TRANSLATIVE PACKING OF A SQUARE WITH SEQUENCES OF SQUARES

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Abstract. Let $S$ be a square and let $S'$ be a square of unit area with a diagonal parallel to a side of $S$. Any (finite or infinite) sequence of homothetic copies of $S$ whose total area does not exceed $\frac{4}{9}$ can be packed translatively into $S'$.

1. Introduction. Let $C, C_1, C_2, \ldots$ be convex bodies in the plane. We say that the sequence $(C_i)$ can be packed translatively into $C$ if there are translations $\sigma_i$ such that $\sigma_iC_i$ are subsets of $C$ with mutually disjoint interiors. We say that the sequence $(C_i)$ permits a translative covering of $C$ if there are translations $\sigma_i$ such that $C \subset \bigcup \sigma_iC_i$. The area of $C$ is denoted by $|C|$.

Let $S$ be a square. Moon and Moser showed in [5] that any sequence of squares homothetic to $S$ can be packed translatively into $S$ provided the total area of the squares in the sequence does not exceed $\frac{1}{2}|S|$. Additionally, any sequence of homothetic copies of $S$ with total area not smaller than $3|S|$ permits a translative covering of $S$. In [4] it is shown that any sequence of homothetic copies of $S$ whose total area is not smaller than $2.5|S'|$ permits a translative covering of $S'$, where $S'$ is a square with a diagonal parallel to a side of $S$. The aim of this paper is to give an analog of this result for packing. We show that if $S'$ is a square with a diagonal parallel to a side of $S$, then any sequence of homothetic copies of $S$ can be packed translatively into $S'$ provided the total area of the copies does not exceed $\frac{4}{9}|S'|$. The bound of $\frac{4}{9}$ cannot be improved upon. The reason is that two homothetic copies of $S$, each of area greater than $\frac{2}{9}|S'|$, cannot be packed translatively into $S'$ (see Fig. 1, left).

Various results concerning packings and coverings are discussed in [1–3].

2. Packing method. Denote by $S'$ a square whose vertices are $(0, -1)$, $(1, 0)$, $(0, 1)$, $(-1, 0)$. Let $S$ be a square with sides parallel to the coordinate axes, let $(S_i)$ be a sequence of homothetic copies of $S$ and let $a_1 \geq a_2 \geq \cdots$, where $a_i$ denotes the side length of $S_i$, for $i = 1, 2, \ldots$.
We describe a method of translative packing of \( S_1, S_2, \ldots \) into \( S' \).

The first square from the sequence is packed into \( S' \) as low as possible, i.e.,

\[
\sigma_1 S_1 = \left\{ (x, y); -\frac{1}{2} a_1 \leq x \leq \frac{1}{2} a_1, -1 + \frac{1}{2} a_1 \leq y \leq -1 + \frac{3}{2} a_1 \right\}
\]

(see Figs. 1 and 2; in Figs. 2–7 each square \( \sigma_i S_i \) is denoted by the integer \( i \), for short).

We will pack \( S_2, S_3, \ldots \) into \( S' \) in layers. Let \(-1 < d < 1 \) and \( h > 0 \). By a \textit{layer} \( L \) of height \( h \) we mean \( \{(x, y); d \leq y \leq d + h\} \); by a \textit{container} we mean the intersection of a layer with \( S' \).

Each container is a polygon. The longest side of this polygon that is parallel to the \( x \)-axis is called the \textit{base} of the container. If there are two such sides, then we mean the lower one. The \textit{height} \( h(K) \) of a container \( K = L \cap S' \) is equal to the height of \( L \). We say that \( S_i \) is \( k \)-\textit{packed} into a container \( K \) if it is packed translatively into \( K \) so that one side of \( \sigma_i S_i \) is contained in the base of \( K \) and, at the same time, no point of the interior of \( K \) lying on the right side of \( \sigma_i S_i \) belongs to \( \sigma_1 S_1 \cup \cdots \cup \sigma_{i-1} S_{i-1} \).

