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TRANSLATIVE PACKING OF A SQUARE WITH SEQUENCES OF SQUARES

BY

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Abstract. Let S be a square and let S' be a square of unit area with a diagonal parallel to a side of S. Any (finite or infinite) sequence of homothetic copies of S whose total area does not exceed $\frac{4}{9}$ can be packed translatively into S'.

1. Introduction. Let $C, C_1, C_2, \dots$ be convex bodies in the plane. We say that the sequence (C_i) can be *packed translatively* into C if there are translations σ_i such that $\sigma_i C_i$ are subsets of C with mutually disjoint interiors. We say that the sequence (C_i) *permits a translative covering* of C if there are translations σ_i such that $C \subset \bigcup \sigma_i C_i$. The area of C is denoted by $|C|$.

Let S be a square. Moon and Moser showed in [5] that any sequence of squares homothetic to S can be packed translatively into S provided the total area of the squares in the sequence does not exceed $\frac{1}{2}|S|$. Additionally, any sequence of homothetic copies of S with total area not smaller than $3|S|$ permits a translative covering of S. In [4] it is shown that any sequence of homothetic copies of S whose total area is not smaller than $2.5|S'|$ permits a translative covering of S', where S' is a square with a diagonal parallel to a side of S. The aim of this paper is to give an analog of this result for packing. We show that if S' is a square with a diagonal parallel to a side of S, then any sequence of homothetic copies of S can be packed translatively into S' provided the total area of the copies does not exceed $\frac{4}{9}|S'|$. The bound of $\frac{4}{9}$ cannot be improved upon. The reason is that two homothetic copies of S, each of area greater than $\frac{2}{9}|S'|$, cannot be packed translatively into S' (see Fig. 1, left).

Various results concerning packings and coverings are discussed in [1–3].

2. Packing method. Denote by S' a square whose vertices are $(0,-1)$, $(1,0)$, $(0,1)$, $(-1,0)$. Let S be a square with sides parallel to the coordinate axes, let (S_i) be a sequence of homothetic copies of S and let $a_1 \ge a_2 \ge \cdots$, where a_i denotes the side length of S_i, for $i = 1, 2, \dots$.

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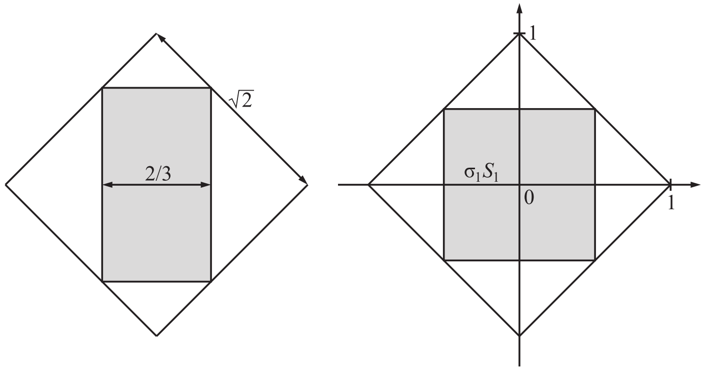

Fig. 1

We describe a method of translative packing of $S_1, S_2, \dots$ into S'.

The first square from the sequence is packed into S' as low as possible, i.e.,

$$\sigma_1 S_1 = \left\{ (x,y);\ -\frac{1}{2}a_1 \le x \le \frac{1}{2}a_1,\ -1+\frac{1}{2}a_1 \le y \le -1+\frac{3}{2}a_1 \right\}$$

(see Figs. 1 and 2; in Figs. 2–7 each square $\sigma_i S_i$ is denoted by the integer i, for short).

We will pack $S_2, S_3, \dots$ into S' in layers. Let $-1 < d < 1$ and $h > 0$. By a *layer* L of height h we mean $\{(x,y);\ d \le y \le d+h\}$; by a *container* we mean the intersection of a layer with S'.

Each container is a polygon. The longest side of this polygon that is parallel to the x-axis is called the *base* of the container. If there are two such sides, then we mean the lower one. The *height* $h(K)$ of a container $K = L \cap S'$ is equal to the height of L. We say that S_i is *k-packed* into a container K if it is packed translatively into K so that one side of $\sigma_i S_i$ is contained in the base of K and, at the same time, no point of the interior of K lying on the right side of $\sigma_i S_i$ belongs to $\sigma_1 S_1 \cup \dots \cup \sigma_{i-1} S_{i-1}$.

