta} \leq c(1 + \|v\|^\rho_{X^\alpha}), \quad v \in X^\alpha. \tag{1.5}$$

To do so we will prove the following result.

THEOREM 1.2. *Suppose that* A *is a sectorial operator in a Banach space* X *with compact resolvent and the assumptions* (1.1), (1.5) *hold. Let* $\zeta \geq 0$ *be such that*

$$\alpha > \zeta \geq \frac{-1 - \beta + \alpha\rho}{\rho - 1}, \qquad \zeta > \alpha - \frac{1}{\rho}. \tag{1.6}$$

Then there exists a positive constant θ_0*, depending only on* $A, F, \alpha, \beta, \zeta$*, such that given any* $\tilde{u}_0 \in X^\zeta$ *and any* $\theta \in (0, \theta_0)$*, there are* $r_0, \delta_0 > 0$ *for which the following conditions hold.*

(i) *For each* $u_0 \in X^\alpha \cap B_{X^\zeta}(\tilde{u}_0, r_0)$*, where* $B_{X^\zeta}(\tilde{u}_0, r_0) = \{u_0 \in X^\zeta : \|u_0 - \tilde{u}_0\|_{X^\zeta} < r_0\}$*, any maximally defined* X^α*-solution* $u(t)$ *of* (1.2)–(1.3) *from Proposition* 1.1 *exists for all* $t \in [0, \delta_0]$ *and satisfies the estimate*

$$\sup_{t \in [0, \delta_0]} t^{\alpha - \zeta} \|u(t)\|_{X^\alpha} \leq \theta. \tag{1.7}$$

(ii) *For each* $u_0 \in B_{X^\zeta}(\tilde{u}_0, r_0)$*, for any sequence* $\{u_{0n}\} \subset X^\alpha$ *that converges to* u_0 *in* X^ζ*, and for any sequence* $\{u_n\}$ *where* u_n*,* $n \in \mathbb{N}$*, is a maximally defined* X^α*-solution of* (1.2) *through* $u_n(0) = u_{0n}$ *resulting from Proposition* 1.1*, there is a subsequence* $\{u_{n_k}\}$ *and a function*

$$\phi \in C([0, \delta_0], X^\zeta) \cap C((0, \delta_0], X^{\alpha+\varepsilon}), \qquad \varepsilon \in [0, 1 + \beta - \alpha), \tag{1.8}$$

such that

$$u_{n_k}(t) \to \phi(t) \quad \textit{in } X^{\alpha+\varepsilon} \textit{ as } k \to \infty \textit{ for each } \varepsilon \in [0, 1 + \beta - \alpha) \tag{1.9}$$

uniformly for t *in compact subintervals of* $(0,\delta_0]$ *and*

$$\sup_{t\in(0,\delta_0]} t^{\alpha-\zeta}\|\phi(t)\|_{X^\alpha}\le\theta, \tag{1.10}$$

$$\lim_{t\to 0^+} t^{\alpha-\zeta}\|\phi(t)\|_{X^\alpha}=0. \tag{1.11}$$

(iii) *The function* ϕ *defined in* (ii) *satisfies the variation of constants formula* (1.4) *in* $[0,\delta_0]$.

Consequently, Theorem 1.2 and Proposition 1.1 yield the following local existence result.

THEOREM 1.3. *If* A *is a sectorial operator in a Banach space* X *with compact resolvent,* F *satisfies* (1.1), (1.5), *and* $\zeta\ge 0$ *is such that* (1.6) *holds, then for each* $\tilde u_0\in X^\zeta$ *there are* $r_0,\delta_0>0$ *such that for each* u_0 *in a ball* $B_{X^\zeta}(\tilde u_0,r_0)$ *there exists a mild solution* u *of the problem* (1.2) *in the class*

$$\mathfrak{C}:=\{\chi\in C([0,\delta_0],X^\zeta)\cap C((0,\delta_0],X^\alpha):\lim_{t\to 0^+} t^{\alpha-\zeta}\|\chi(t)\|_{X^\alpha}=0\}. \tag{1.12}$$

Furthermore, u *has a continuation (denoted the same) onto an interval* $[0,\tau_{u_0})$ *such that*

$$\text{either}\quad \tau_{u_0}=\infty\quad\text{or}\quad \limsup_{t\to\tau_{u_0}^-}\|u(t)\|_{X^\alpha}=\infty.$$

Under a stronger assumption on the nonlinear term,

$$\exists_{\rho>1}\exists_{C>0}\,\|F(v)-F(w)\|_{X^\beta}\le C\|v-w\|_{X^\alpha}(1+\|v\|_{X^\alpha}^{\rho-1}+\|w\|_{X^\alpha}^{\rho-1}),\quad v,w\in X^\alpha, \tag{1.13}$$

the problem (1.2) will be locally uniquely solvable in X^ζ with ζ as in (1.6). The following conclusion, which goes back to the results of [2], is a consequence of Theorem 1.2 and (1.13).

COROLLARY 1.4. *If* A *is a sectorial operator in a Banach space* X *with compact resolvent,* F *satisfies* (1.1), (1.13), *and* $\zeta\ge 0$ *is such that* (1.6) *holds, then Theorem* 1.3 *applies and the solution of the problem* (1.2)–(1.3) *in the class* $\mathfrak{C}$ *is unique.*

REMARK 1.5. *Under the assumptions of Corollary* 1.4, *for suitably chosen* r_0 *and* δ_0, *there are* $L,\tilde L>0$ *such that for each* $u_0\in B_{X^\zeta}(\tilde u_0,r_0)$ *the corresponding solutions* $u,\tilde u$ *of* (1.2) *with* $u(0)=u_0$ *and* $\tilde u(0)=\tilde u_0$ *satisfy*

$$\|u(t)-\tilde u(t)\|_{X^\alpha}\le Lt^{\zeta-\alpha}\|u_0-\tilde u_0\|_{X^\zeta},\quad t\in(0,\delta_0], \tag{1.14}$$

$$\|u(t)-\tilde u(t)\|_{X^\zeta}\le \tilde L\|u_0-\tilde u_0\|_{X^\zeta},\quad t\in[0,\delta_0]. \tag{1.15}$$

To compare our results with those in the references we remark that in [2] the problem (1.2) has been considered under the assumption that F belongs to the class $\mathcal{F}[X^1, X^0, \varepsilon, \rho, \gamma(\varepsilon)]$ of ε-regular maps such that

$$\|F(\phi_1) - F(\phi_2)\|_{X^{\gamma(\varepsilon)}} \le C\|\phi_1 - \phi_2\|_{X^{1+\varepsilon}} \, (1 + \|\phi_1\|^{\rho-1}_{X^{1+\varepsilon}} + \|\phi_2\|^{\rho-1}_{X^{1+\varepsilon}}) \tag{1.16}$$

for $\phi_1, \phi_2 \in X^{1+\varepsilon}$, with constants $\rho > 1$, $\varepsilon \in (0, 1/\rho)$, $\gamma(\varepsilon) \in [\rho\varepsilon, 1)$ and $C > 0$.

Note that (1.16) implies (1.13) and hence also (1.5) with $\alpha = 1 + \varepsilon$ and $\beta = \gamma(\varepsilon)$. In the latter case when $\gamma(\varepsilon) = \varepsilon\rho$ we can also allow $\zeta = 1$ in (1.6), which corresponds to the critical case described in [2]. Thus, for semilinear problems with sectorial operators possessing compact resolvents, Theorems 1.2 and 1.3 generalize earlier considerations of [2] to the case of non-Lipschitz nonlinearities.

The proofs of Theorems 1.2, 1.3 and Corollary 1.4 will be given in Section 2. In the closing Section 3 some applications involving $2m$th order parabolic problems and strongly damped wave equations will be discussed.

2. Abstract results. We now proceed with the proofs of the results reported in Section 1. We start from the following two propositions, which go back to the results in [2, 11].