Let

\[
L_2 = \left\{ (x, y); -1 + \frac{3}{2} a_1 \leq y \leq -1 + \frac{3}{2} a_1 + a_2 \right\}
\]

and let \( K_2 = L_2 \cap S' \). We declare that this container is \textit{basic} and \textit{2-open}. We \( k \)-pack the second square from the sequence into \( K_2 \) as far to the left as possible (see \( S_2 \) in Fig. 2).

For each \( i \geq 3 \) we proceed as follows. Assume that the translations \( \sigma_1, \ldots, \sigma_{i-1} \) have already been provided, that the \((i - 1)\)-open containers have been defined and that the basic containers \( K(j) \), for some \( j < i \), are defined.
1. If there is an \((i - 1)\)-open container \(K\) into which \(S_i\) can be \(k\)-packed and if \(a_i \geq \frac{1}{2}h(K)\), then each \((i - 1)\)-open container is \(i\)-open. Denote by \(K(i)\) the lowest \(i\)-open container into which \(S_i\) can be \(k\)-packed. We \(k\)-pack \(S_i\) into \(K(i)\) as far to the left as possible (see \(S_4, S_5\) and \(S_7\) in Fig. 2).

2. If there is an \((i - 1)\)-open container \(K\) into which \(S_i\) can be \(k\)-packed and if \(a_i < \frac{1}{2}h(K)\), then let \(m\) be an integer such that \(2^{-m-1}h(K) < a_i \leq 2^{-m}h(K)\). Each \((i - 1)\)-open container is divided into \(2^m\) containers of height \(2^{-m}h(K)\). Only the newly created containers of height \(2^{-m}h(K)\) are \(i\)-open. Denote by \(K(i)\) the lowest \(i\)-open container into which \(S_i\) can be \(k\)-packed. We \(k\)-pack \(S_i\) into \(K(i)\) as far to the left as possible (see \(S_3\) in Fig. 2).

3. If there is no \((i - 1)\)-open container \(K\) into which \(S_i\) can be \(k\)-packed, then we create a new layer \(L(i)\) of height \(a_i\) directly above the highest layer. We declare that the container \(K(i) = L(i) \cap S'\) is basic. Moreover, only \(K(i)\) is \(i\)-open. We \(k\)-pack \(S_i\) into \(K(i)\) as far to the left as possible (see \(S_6\) in Fig. 2).

**3. Packing density in basic containers.** In this section we show that a large part of each basic container is filled with packed squares.

**Lemma.** Assume that \(K\) is a basic container, that \(S_p\) is the first square from the sequence packed into \(K\), that \(S_{q+1}\) is the first square which cannot be packed into \(K\) by the method presented in Section 2 and that \(q \geq p + 1\). Then the total area of the squares packed into \(K\) is greater than \(\frac{4}{9}|K|\).

**Proof.** Consider two cases depending on the size of the last square packed into \(K\).

**Case 1:** \(a_q \geq \frac{1}{2}a_p\). Let \(R\) be the set of points of \(K\) lying between the right side of \(\sigma_pS_p\) and the straight line containing the left side of \(\sigma_qS_q\) (see
Fig. 3. Obviously,

\[ \sum_{i=p+1}^{q-1} |S_i| \geq \frac{1}{2} |R| \]

(if \( q = p + 1 \), then \( R = \emptyset \) and the sum on the left-hand side of this inequality is meant to be zero).

We show that

\[ |S_p| + |S_q| > \frac{4}{9} |K \setminus R|. \]

First consider the case where \( K \) is a trapezoid. Since \( \left( \frac{3}{2}a_q - a_p \right)^2 \geq 0 \) it follows that

\[ 3a_qa_p - a_p^2 \leq \frac{9}{4}a_q^2. \]

As a consequence,

\[ |K \setminus R| < 2a_p^2 + (3a_q - a_p)a_p < \frac{9}{4}a_p^2 + \frac{9}{4}a_q^2 = \frac{9}{4}(|S_p| + |S_q|) \]

(see Fig. 3 (left), where \( v < 3a_q - a_p \)).

Now consider the case where \( K \) is a hexagon. Denote by \( b \) and \( c \) the length of the sides of \( K \) parallel to the \( x \)-axis and let \( t = a_p - \frac{1}{2}|b - c| \) (see Fig. 4).