Proposition 2.1. *Let A be a sectorial operator in a Banach space X. Then*

(i) *for $\sigma_0 > 0$ and $\tau_0 > 0$ there is a positive constant M such that for any $0 \le \gamma \le \sigma \le \sigma_0$ and all $v \in X^\gamma$ and $t \in (0, \tau_0]$,*

$$\|e^{-At}v\|_{X^\sigma} \le Mt^{\gamma-\sigma}\|v\|_{X^\gamma}, \tag{2.1}$$

(ii) *given $\sigma > \gamma \ge 0$ and a subset J of X^γ precompact in X^γ, we have*

$$\lim_{t\to 0^+} \sup_{v\in J} \|t^{\sigma-\gamma}e^{-At}v\|_{X^\sigma} = 0, \tag{2.2}$$

(iii) *given $\sigma > \gamma \ge 0$ and $\tilde{v} \in X^\gamma$ we also have*

$$\forall_{\varepsilon>0}\ \exists_{r>0}\ \exists_{\delta>0}\ \forall_{v\in B_{X^\gamma}(\tilde{v},r)}\ \forall_{t\in(0,\delta]} \quad \|t^{\sigma-\gamma}e^{-At}v\|_{X^\sigma} < \varepsilon. \tag{2.3}$$

Proof. Part (i) follows from [11, Theorem 1.4.3].

Now, if J is precompact in X^γ, for any $\varepsilon > 0$ there exist $k \in \mathbb{N}$ and $v_1, \dots, v_k \in X^\gamma$ such that

$$J \subset \mathrm{cl}_{X^\gamma} J \subset \bigcup_{j=1}^{k} B_{X^\gamma}\Big(v_j, \frac{\varepsilon}{4M}\Big).$$

Therefore, by density of X^σ in X^γ,

$$J \subset \bigcup_{j=1}^{k} B_{X^\gamma}\Big(\tilde{v}_j, \frac{\varepsilon}{2M}\Big) \quad \text{with certain } \tilde{v}_1, \dots, \tilde{v}_k \in X^\sigma \subset X^\gamma, \tag{2.4}$$

Also, for $j = 1, \dots, k$ there exists $\delta_j \in (0,1)$ such that

$$t^{\sigma-\gamma}\|e^{-At}\tilde{v}_j\|_{X^\sigma} \le t^{\sigma-\gamma}M\|\tilde{v}_j\|_{X^\sigma} < \varepsilon/2 \quad \text{for } t < \delta_j. \tag{2.5}$$

Combining (2.1) and (2.4)–(2.5), we find that if $v \in J$ then for $t < \delta = \min\{\delta_1, \dots, \delta_k\}$,

$$\begin{aligned} t^{\sigma-\gamma}\|e^{-At}v\|_{X^\sigma} &\le t^{\sigma-\gamma}(\|e^{-At}\tilde{v}_j\|_{X^\sigma} + \|e^{-At}(v-\tilde{v}_j)\|_{X^\sigma}) \\ &< \varepsilon/2 + M\|v-\tilde{v}_j\|_{X^\gamma} < \varepsilon, \end{aligned}$$

which proves (ii).

Finally, if (iii) were false then one could choose $\tilde{v} \in X^\gamma$, $\varepsilon > 0$ and sequences $v_n \overset{X^\gamma}{\to} \tilde{v}$ and $t_n \to 0^+$ such that $\|t_n^{\sigma-\gamma}e^{-At_n}v_n\|_{X^\sigma} \ge \varepsilon$, which contradicts (ii).

Proposition 2.2. *If A is a sectorial operator in a Banach space X, (1.1), (1.5) hold and $u \in C([0,\tau], X^\alpha)$ for some $\tau > 0$, then for $\zeta \ge 0$ satisfying (1.6) and each $0 \le t \le \tau$ the following estimate holds:*

$$\begin{aligned} &t^{\alpha-\zeta}\int_0^t \|e^{-A(t-s)}F(u(s))\|_{X^\alpha}\,ds \\ &\quad\le \frac{cMt^{1+\beta-\zeta}}{1+\beta-\alpha} + cMt^{1+\beta-\zeta-\rho(\alpha-\zeta)}\lambda^\rho(t)\mathbf{B}(1+\beta-\alpha, 1-\rho(\alpha-\zeta)) \end{aligned}$$

where

$$\lambda(t) := \sup_{s\in(0,t]} s^{\alpha-\zeta}\|u(s)\|_{X^\alpha} \tag{2.6}$$

and $\mathbf{B}$ is the beta function, $\mathbf{B}(a,b) = \int_0^1 (1-x)^{a-1}x^{b-1}\,dx$ for $a,b > 0$.

Proof. We start from the remark that $1+\beta-\alpha > 0$ by assumption (1.1), and consequently $1+\beta-\zeta > 0$ as $\alpha > \zeta$ in (1.6). Condition (1.6) also implies that $1+\beta-\zeta-\rho(\alpha-\zeta) \ge 0$ and $1-\rho(\alpha-\zeta) > 0$.

Using the estimate

$$\|e^{-A(t-s)}\|_{\mathcal{L}(X^\beta,X^\alpha)} \le M(t-s)^{\beta-\alpha}, \qquad \tau \ge t > s \ge 0,$$

observe that, by (1.5),

$$\begin{aligned} t^{\alpha-\zeta}\int_0^t \|e^{-A(t-s)}F(u(s))\|_{X^\alpha}\,ds &\le t^{\alpha-\zeta}M\int_0^t (t-s)^{\beta-\alpha}\|F(u(s))\|_{X^\beta}\,ds \\ &\le t^{\alpha-\zeta}cM\int_0^t (t-s)^{\beta-\alpha}(1+\|u(s)\|_{X^\alpha}^\rho)\,ds \\ &= \frac{cM}{1+\beta-\alpha}t^{1+\beta-\zeta} + t^{\alpha-\zeta}cM\int_0^t (t-s)^{\beta-\alpha}\|u(s)\|_{X^\alpha}^\rho\,ds. \end{aligned}$$

Now we estimate the second term on the right hand side:

$$
\begin{aligned}
t^{\alpha-\zeta} cM \int_0^t (t-s)^{\beta-\alpha} \|u(s)\|^{\rho}_{X^\alpha}\, ds \\
&= t^{\alpha-\zeta} cM \int_0^t (t-s)^{\beta-\alpha} s^{-\rho(\alpha-\zeta)} \|s^{\alpha-\zeta} u(s)\|^{\rho}_{X^\alpha}\, ds \\
&\le t^{\alpha-\zeta} cM \lambda^{\rho}(t) \int_0^t (t-s)^{\beta-\alpha} s^{-\rho(\alpha-\zeta)}\, ds \\
&= cM t^{1+\beta-\zeta-\rho(\alpha-\zeta)} \lambda^{\rho}(t) \int_0^1 (1-s)^{\beta-\alpha} s^{-\rho(\alpha-\zeta)}\, ds \\
&= cM t^{1+\beta-\zeta-\rho(\alpha-\zeta)} \lambda^{\rho}(t) \mathbf{B}(1+\beta-\alpha, 1-\rho(\alpha-\zeta)). \ \blacksquare
\end{aligned}
$$

We now recall from [9, 7.5.7] the following compactness criterion.

PROPOSITION 2.3. *Suppose that $\mathcal{X}$ is a Banach space and $\mathcal{Y}$ is a compact metric space. A necessary and sufficient condition for $\mathcal{G} \subset C(\mathcal{Y}, \mathcal{X})$ to be relatively compact in $C(\mathcal{Y}, \mathcal{X})$ is that $\mathcal{G}$ is equicontinuous and, for each $y \in \mathcal{Y}$, the set $\{g(y) : y \in \mathcal{Y}\}$ is relatively compact in $\mathcal{X}$.*

Proof of Theorem 1.2. The proof will be given in a sequence of lemmas.
We define

$$D := cM\mathbf{B}(1+\beta-\alpha, 1-\rho(\alpha-\zeta)), \tag{2.7}$$

$$\theta_0 := (2D)^{-1/(\rho-1)}, \tag{2.8}$$

where $\alpha, \beta, \zeta, \rho, c$ are as in the assumptions of Theorem 1.2 and M is chosen by application of Proposition 2.1(i) with $\tau_0 = 1$ and $\sigma_0 = \beta + 1$, that is, uniformly for the unit time interval and for the portion of the fractional power scale considered below.

Also, fix $\tilde{u}_0 \in X^\zeta$ and choose $\theta > 0$ such that

$$\theta_0 = (2D)^{-1/(\rho-1)} > \theta. \tag{2.9}$$

Then, with the aid of (2.3), choose $r_0 > 0$ and $\delta_0 \in (0, 1]$ for which

$$\sup_{u_0 \in B_{X^\zeta}(\tilde{u}_0, r_0)} t^{\alpha-\zeta} \|e^{-At} u_0\|_{X^\alpha} < \tfrac{1}{4}\theta \quad \text{for } t \in (0, \delta_0] \tag{2.10}$$

and

$$\frac{cM}{1+\beta-\alpha} \delta_0^{1+\beta-\alpha} < \tfrac{1}{4}\theta. \tag{2.11}$$

With the above set-up we now prove the following lemmas.

LEMMA 2.4. *If $u_0 \in X^\alpha \cap B_{X^\zeta}(\tilde{u}_0, r_0)$ and $u \in C([0, \tau_{u_0}), X^\alpha)$ is a maximally defined X^α-solution of (1.2)–(1.3) through u_0 defined on its maximal* *interval of existence* $[0,\tau_{u_0})$ *as in Proposition* 1.1, *then*

$$\tau_{u_0} > \delta_0 \tag{2.12}$$

and

$$\forall_{\tau\in[0,\delta_0)} \quad \tau^{\alpha-\zeta}\|u(\tau)\|_{X^\alpha} < \theta. \tag{2.13}$$

Proof. Writing the variation of constants formula associated with (1.2) we know via Proposition 2.2 that, as long as the solution u exists,

$$\begin{aligned} t^{\alpha-\zeta}\|u(t)\|_{X^\alpha} &\le t^{\alpha-\zeta}\|e^{-At}u_0\|_{X^\alpha} + t^{\alpha-\zeta}\Big\|\int_0^t e^{-A(t-s)}F(u(s))\,ds\Big\|_{X^\alpha} \\ &\le t^{\alpha-\zeta}\|e^{-At}u_0\|_{X^\alpha} + \frac{cM}{1+\beta-\alpha}t^{1+\beta-\zeta} \\ &\quad + cMt^{1+\beta-\zeta-\rho(\alpha-\zeta)}\lambda^\rho(t)\mathbf{B}(1+\beta-\alpha, 1-\rho(\alpha-\zeta)), \end{aligned} \tag{2.14}$$

where $\lambda(t)$ is given in (2.6). From (2.14) and (2.10)–(2.11) we next have

$$t^{\alpha-\zeta}\|u(t)\|_{X^\alpha} < \tfrac{1}{2}\theta + D(\sup_{s\in[0,\tau]}\{s^{\alpha-\zeta}\|u(s)\|_{X^\alpha}\})^\rho \quad \text{for } 0 \le t \le \tau < \min\{\delta_0,\tau_{u_0}\},$$

and consequently,

$$\lambda(\tau) = \sup_{s\in[0,\tau]} s^{\alpha-\zeta}\|u(s)\|_{X^\alpha} \le \tfrac{1}{2}\theta + D\lambda^\rho(\tau) \quad \text{for } 0 \le \tau < \min\{\delta_0,\tau_{u_0}\}. \tag{2.15}$$

Note that $\lambda(0) = 0$ and $\lambda(s)$ is continuous with respect to $s \in [0,\tau_{u_0})$ as $u \in C([0,\tau_{u_0}), X^\alpha)$, which ensures that the set

$$I_{u_0} = \{\tau \in [0, \min\{\delta_0,\tau_{u_0}\}) : \forall_{s\in[0,\tau]}\ s^{\alpha-\zeta}\|u(s)\|_{X^\alpha} < \theta\}$$

is nonvoid.

Suppose now that (2.12) is not true, so $\min\{\delta_0,\tau_{u_0}\} = \tau_{u_0}$ is finite and, by Proposition 1.1, $\limsup_{s\to\tau_{u_0}^-}\|u(s)\|_{X^\alpha} = \infty$. Defining

$$\tilde{\tau}_{u_0} := \sup I_{u_0}$$

we have $\tilde{\tau}_{u_0} < \tau_{u_0}$, $\sup_{s\in[0,\tilde{\tau}_{u_0}]} s^{\alpha-\zeta}\|u(s)\|_{X^\alpha} = \theta$ and, from (2.9), (2.15),

$$\sup_{s\in[0,\tilde{\tau}_{u_0}]} s^{\alpha-\zeta}\|u(s)\|_{X^\alpha} = \theta \le \tfrac{1}{2}\theta + D\theta^\rho. \tag{2.16}$$

Since for θ satisfying (2.9) it is evident that (2.16) yields $\theta \le \frac{1}{2}\theta + D\theta^\rho < \theta$, we reach a contradiction and thus (2.12) is true.

From the earlier reasoning, any X^α-solution $u \in C([0,\tau_{u_0}), X^\alpha)$ of (1.2)–(1.3) through $u_0 \in X^\alpha \cap B_{X^\zeta}(\tilde{u}_0, r_0)$ from Proposition 1.1 actually cannot cease to exist before time δ_0; in particular $\min\{\delta_0, \tau_{u_0}\} = \delta_0$ for each $u_0 \in X^\alpha \cap B_{X^\zeta}(\tilde{u}_0, r_0)$.

Now, if the condition (2.13) fails, then there are $\tau \in [0, \delta_0)$ and $u_0 \in X^\alpha \cap B_{X^\zeta}(\tilde{u}_0, r_0)$ such that $\tau^{\alpha-\zeta}\|u(\tau)\|_{X^\alpha} = \theta$ and $t^{\alpha-\zeta}\|u(t)\|_{X^\alpha} < \theta$ for $t \in [0, \tau)$. Consequently, as before we get from (2.15)

$$\sup_{s\in[0,\tau]} s^{\alpha-\zeta}\|u(s)\|_{X^\alpha} = \theta \le \tfrac{1}{2}\theta + D\theta^\rho < \theta,$$

which is absurd.

Lemma 2.4 is thus proved. ∎

This completes the proof of part (i). For the rest of the proof of Theorem 1.2 let us fix $u_0 \in B_{X^\zeta}(\tilde{u}_0, r_0)$ and choose any $\{u_{0n}\} \subset X^\alpha$ that converges to u_0 in X^ζ. Since $\mathcal{FPS}$ is compactly embedded, without loss of generality we may assume that $u_{0n} \in B_{X^\zeta}(\tilde{u}_0, r_0)$ for each $n \in \mathbb{N}$.

In what follows we write u_n for a maximally defined X^α-solution of (1.2) with $u_n(0) = u_{0n}$ resulting from Proposition 1.1. Since, via Lemma 2.4, the domain of definition of each u_n actually contains $[0, \delta_0]$, we will establish in this interval suitable properties of the family of maps

$$\{u_n\} =: H.$$

Lemma 2.5. *The family H has the property*

$$\forall_{\varepsilon>0}\ \exists_{h\in(0,\delta_0)}\ \forall_{n\in\mathbb{N}}\ \forall_{\tau\in[0,h)} \quad \tau^{\alpha-\zeta}\|u_n(\tau)\|_{X^\alpha} < \varepsilon. \tag{2.17}$$

Proof. For each $\varepsilon > 0$ there exists $h \in (0, \delta_0]$ for which

$$cM(1+\beta-\alpha)^{-1}h^{1+\beta-\alpha} < \varepsilon/4$$

and, via (2.2), also

$$\sup_{n\in\mathbb{N}} t^{\alpha-\zeta}\|e^{-At}u_{0n}\|_{X^\alpha} < \varepsilon/4 \quad \text{for } t \in [0, h).$$

Recalling (2.7), (2.14) we have

$$t^{\alpha-\zeta}\|u_n(t)\|_{X^\alpha} < \varepsilon/2 + D\Big(\sup_{s\in[0,\tau]}\{s^{\alpha-\zeta}\|u_n(s)\|_{X^\alpha}\}\Big)^\rho \quad \text{for } 0 \le t \le \tau < h,$$

and with the aid of (2.13) we obtain

$$\begin{aligned}\sup_{s\in[0,\tau]} & s^{\alpha-\zeta}\|u_n(s)\|_{X^\alpha} \\ &\le \varepsilon/2 + D\theta^{\rho-1}\sup_{s\in[0,\tau]}\{s^{\alpha-\zeta}\|u_n(s)\|_{X^\alpha}\} \quad \text{for } 0 \le \tau < h.\end{aligned} \tag{2.18}$$

From (2.9), (2.18), we now get

$$\sup_{s\in[0,\tau]} s^{\alpha-\zeta}\|u_n(s)\|_{X^\alpha} < \varepsilon \quad \text{for } 0 \le \tau < h.$$

Hence $\tau^{\alpha-\zeta}\|u_n(\tau)\|_{X^\alpha} < \varepsilon$ for all $n \in \mathbb{N}$ and $\tau \in [0, h)$, and the proof is complete. ∎