If \( t \leq a_q \), then we argue as in the case where \( K \) is a trapezoid (see Fig. 3, right).

If \( t > a_q \), then

\[ |K \setminus R| < 2a_p^2 - \left( \frac{t}{\sqrt{2}} \right)^2 + a_p(2a_q + t - a_p) = a_p^2 + 2a_p a_q - \frac{1}{2}t^2 + a_p t \]
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(see Fig. 4, where \( w < 2a_q + t - a_p \)). Consequently,

\[ |K \setminus R| < a_p^2 + 2a_pa_q - \frac{1}{2}a_p^2 + a_q^2 = \frac{3}{2}a_p^2 + 2a_pa_q. \]

Since

\[ \frac{9}{4}a_p^2 - \frac{3}{2}a_q^2 - 2a_pa_q + \frac{9}{4}a_q^2 > \frac{3}{4}a_p^2 - \frac{3\sqrt{3}}{2}a_p a_q + \frac{9}{4}a_q^2 = \left( \frac{\sqrt{3}}{2}a_p - \frac{3}{2}a_q \right)^2 \geq 0 \]

it follows that

\[ |K \setminus R| < \frac{9}{4}a_p^2 + \frac{9}{4}a_q^2 = \frac{9}{4}(|S_p| + |S_q|). \]

We conclude from (1) and (2) that

\[ \sum_{i=p}^{q} |S_i| > \frac{4}{9} |K|. \]

**Case 2:** \( a_q < \frac{1}{2}a_p \). Let \( m \) be an integer such that \( 2^{-m-1}a_p < a_q \leq 2^{-m}a_p \). Denote by \( K_q(1), \ldots, K_q(2^m) \) the \( q \)-open containers of height \( 2^{-m}a_p \) obtained by partitioning \( K \). For each \( i \in \{1, \ldots, 2^m\} \) denote by \( s_q(i) \) the maximum value of the \( x \)-coordinate on \( (\sigma_1S_1 \cup \cdots \cup \sigma_qS_q) \cap \text{Int} \ K_q(i) \). Let \( R_q(i) \) be the set of points of \( K_q(i) \) lying between the right side of \( \sigma_pS_p \) and the straight line \( x = s_q(i) \) and let \( R_q = \bigcup_{i=1}^{2^m} R_q(i) \). By the description of the packing method we deduce that

\[ \sum_{i=p+1}^{q} |S_i| \geq \frac{1}{2} |R_q| \]

(see Fig. 5). Moreover,

\[ |K \setminus R_q| < \frac{3}{2}a_p^2 + 2^m \cdot \frac{3}{2} \cdot (2^{-m}a_p)^2 = \frac{3}{2}a_p^2 + \frac{3}{2} \cdot 2^{-m}a_p^2 \leq a_p^2 \left( \frac{3}{2} + \frac{3}{4} \right) = \frac{9}{4}|S_p|. \]

Consequently,

\[ \sum_{i=p}^{q} |S_i| > \frac{4}{9} |K|. \]

\[ \blacksquare \]
4. The main result

THEOREM. Assume that $S$ is a square and that $S'$ is a square with a diagonal parallel to a side of $S$. Any (finite or infinite) sequence of homothetic copies of $S$ can be packed translatively into $S'$ provided the total area of the copies does not exceed $\frac{4}{9} |S'|$.

Proof. Due to the affine invariant nature of the problem we can assume that the vertices of $S'$ are $(0, -1), (1, 0), (0, 1), (-1, 0)$. Let $(S_i)$ be a sequence of homothetic copies of $S$ and let $\sum |S_i| \leq \frac{4}{9} |S'|$. Denote by $a_i$ the side length of $S_i$ for $i = 1, 2, \ldots$. Without loss of generality we can assume that $a_1 \geq a_2 \geq \cdots$.

We show that $S_1, S_2, \ldots$ can be packed translatively into $S'$.