LEMMA 2.6 (Equicontinuity of H). *For each* $\varepsilon \in (0, 1+\beta-\alpha)$ *and* $k \in \mathbb{N}$ *satisfying* $1/k < \delta_0$ *the following condition holds:*

$$\forall_{\nu>0}\ \exists_{\eta>0}\ \forall_{t_1,t_2\in[1/k,\delta_0]}\ \forall_{n\in\mathbb{N}} \quad |t_2 - t_1| < \eta \Rightarrow \|u_n(t_2) - u_n(t_1)\|_{X^{\alpha+\varepsilon}} < \nu. \tag{2.19}$$

Proof. Fix $k > \delta_0^{-1}$, $\varepsilon \in (0, 1+\beta-\alpha)$, and let $1/k \le t_1 < t_2 \le \delta_0$. From the variation of constants formula we have

$$\begin{aligned}\|u_n(t_2) - u_n(t_1)\|_{X^{\alpha+\varepsilon}} &\le \|(e^{-At_2} - e^{-At_1})u_{0n}\|_{X^{\alpha+\varepsilon}} \\ &\quad + \Big\| \int_0^{t_1} (e^{-A(t_2-s)} - e^{-A(t_1-s)}) F(u_n(s))\, ds \Big\|_{X^{\alpha+\varepsilon}} \\ &\quad + \Big\| \int_{t_1}^{t_2} e^{-A(t_2-s)} F(u_n(s))\, ds \Big\|_{X^{\alpha+\varepsilon}} =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3\end{aligned} \tag{2.20}$$

and we next estimate each term $\mathcal{J}_j$, $j = 1, 2, 3$.

For the last term, using (1.5) and (2.13) we obtain

$$\begin{aligned}\mathcal{J}_3 &\le \int_{t_1}^{t_2} \|e^{-A(t_2-s)} F(u_n(s))\|_{X^{\alpha+\varepsilon}}\, ds \\ &\le M \int_{t_1}^{t_2} (t_2 - s)^{\beta-\alpha-\varepsilon} \|F(u_n(s))\|_{X^\beta}\, ds \\ &\le cM \int_{t_1}^{t_2} (t_2 - s)^{\beta-\alpha-\varepsilon} (1 + \|u_n(s)\|_{X^\alpha}^\rho)\, ds \\ &= cM \int_{t_1}^{t_2} (t_2 - s)^{\beta-\alpha-\varepsilon}\, ds \\ &\quad + cM \int_{t_1}^{t_2} (t_2 - s)^{\beta-\alpha-\varepsilon} s^{\rho(\zeta-\alpha)} (s^{\alpha-\zeta} \|u_n(s)\|_{X^\alpha})^\rho\, ds \\ &\le cM \frac{1}{1+\beta-\alpha-\varepsilon} (t_2 - t_1)^{1+\beta-\alpha-\varepsilon} \\ &\quad + c\theta^\rho M \int_{t_1}^{t_2} (t_2 - s)^{\beta-\alpha-\varepsilon} s^{\rho(\zeta-\alpha)}\, ds \\ &\le cM \frac{1}{1+\beta-\alpha-\varepsilon} (t_2 - t_1)^{1+\beta-\alpha-\varepsilon} \\ &\quad + c\theta^\rho M \frac{k^{\rho(\alpha-\zeta)}}{1+\beta-\alpha-\varepsilon} (t_2 - t_1)^{1+\beta-\alpha-\varepsilon}.\end{aligned} \tag{2.21}$$

Now we fix $\delta > 0$ such that $1+\beta-\alpha-\delta-\varepsilon > 0$ and we will use the estimate (1.5) together with

$$\|(e^{-At}-I)w\|_X \le c(\delta)t^{\delta}\|w\|_{X^\delta}, \quad t\in[0,1],\ w\in X^\delta, \tag{2.22}$$

(see [11, Theorem 1.4.3]) to get

$$\begin{aligned}
\mathcal{J}_2 &\le \int_0^{t_1} \|[e^{-A(t_2-t_1)}-I]e^{-A(t_1-s)}F(u_n(s))\|_{X^{\alpha+\varepsilon}}\,ds\\
&\le c(\delta)(t_2-t_1)^{\delta}\int_0^{t_1}\|e^{-A(t_1-s)}F(u_n(s))\|_{X^{\alpha+\delta+\varepsilon}}\,ds\\
&\le c(\delta)M(t_2-t_1)^{\delta}\int_0^{t_1}(t_1-s)^{\beta-\alpha-\delta-\varepsilon}\|F(u_n(s))\|_{X^\beta}\,ds\\
&\le cc(\delta)M(t_2-t_1)^{\delta}\int_0^{t_1}(t_1-s)^{\beta-\alpha-\delta-\varepsilon}(1+\|u_n(s)\|^{\rho}_{X^\alpha})\,ds\\
&\le cc(\delta)M(t_2-t_1)^{\delta}\int_0^{t_1}(t_1-s)^{\beta-\alpha-\delta-\varepsilon}\,ds\\
&\quad + cc(\delta)M(t_2-t_1)^{\delta}\int_0^{t_1}(t_1-s)^{\beta-\alpha-\delta-\varepsilon}s^{\rho(\zeta-\alpha)}(s^{\alpha-\zeta}\|u_n(s))\|_{X^\alpha})^{\rho}\,ds.
\end{aligned}$$

Consequently, using (2.13) we have

$$\begin{aligned}
\mathcal{J}_2 &\le cc(\delta)M(t_2-t_1)^{\delta}\bigg(\frac{t_1^{1+\beta-\alpha-\delta-\varepsilon}}{1+\beta-\alpha-\delta-\varepsilon}+\theta^{\rho}\int_0^{t_1}(t_1-s)^{\beta-\alpha-\delta-\varepsilon}s^{\rho(\zeta-\alpha)}\,ds\bigg)\\
&\le cc(\delta)M(t_2-t_1)^{\delta}\frac{\delta_0^{1+\beta-\alpha-\delta-\varepsilon}}{1+\beta-\alpha-\delta-\varepsilon}\\
&\quad + cc(\delta)\theta^{\rho}M(t_2-t_1)^{\delta}t_1^{1+\beta-\alpha-\delta-\varepsilon+\rho(\zeta-\alpha)}\mathbf{B}(1+\beta-\alpha-\delta-\varepsilon,1+\rho(\zeta-\alpha)),
\end{aligned}\tag{2.23}$$

where

$$t_1^{1+\beta-\alpha-\delta-\varepsilon+\rho(\zeta-\alpha)} \le \delta_0^{1+\beta-\alpha-\delta-\varepsilon}k^{\rho(\alpha-\zeta)}.$$

Estimating $\mathcal{J}_1$ we use (2.22) to get

$$\begin{aligned}
\mathcal{J}_1 &= \|(e^{-A(t_2-t_1)}-I)e^{-At_1}u_{0n}\|_{X^{\alpha+\varepsilon}} \le c(\delta)(t_2-t_1)^{\delta}\|e^{-At_1}u_{0n}\|_{X^{\alpha+\delta+\varepsilon}}\\
&\le c(\delta)M(t_2-t_1)^{\delta}t_1^{\zeta-\alpha-\delta-\varepsilon}\|u_{0n}\|_{X^\zeta} \le c(\delta)M(t_2-t_1)^{\delta}k^{-\zeta+\alpha+\delta+\varepsilon}\|u_{0n}\|_{X^\zeta}.
\end{aligned}\tag{2.24}$$

Thus, for every $k\in\mathbb{N}$, $t_1,t_2\in[1/k,\delta_0]$ and for each ν we infer from (2.20)–(2.24) that

$$\|u_n(t_2)-u_n(t_1)\|_{X^{\alpha+\varepsilon}} \le \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3 < \nu$$

whenever $n \in \mathbb{N}$ and $|t_1 - t_2| \le \eta(\nu)$, where $\eta(\nu)$ is a multiple of $\nu^{1/\kappa}$ and $\kappa = \min\{\delta, 1+\beta-\alpha-\varepsilon\}$. Consequently, the family $H = \{u_n : n \in \mathbb{N}\}$ can be viewed as an equicontinuous subfamily of $C([1/k, \delta_0], X^{\alpha+\varepsilon})$ for each $k \in \mathbb{N}$, which completes the proof of Lemma 2.6. ∎

For each $t \in (0, \delta_0]$ we now define the set

$$H(t) = \{u_n(t) : n \in \mathbb{N}\}$$

and prove that $H(t)$ is precompact in $X^{\alpha+\varepsilon}$ whenever $\varepsilon \in (0, 1+\beta-\alpha)$. Since by assumption the embedding

$$X^{\sigma} \subset X^{\tilde{\sigma}}, \qquad \sigma > \tilde{\sigma} \ge 0,$$

is compact, this will be a consequence of the boundedness of the solutions derived in the lemma below.