Suppose that it is impossible to pack $S_1, S_2, \ldots$ into $S'$ by the method described in Section 2. Let $S_z$ be the first square which cannot be packed into $S'$. Denote by $K_{l+1}$ the set of the points of $S'$ with $y$-coordinate not greater than $-1 + \frac{3}{2} a_1$. All basic containers are denoted by $K_2, \ldots, K_{l+1}$ in such a way that $K_i$ is higher than $K_j$ provided $i > j$ ($l = 3$ and $z = 8$ in Fig. 2). Moreover, let $K_i^+$ be the set of points of $S'$ lying above the base of $K_i$. Into $K_{l+1} = K(z)$ no square has been packed—this container is $z$-open, but it is impossible to pack translatively $S_z$ into $K_{l+1}$.

First we show that $l \geq 2$. Since $|S_1| \leq \frac{4}{9} |S'| < \frac{1}{2} |S'|$ it follows that $l \geq 1$ ($|S_1| = \frac{1}{2} |S|$ in Fig. 1, right). If $l = 1$, then $\frac{3}{2} a_1 + \frac{3}{2} a_2 > 2$ (see Fig. 1 (left), where $\frac{3}{2} a_1 + \frac{3}{2} a_2 = 2$). Consequently,

$$a_1^2 + a_2^2 > a_1^2 + \left( \frac{4}{3} - a_1 \right)^2 \geq \frac{8}{9} = \frac{4}{9} |S'|,$$

which is a contradiction.

Obviously,

$$S' = K_1^+ \cup K_2 \cup \cdots \cup K_{l-1} \cup K_l^+. \tag{3}$$

Observe that

$$|S_1| \geq \frac{4}{9} |K_1^+| \tag{4}$$

(see Fig. 6, left).

Denote by $\sigma_r S_r$ the first square packed into $K_l$. We show that

$$\sum_{i=r}^{z} |S_i| > \frac{4}{9} |K_l^+|. \tag{5}$$

Let $T_l$ be the smallest right-angled isosceles triangle containing $K_l^+$. Obviously, if $K_l$ is a trapezoid, then $T_l = K_l^+$. Denote by $b_l$ the length of the hypotenuse of $T_l$ and denote by $b_{l+1}$ the length of the base of $K_{l+1}$.
Observe that $z \leq r + 2$. If $z \geq r + 3$, then $2a_r + a_{r+1} + 2a_{r+2} \leq b_l$ (see Fig. 6, right). Since $2a_r + 3a_{r+2} \leq b_l$ and $b_{l+1} = b_l - 2a_r$ it follows that $3a_{r+2} \leq b_{l+1}$, i.e., $S_z$ can be packed into $K_{l+1}$, which is a contradiction.

There are two possibilities: either $z = r + 1$ or $z = r + 2$.

If $z = r + 1$, then $2a_r + 2a_z > b_l$ (see Fig. 7, left). Consequently,
\[
|S_r| + |S_z| > a_r^2 + \left( \frac{1}{2} b_l - a_r \right)^2 = 2a_r^2 - a_r b_l + \frac{1}{4} b_l^2 \geq \frac{1}{8} b_l^2 \geq \frac{1}{2} |K_l^+| > \frac{4}{9} |K_l^+|.
\]

If $z = r + 2$, then $2a_r + a_{r+1} + 2a_z > b_l$ (see Fig. 7, right). Consequently,
\[
|S_r| + |S_{r+1}| + |S_z| = a_r^2 + a_{r+1}^2 + a_z^2 > a_r^2 + a_{r+1}^2 + \left( \frac{1}{2} b_l - a_r - \frac{1}{2} a_{r+1} \right)^2.
\]

By using the standard method of finding the minimum of a function of two variables it is easy to check that this value is not less than $\frac{1}{9} b_l^2 \geq \frac{4}{9} |K_l^+|$.

It is easy to see that if $j \in \{2, \ldots, l\}$ and if only one square is packed into $K_j$, then $j = l$ (as in Fig. 7, left). Consequently, at least two squares are packed into $K_j$ for $j = 2, \ldots, l - 1$. By (3)–(5) and by the Lemma we deduce that
\[
\sum_{i=1}^{z} |S_i| > \frac{4}{9} (|K_1^+| + |K_2| + \cdots + |K_{l-1}| + |K_l^+|) = \frac{4}{9} |S'|,
\]
which is a contradiction. ■
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REFERENCES


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