Lemma 2.7 (Boundedness of $H(t)$). *For each $\varepsilon \in (0, 1+\beta-\alpha)$ and every $t \in (0, \delta_0]$, $H(t)$ is a bounded subset of the space $X^{\alpha+\varepsilon}$.*

Proof. Choose any $t \in (0, \delta_0]$ and $\varepsilon > 0$ such that $1+\beta-\alpha-\varepsilon > 0$. Note that

$$\begin{aligned}
\|u_n(t)\|_{X^{\alpha+\varepsilon}} &\le \|e^{-At}u_{0n}\|_{X^{\alpha+\varepsilon}} + \Big\| \int_0^t e^{-A(t-s)} F(u_n(s))\, ds \Big\|_{X^{\alpha+\varepsilon}} \\
&\le M t^{-\alpha-\varepsilon+\zeta} \|u_{0n}\|_{X^{\zeta}} + M \int_0^t (t-s)^{\beta-\alpha-\varepsilon} \|F(u_n(s))\|_{X^{\beta}}\, ds \\
&\le M t^{-\alpha-\varepsilon+\zeta} \|u_{0n}\|_{X^{\zeta}} + cM \int_0^t (t-s)^{\beta-\alpha-\varepsilon} (1 + \|u_n(s)\|_{X^{\alpha}}^{\rho})\, ds
\end{aligned}$$

and hence, via (2.13),

$$\begin{aligned}
\|u_n(t)\|_{X^{\alpha+\varepsilon}} &\le M t^{-\alpha-\varepsilon+\zeta} \|u_{0n}\|_{X^{\zeta}} + cM \int_0^t (t-s)^{\beta-\alpha-\varepsilon}\, ds \\
&\quad + cM \int_0^t (t-s)^{\beta-\alpha-\varepsilon} s^{\rho(\zeta-\alpha)} (s^{\alpha-\zeta} \|u_n(s)\|_{X^{\alpha}})^{\rho}\, ds \\
&\le M t^{-\alpha-\varepsilon+\zeta} \|u_{0n}\|_{X^{\zeta}} \\
&\quad + cM \int_0^t (t-s)^{\beta-\alpha-\varepsilon}\, ds + c\theta^{\rho} M \int_0^t (t-s)^{\beta-\alpha-\varepsilon} s^{\rho(\zeta-\alpha)}\, ds \\
&\le M t^{-\alpha-\varepsilon+\zeta} \|u_{0n}\|_{X^{\zeta}} + cM \frac{t^{1+\beta-\alpha-\varepsilon}}{1+\beta-\alpha-\varepsilon} \\
&\quad + c\theta^{\rho} M t^{1+\beta-\alpha-\varepsilon-\rho(\alpha-\zeta)} \mathbf{B}(1+\beta-\alpha-\varepsilon, 1-\rho(\alpha-\zeta)).
\end{aligned}$$

Since $\{u_{0n}\}$ is convergent in X^{ζ}, it is also bounded, which completes the proof of Lemma 2.7. ∎

We are now ready to construct a mild solution of (1.2) through the initial condition $u_0 \in B_{X^\varsigma}(\tilde{u}_0, r_0)$.

LEMMA 2.8. *There exists a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that, for each* $\varepsilon \in [0, 1+\beta-\alpha)$, $\{u_{n_k}\}$ *converges in* $X^{\alpha+\varepsilon}$ *almost uniformly on* $(0,\delta_0]$ *to a function* $\phi \in C((0,\delta_0], X^{\alpha+\varepsilon})$ *satisfying* (1.10).

Proof. Let

$$\varepsilon_k = 1+\beta-\alpha-1/k, \quad \text{where} \quad k \geq \max\{(1+\beta-\alpha)^{-1}, \delta_0^{-1}\} =: k^*.$$

From Proposition 2.3 and Lemmas 2.6, 2.7 we conclude that for each such $k \in \mathbb{N}$ the set H is precompact in $C([1/k,\delta_0], X^{\alpha+\varepsilon_k})$. Hence

$$\forall_{k\geq k^*}\exists_{\{n_l^{(k)}\}_{l\in\mathbb{N}}}\exists_{\phi_k\in C([1/k,\delta_0],X^{\alpha+\varepsilon_k})}\forall_{t\in[1/k,\delta_0]} \quad u_{n_l^{(k)}}(t) \xrightarrow{X^{\alpha+\varepsilon_k}} \phi_k(t). \tag{2.25}$$

We remark that $\{n_l^{(k)}\}$ in (2.25) can be chosen in such a manner that if $k_1 < k_2$, then for all $i \in \mathbb{N}$ we have $n_i^{(k_2)} \in \{n_j^{(k_1)} : j \in \mathbb{N}\}$. Consequently, $\phi_{k_2}|_{[1/k_1,\delta_0]}(t) = \phi_{k_1}(t)$ for $t \in [1/k_1, \delta_0]$.

We also remark that, by a standard diagonal argument, there exists a subsequence $\{u_{n_m}\}$ of $\{u_n\}$, and a function ϕ defined on $(0,\delta_0]$ with values in $X^{\alpha+\varepsilon}$, such that, whenever $\delta \in (0,\delta_0)$ and $\varepsilon \in [0,1+\beta-\alpha)$,

$$u_{n_m}(t) \to \phi(t) \quad \text{in } X^{\alpha+\varepsilon} \text{ uniformly for } t \in [\delta,\delta_0]. \tag{2.26}$$

Since from (2.13) we have

$$\forall_{t\in[0,\delta_0]}\ \forall_{m\in\mathbb{N}} \quad t^{\alpha-\zeta}\|u_{n_m}(t)\|_{X^\alpha} \leq \theta, \tag{2.27}$$

passing to the limit as $m \to \infty$ we obtain

$$\forall_{t\in(0,\delta_0]} \quad t^{\alpha-\zeta}\|\phi(t)\|_{X^\alpha} \leq \theta, \tag{2.28}$$

which completes the proof of Lemma 2.8. ∎

In what follows we show that, extending ϕ to the interval $[0,\delta_0]$ by

$$\phi(0) = u_0$$

and writing again ϕ for this extension, we have

$$\phi \in C([0,\delta_0], X^\zeta).$$

In fact, since ϕ is continuous on $(0,\delta_0]$ as a map with values in X^α, it suffices to prove the right hand continuity of ϕ at $t=0$ in the X^ζ norm.

LEMMA 2.9. *The function* ϕ *in Lemma* 2.8 *satisfies* (1.11) *and*

$$\|\phi(t) - u_0\|_{X^\varsigma} \to 0 \quad \text{as } t \to 0^+.$$

Proof. Firstly, from (2.17) and (2.26), passing to the limit as $k \to \infty$ we infer that

$$\forall_{\varepsilon>0}\ \exists_{h\in(0,\delta_0)}\ \forall_{s\in(0,h]} \quad s^{\alpha-\zeta}\|\phi(s)\|_{X^\alpha} \leq \varepsilon. \tag{2.29}$$

Secondly, we use (1.5) to estimate the difference $U_{n_k}(t) := \|u_{n_k} - u_0\|_{X^\zeta}$. If $\zeta \geq \beta$ then

$$\begin{aligned} U_{n_k}(t) &\leq \|e^{-At}u_{0n_k} - u_0\|_{X^\zeta} + \int_0^t \|e^{-A(t-s)}F(u_{n_k}(s))\|_{X^\zeta}\, ds \\ &\leq \|e^{-At}u_{0n_k} - u_0\|_{X^\zeta} + M\int_0^t (t-s)^{\beta-\zeta}\|F(u_{n_k}(s))\|_{X^\beta}\, ds \\ &\leq \|e^{-At}u_{0n_k} - u_0\|_{X^\zeta} + cM\int_0^t (t-s)^{\beta-\zeta}(1 + \|u_{n_k}(s)\|^\rho_{X^\alpha})\, ds \\ &\leq \|e^{-At}u_{0n_k} - u_0\|_{X^\zeta} + cM\int_0^t (t-s)^{\beta-\zeta}\, ds \\ &\quad + cM\int_0^t (t-s)^{\beta-\zeta}s^{\rho(\zeta-\alpha)}(\sup_{s\in(0,t)} s^{\alpha-\zeta}\|u_{n_k}(s)\|_{X^\alpha})^\rho\, ds,\ t \in [0,\delta_0], \end{aligned}$$

so that for $\tilde{D} := cM\mathbf{B}(1+\beta-\zeta, 1-\rho(\alpha-\zeta))$ we have

$$\begin{aligned} U_{n_k}(t) \leq \|e^{-At}u_{0n_k} - u_0\|_{X^\zeta} + cM\frac{t^{1+\beta-\zeta}}{1+\beta-\zeta} \\ + \tilde{D}(\sup_{s\in(0,t)} s^{\alpha-\zeta}\|u_{n_k}(s)\|_{X^\alpha})^\rho, \qquad t \in [0,\delta_0]. \end{aligned} \tag{2.30}$$

We remark that if $\beta > \zeta$ then there exists $\tilde{D} > 0$ such that the estimate of the form (2.30) holds as well.

Recalling that $1+\beta-\zeta > 1+\beta-\alpha > 0$, u_{0n_k} converges to u_0 in X^ζ, $\{e^{-At}\}$ is a C^0-semigroup and (2.29) holds, we infer from (2.30) that

$$\forall_{\varepsilon>0}\ \exists_{\delta\in(0,\delta_0)}\ \exists_{k_0>0}\ \forall_{k\geq k_0}\ \forall_{t\in(0,\delta]} \quad U_{n_k}(t) \leq \varepsilon.$$

Then passing to the limit as $k \to \infty$ we get

$$\forall_{\varepsilon>0}\ \exists_{\delta\in(0,\delta_0)}\ \forall_{t\in(0,h]} \quad \|\phi(t) - u_0\|_{X^\zeta} \leq \varepsilon,$$

which completes the proof of Lemma 2.9. ■

Part (ii) is thus proved and we now show that ϕ satisfies in $[0,\tau_0]$ the variation of constants formula associated with (1.2)–(1.3).

LEMMA 2.10. *The function ϕ from Lemma* 2.8 *satisfies*

$$\phi(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(\phi(s))\, ds, \qquad t \in [0,\delta_0]. \tag{2.31}$$

Proof. Note that if $\zeta > \beta$, then for $s \in (0,t) \subset (0,\delta_0]$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}(2.32)\qquad &\|e^{-A(t-s)}[F(u_n(s)) - F(\phi(s))]\|_{X^\zeta} \\ &\qquad \le \frac{M}{(t-s)^{\zeta-\beta}}\|F(u_n(s)) - F(\phi(s))\|_{X^\beta}.\end{aligned}$$

Note also that $u_{n_k}(s) \xrightarrow{X^\alpha} \phi(s)$ in $(0,\delta_0]$ for a suitable subsequence $\{u_{n_k}\}$ and hence, as $F \in C(X^\alpha, X^\beta)$, the right hand side of (2.32) tends to zero for each $s \in (0,t)$.

From (1.5), (2.27) and (2.28) we next have

$$\begin{aligned}&\|e^{-A(t-s)}[F(u_{n_k}(s)) - F(\phi(s))]\|_{X^\zeta} \\ &\quad \le \frac{M}{(t-s)^{\zeta-\beta}}(\|F(u_{n_k}(s))\|_{X^\beta} + \|F(\phi(s))\|_{X^\beta}) \\ &\quad \le \frac{cM}{(t-s)^{\zeta-\beta}}[2 + \|u_{n_k}(s)\|^\rho_{X^\alpha} + \|\phi(s)\|^\rho_{X^\alpha}] \\ &\quad \le \frac{2cM}{(t-s)^{\zeta-\beta}} + \frac{cM}{(t-s)^{\zeta-\beta}s^{\rho(\alpha-\zeta)}}(\|s^{\alpha-\zeta}u_{n_k}(s)\|^\rho_{X^\alpha} + \|s^{\alpha-\zeta}\phi(s)\|^\rho_{X^\alpha}) \\ &\quad \le \frac{2cM}{(t-s)^{\zeta-\beta}} + \frac{2c\theta^\rho M}{(t-s)^{\zeta-\beta}s^{\rho(\alpha-\zeta)}} =: h(s),\end{aligned}$$

which ensures that for each $t \in (0,\delta_0]$ the left hand side of (2.32) is, as a function of $s \in (0,t)$, bounded uniformly for $n \in \mathbb{N}$ by a function $h(s)$ integrable over $(0,t)$.

Using Lebesgue's dominated convergence theorem we now conclude that for all $t \in (0,\delta_0]$ we have

$$(2.33)\qquad \int_0^t e^{-A(t-s)}F(u(s,u_{0n_k}))\,ds \xrightarrow{X^\zeta} \int_0^t e^{-A(t-s)}F(\phi(s))\,ds \quad \text{as } k \to \infty.$$

Similar considerations lead to (2.33) also when $\zeta \le \beta$, so (2.31) holds and Lemma 2.10 is proved. ■

With Lemmas 2.4–2.10 the proof of Theorem 1.2, and thus also of Theorem 1.3, is now straightforward.

Proof of Corollary 1.4. First note that (1.13) implies (1.5) with $c = 2C + \|F(0)\|_{X^\beta}$ so that Theorem 1.3 applies.

Suppose now that for $i = 1,2$ there are given functions $u_i \in C((0,\tau_i), X^\alpha)$ such that $u_i(0) = u_0 \in X^\zeta$, and u_i satisfies on a certain interval $(0,\tau_0] \subset (0,\tau_i)$ the variation of constants formula associated with (1.2)–(1.3) and

$\lim_{t\to 0^+} t^{\alpha-\zeta}\|u_i(t)\|_{X^\alpha} = 0$. The latter implies

$$\forall_{\varepsilon\in(0,\tau_0)}\ \exists_{\delta_\varepsilon\in(0,\varepsilon)}\ \forall_{s\in(0,\delta_\varepsilon)} \qquad s^{\alpha-\zeta}(\|u_1(s)\|_{X^\alpha}+\|u_2(s)\|_{X^\alpha})\varepsilon^{1/(\rho-1)}, \tag{2.34}$$

and with the aid of (1.13) we then get

$$\begin{aligned}\|u_1(t)-u_2(t)\|_{X^\alpha} &\le \int_0^t \|e^{-A(t-s)}[F(u_1(s))-F(u_2(s))]\|_{X^\alpha}\,ds \\ &\le CM\int_0^t (t-s)^{\beta-\alpha}\|u_1(s)-u_2(s)\|_{X^\alpha}(1+\|u_1(s)\|^{\rho-1}_{X^\alpha}+\|u_2(s)\|^{\rho-1}_{X^\alpha})\,ds \\ &\le CM\int_0^t (t-s)^{\beta-\alpha}\|u_1(s)-u_2(s)\|_{X^\alpha}\,ds \\ &\quad+\varepsilon CM\int_0^t (t-s)^{\beta-\alpha}s^{-(\rho-1)(\alpha-\zeta)}\|u_1(s)-u_2(s)\|_{X^\alpha}\,ds,\ t\in(0,\delta_\varepsilon).\end{aligned} \tag{2.35}$$

Define next

$$z(t) := \sup_{s\in(0,t)} s^{\alpha-\zeta}\|u_1(s)-u_2(s)\|_{X^\alpha}, \qquad t\in(0,\delta_\varepsilon),$$

$$\hat D := CM\max\{\mathbf{B}(1+\beta-\alpha,1+\zeta-\alpha),\mathbf{B}(1+\beta-\alpha,1-\rho(\alpha-\zeta))\}.$$

From (1.6) we have $1+\zeta-\alpha>0$ and $1+\beta-\zeta-\rho(\alpha-\zeta)\ge 0$, and from (2.35) we obtain

$$\begin{aligned}&t^{\alpha-\zeta}\|u_1(t)-u_2(t)\|_{X^\alpha} \\ &\quad\le \Big(\int_0^t (t-s)^{\beta-\alpha}s^{\zeta-\alpha}\,ds+\varepsilon\int_0^t (t-s)^{\beta-\alpha}s^{-\rho(\alpha-\zeta)}\,ds\Big)CMt^{\alpha-\zeta}z(t) \\ &\quad= (t^{1+\beta-\alpha}\mathbf{B}(1+\beta-\alpha,1+\zeta-\alpha) \\ &\qquad+\varepsilon t^{1+\beta-\zeta-\rho(\alpha-\zeta)}\mathbf{B}(1+\beta-\alpha,1-\rho(\alpha-\zeta)))CMz(t) \\ &\quad= (t^{1+\beta-\alpha}+\varepsilon)\hat Dz(t)\le(\delta_\varepsilon^{1+\beta-\alpha}+\varepsilon)\hat Dz(\tau) \quad \text{for } 0<t\le\tau<\delta_\varepsilon.\end{aligned}$$

Consequently, for each $\tau\in(0,\delta_\varepsilon)$, we have

$$0\le z(\tau)\le(\delta_\varepsilon^{1+\beta-\alpha}+\varepsilon)\hat Dz(\tau),$$

and choosing in (2.34) $\varepsilon\in(0,\tau_0)$ such that $(\varepsilon^{1+\beta-\alpha}+\varepsilon)\hat D<1$, we conclude that

$$z(\tau)=0, \qquad \tau\in(0,\delta_\varepsilon).$$

The solution is thus locally unique and Corollary 1.4 follows. ∎

Proof of Remark 1.5. Recall that now (1.5) holds with $c := C+\|F(0)\|_{X^\beta}$, fix $\tilde u_0\in X^\zeta$ and let $\theta=(8D)^{-1/(\rho-1)}$ in (2.9). Then choose $r_0>0$ and

$\delta_0 \in (0,1]$ as in (2.10)–(2.11) and restrict δ_0 by the additional condition

$$CM\delta_0^{1+\beta-\alpha}\mathbf{B}(1+\beta-\alpha, 1+\zeta-\alpha) \le 1/4. \tag{2.36}$$

Note that the unique solution u through any point of $u_0 \in B_{X^\zeta}(\tilde{u}_0, r)$ considered in Corollary 1.4 will satisfy (1.10), that is,

$$\sup_{t\in(0,\delta_0]} t^{\alpha-\zeta}\|u(t)\|_{X^\alpha} \le \theta,$$

as this solution can be constructed via Theorem 1.2.

With the above set-up for each $t \in [0,\delta_0]$ we get

$$\begin{aligned}
&\|u(t)-\tilde{u}(t)\|_{X^\alpha} \\
&\quad\le \|e^{-At}(u_0-\tilde{u}_0)\|_{X^\alpha} + \int_0^t \|e^{-A(t-s)}[F(u(s))-F(\tilde{u}(s))]\|_{X^\alpha}\,ds \\
&\quad\le Mt^{\zeta-\alpha}\|u_0-\tilde{u}_0\|_{X^\zeta} \\
&\qquad + CM\int_0^t (t-s)^{\beta-\alpha}\|u(s)-\tilde{u}(s)\|_{X^\alpha}(1+\|u(s)\|_{X^\alpha}^{\rho-1}+\|\tilde{u}(s)\|_{X^\alpha}^{\rho-1})\,ds \\
&\quad\le Mt^{\zeta-\alpha}\|u_0-\tilde{u}_0\|_{X^\zeta} + CM\int_0^t (t-s)^{\beta-\alpha}\|u(s)-\tilde{u}(s)\|_{X^\alpha}\,ds \\
&\qquad + 2C\theta^{\rho-1}M\int_0^t (t-s)^{\beta-\alpha}s^{-(\rho-1)(\alpha-\zeta)}\|u(s)-\tilde{u}(s)\|_{X^\alpha}\,ds,
\end{aligned}$$

and consequently, by (1.6),

$$\begin{aligned}
&t^{\alpha-\zeta}\|u(t)-\tilde{u}(t)\|_{X^\alpha} \\
&\quad\le M\|u_0-\tilde{u}_0\|_{X^\zeta} \\
&\qquad + CMt^{1+\beta-\alpha}\mathbf{B}(1+\beta-\alpha, 1+\zeta-\alpha)\sup_{s\in(0,t)} s^{\alpha-\zeta}\|u(s)-\tilde{u}(s)\|_{X^\alpha} \\
&\qquad + 2C\theta^{\rho-1}Mt^{1+\beta-\zeta-\rho(\alpha-\zeta)} \\
&\qquad \times \mathbf{B}(1+\beta-\alpha, 1-\rho(\alpha-\zeta))\sup_{s\in(0,t)} s^{\alpha-\zeta}\|u(s)-\tilde{u}(s)\|_{X^\alpha} \\
&\quad\le M\|u_0-\tilde{u}_0\|_{X^\zeta} + (CMt^{1+\beta-\alpha}\mathbf{B}(1+\beta-\alpha, 1+\zeta-\alpha)+2D\theta^{\rho-1}) \\
&\qquad \times \sup_{s\in(0,t)} s^{\alpha-\zeta}\|u(s)-\tilde{u}(s)\|_{X^\alpha}.
\end{aligned}$$

Since $\theta = (8D)^{-1/(\rho-1)}$ and (2.36) holds we actually have

$$t^{\alpha-\zeta}\|u(t)-\tilde{u}(t)\|_{X^\alpha} \le M\|u_0-\tilde{u}_0\|_{X^\zeta} + \frac{1}{2}\sup_{s\in(0,\delta_0)} s^{\alpha-\zeta}\|u(s)-\tilde{u}(s)\|_{X^\alpha}$$

for $t \in (0,\delta_0]$, which ensures that (1.14) holds with $L = 2M$.

Now we write again the variation of constants formula and use (1.14) to obtain

$$
\begin{aligned}
\|u(t)-\tilde{u}(t)\|_{X^\zeta} &\le \|e^{-At}(u_0-\tilde{u}_0)\|_{X^\zeta} + \int_0^t \|e^{-A(t-s)}[F(u(s))-F(\tilde{u}(s))]\|_{X^\zeta}\,ds \\
&\le M\|u_0-\tilde{u}_0\|_{X^\zeta} \\
&\quad + CM\int_0^t (t-s)^{\beta-\zeta}\|u(s)-\tilde{u}(s)\|_{X^\alpha}(1+\|u(s)\|_{X^\alpha}^{\rho-1}+\|\tilde{u}(s)\|_{X^\alpha}^{\rho-1})\,ds \\
&\le M\|u_0-\tilde{u}_0\|_{X^\zeta} + CLM\int_0^t (t-s)^{\beta-\zeta}s^{\zeta-\alpha}\|u_0-\tilde{u}_0\|_{X^\zeta}\,ds \\
&\quad + 2C\theta^{\rho-1}LM\int_0^t (t-s)^{\beta-\zeta}s^{-\rho(\alpha-\zeta)}\|u_0-\tilde{u}_0\|_{X^\zeta}\,ds \\
&= M\big(1+CL\mathbf{B}(1+\beta-\zeta,1+\zeta-\alpha) \\
&\quad + 2C\theta^{\rho-1}L\mathbf{B}(1+\beta-\zeta,1-\rho(\alpha-\zeta))\big)\|u_0-\tilde{u}_0\|_{X^\zeta} \\
&=: \tilde{L}\|u_0-\tilde{u}_0\|_{X^\zeta}, \quad t\in(0,\delta_0).
\end{aligned}
$$

The proof is complete. ∎

REMARK 2.11. Note that, due to [11] (see [8] and [6, Appendix]), the solution u in Corollary 1.4 has further regularity properties:

$$
u(t)\in C([0,\tilde{\tau}],X^\zeta)\cap C((0,\tilde{\tau}],X^{\beta+1})\cap C^1((0,\tilde{\tau}],X^{\alpha+\varepsilon}), \quad \varepsilon\in[0,1+\beta-\alpha), \tag{2.37}
$$

and $u(t)$ satisfies both relations in (1.2).

3. Examples. In this section we discuss a few applications of the abstract results to sample problems involving non-Lipschitz nonlinearities and critical exponents.

EXAMPLE 3.1. Consider first

$$
\begin{cases} u_t-\Delta u = f(u), & t>0,\ x\in\Omega, \\ u(t,x)=0, & t>0,\ x\in\partial\Omega, \\ u(0,x)=u_0(x), & x\in\Omega, \end{cases} \tag{3.1}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $f:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying

$$
\exists_{\rho>1}\ \forall_{s\in\mathbb{R}} \quad |f(s)|\le c(1+|s|^\rho). \tag{3.2}
$$

In this example, A is defined by the negative Laplacian in $X=L^p(\Omega)$, $p>1$, with the domain $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$.

Whenever $N(\rho-1)/(2p\rho) < 1$ we find that (1.1) satisfied with $\alpha = N(\rho-1)/(2p\rho)$, $\beta = 0$ as a consequence of the Sobolev embedding

$$X^\alpha = W_0^{2\alpha,p}(\Omega) \subset L^{p\rho}(\Omega).$$

Being now interested in solvability of (3.1) with initial conditions in $W_0^{1,p}(\Omega)$ we refer to (1.6) and obtain $1/2 = \zeta = -1/(\rho-1) + N/(2p)$, from which the exponent ρ can be derived. Hence we conclude that Theorem 1.3 applies with $X^\zeta = X^{1/2}$ and $\rho = (N+p)/(N-p)$ when $N > p$.

Note that this ρ is a critical exponent for local solvability of (3.1) in $W_0^{1,p}(\Omega)$ (see [2]) and that the results in [2] are generalized here to non-Lipschitz nonlinearities f satisfying merely continuity and growth assumptions.

We remark that for $p = 2$ and $N = 3$ we have $\rho = 5$ and that for initial data in $H_0^1(\Omega)$ the approach of [11] does not apply with such growth even for Lipschitz continuous right hand sides, as for a sample power function s^5 one has $|\phi|^5 \in H^{-1}(\Omega)$ for $\phi \in H_0^1(\Omega)$.

Example 3.2. More generally, consider a higher order initial-boundary value problem

$$\begin{cases} u_t + \sum_{|\sigma|\le 2m} a_\sigma(x) D^\sigma u = f(u), & \\ & t > 0,\ x \in \Omega \subset \mathbb{R}^N,\ N > 2m > 2, \\ B_0 u = \cdots = B_{m-1} u = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0 \in H^m_{2,\{B_j\}}(\Omega), & \end{cases} \tag{3.3}$$

(see [6]), where Ω is a bounded smooth domain in $\mathbb{R}^N$, $a_\sigma \in C(\overline{\Omega})$ for $|\sigma| = 2m$, $a_\sigma \in L^\infty(\Omega)$ for $|\sigma| < 2m$, and

$$B_j = \sum_{|\sigma|\le m_j} b_{\sigma j}(x) D^\sigma, \qquad j = 0,\ldots,m-1,$$

are boundary operators with coefficients $b_{\sigma j} \in C^{2m-m_j}(\partial\Omega)$ such that (3.3) falls into the class of abstract parabolic problems of the form (1.2) (see [10, Theorem 19.4, p. 78] and the assumptions therein).

In this example A is considered in $X = L^p(\Omega)$ with the domain $H^{2m}_{p,\{B_j\}}(\Omega)$ and corresponds to a regular elliptic boundary value problem $(L, \{B_j\}, \Omega)$, where $Lu = \sum_{|\sigma|\le 2m} a_\sigma(x) D^\sigma u$ is the operator appearing in the linear main part of (3.3) and the spaces $H^{2m}_{p,\{B_j\}}(\Omega)$ are defined as in [14, Chapter 4].

Suppose that $f \in C(\mathbb{R},\mathbb{R})$ satisfies (3.2) and that $N(\rho-1)/(2mp\rho) < 1$. Then (1.1) is satisfied with $\alpha = N(\rho-1)/(2mp\rho)$ and $\beta = 0$ by the embedding

$$X^{N(\rho-1)/(2mp\rho)} \subset H_p^{N(\rho-1)/(p\rho)}(\Omega) \subset L^{p\rho}(\Omega),$$

Being now interested in local solvability of (1.6) with initial conditions in $H^m_{p,\{B_j\}}(\Omega)$ we observe that (1.6) now implies for the critical exponent ρ the relation $1/2 = \zeta = -1/(\rho-1) + N/(2pm)$. We thus conclude that Theorem 1.3 applies in this example with $X^\zeta = X^{1/2}$ and $\rho = (N+mp)/(N-mp)$ when $N > mp$. This generalizes the considerations of [6] that have been carried out in the Hilbert setting and for Lipschitz continuous nonlinearities.

Example 3.3. Consider the initial boundary value problem for the wave equation with a structural damping

$$(3.4)\qquad \begin{cases} u_{tt} + \eta(-\Delta)^{1/2}u_t + (-\Delta)u = f(u), & t > 0,\ x \in \Omega,\\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), & x \in \Omega,\\ u(t,x) = 0, \quad t \geq 0,\ x \in \partial\Omega, \end{cases}$$

(see [3–7]), where Ω is a bounded smooth domain in $\mathbb{R}^N$, $\eta > 0$ and $f \in C(\mathbb{R},\mathbb{R})$ satisfies (3.2).

Denoting by Λ the negative Laplacian in $E = L^2(\Omega)$ with the domain $D(\Lambda) = H^2(\Omega) \cap H^1_0(\Omega)$ recall that the problem (3.4) can be viewed in the form (1.2) as

$$(3.5)\qquad \frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + A\begin{bmatrix} u \\ v \end{bmatrix} = F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right), \quad t > 0, \quad \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in X^{1/2}_p \times X_p,$$

where

$$A = \begin{bmatrix} 0 & -I \\ \Lambda & 2\eta\Lambda^{1/2} \end{bmatrix}$$

is a sectorial positive operator in $X^0 = L^2(\Omega) \times H^{-1}(\Omega)$ with the domain $X^1 = H^1_0(\Omega) \times L^2(\Omega)$ (see [3]) and $F\big(\big[\begin{smallmatrix} u \\ v \end{smallmatrix}\big]\big) = \big[\begin{smallmatrix} 0 \\ f(u) \end{smallmatrix}\big]$.

Next, denoting by $\{E^\sigma : \sigma \in \mathbb{R}\}$ the fractional power scale associated with Λ in E recall also from [3] that

$$(3.6)\qquad X^\sigma = E^{\sigma/2} \times E^{(\sigma-1)/2}, \quad \sigma \in [0,2].$$

We now choose

$$(3.7)\qquad \alpha = 1 + 1/(2\rho), \quad \beta = 1/2, \quad \zeta = 1,$$

which satisfy (1.6), and with the aid of (3.2), (3.6) we obtain

$$\begin{aligned} \left\| F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)\right\|_{X^{1/2}} &= \|f(u)\|_{E^{-1/4}} \leq c_1\|f(u)\|_{L^{2N/(N+1)}(\Omega)} \\ &\leq c_1 c(\|1\|_{L^{2N/(N+1)}(\Omega)} + \|u\|^\rho_{L^{2N/(N+1)}(\Omega)}). \end{aligned}$$

Since $E^{1/2+1/(4\rho)} \subset H^{1+1/(2\rho)}(\Omega)$, (1.5) will hold with the parameters as in (3.7) provided that $H^{1+1/(2\rho)}(\Omega)$ is embedded in $L^{2N\rho/(N+1)}(\Omega)$, which, assuming $N > 2$, translates into the condition

$$\frac{2N\rho}{N+1} \leq \frac{2N\rho}{\rho(N-2)-1}.$$

Thus for the problem (3.4) Theorem 1.3 applies with $\rho = (N+2)/(N-2)$ and α, β, ζ as in (3.7).

Similar considerations apply in the L^p-setting and can also be carried out for more general strongly damped wave equations of the form

$$\begin{cases} u_{tt} + 2\eta A_L^\theta u_t + A_L u = f(u) + g(u_t), & t > 0,\, x \in \Omega, \\ u(0) = u_0 \in X^{1/2}, \quad u_t(0) = v_0 \in X, \end{cases} \tag{3.8}$$

where $\theta \in [1/2, 1)$ and (η, θ, A_L) is an admissible triple (see [7, Definition 1.1]) corresponding to a regular elliptic boundary value problem $(L, \{B_j\}, \Omega)$ as in Example 3.2.

Applying Theorem 1.3 we infer that (3.8) has a local solution through each point $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^m_{p,\{B_j\}}(\Omega) \times L^p(\Omega)$ provided that $f, g \in C(\mathbb{R}, \mathbb{R})$, $N > mp$, $p \geq 2$ and

$$\begin{aligned} |f(s)| &\leq c(1 + |s|^{(N+mp)/(N-mp)}), && s \in \mathbb{R}, \\ |g(s)| &\leq c(1 + |s|^{(N+2mp\theta)/N}), && s \in \mathbb{R}, \end{aligned}$$

which extends the results of [7, Theorem 1.2] to non-Lipschitz nonlinearities.

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