Let

$$L_2 = \left\{ (x,y);\ -1+\frac{3}{2}a_1 \le y \le -1+\frac{3}{2}a_1 + a_2 \right\}$$

and let $K_2 = L_2 \cap S'$. We declare that this container is *basic* and 2-*open*. We k-pack the second square from the sequence into K_2 as far to the left as possible (see S_2 in Fig. 2).

For each $i \ge 3$ we proceed as follows. Assume that the translations $\sigma_1, \dots, \sigma_{i-1}$ have already been provided, that the $(i-1)$-open containers have been defined and that the basic containers $K(j)$, for some $j < i$, are defined.

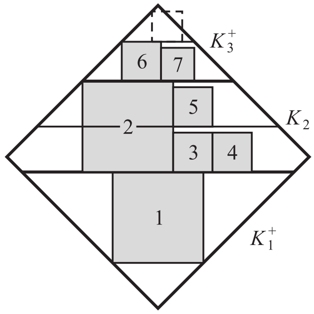

Fig. 2

1. If there is an $(i-1)$-open container K into which S_i can be k-packed and if $a_i \geq \frac{1}{2}h(K)$, then each $(i-1)$-open container is i-*open*. Denote by $K(i)$ the lowest i-open container into which S_i can be k-packed. We k-pack S_i into $K(i)$ as far to the left as possible (see S_4, S_5 and S_7 in Fig. 2).
2. If there is an $(i-1)$-open container K into which S_i can be k-packed and if $a_i < \frac{1}{2}h(K)$, then let m be an integer such that $2^{-m-1}h(K) < a_i \leq 2^{-m}h(K)$. Each $(i-1)$-open container is divided into 2^m containers of height $2^{-m}h(K)$. Only the newly created containers of height $2^{-m}h(K)$ are i-*open*. Denote by $K(i)$ the lowest i-open container into which S_i can be k-packed. We k-pack S_i into $K(i)$ as far to the left as possible (see S_3 in Fig. 2).
3. If there is no $(i-1)$-open container K into which S_i can be k-packed, then we create a new layer $L(i)$ of height a_i directly above the highest layer. We declare that the container $K(i) = L(i) \cap S'$ is *basic*. Moreover, only $K(i)$ is i-open. We k-pack S_i into $K(i)$ as far to the left as possible (see S_6 in Fig. 2).

3. Packing density in basic containers. In this section we show that a large part of each basic container is filled with packed squares.

LEMMA. *Assume that K is a basic container, that S_p is the first square from the sequence packed into K, that S_{q+1} is the first square which cannot be packed into K by the method presented in Section 2 and that $q \geq p+1$. Then the total area of the squares packed into K is greater than $\frac{4}{9}|K|$.*

Proof. Consider two cases depending on the size of the last square packed into K.

CASE 1: $a_q \geq \frac{1}{2}a_p$. Let R be the set of points of K lying between the right side of $\sigma_p S_p$ and the straight line containing the left side of $\sigma_q S_q$ (see

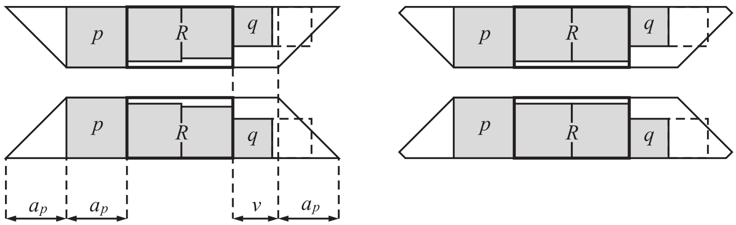

Fig. 3

Fig. 3). Obviously,

$$\sum_{i=p+1}^{q-1} |S_i| \geq \frac{1}{2}|R| \tag{1}$$

(if $q = p+1$, then $R = \emptyset$ and the sum on the left-hand side of this inequality is meant to be zero).

We show that

$$|S_p| + |S_q| > \frac{4}{9}|K \setminus R|. \tag{2}$$

First consider the case where K is a trapezoid. Since $\left(\frac{3}{2}a_q - a_p\right)^2 \geq 0$ it follows that

$$3a_q a_p - a_p^2 \leq \frac{9}{4}a_q^2.$$

As a consequence,

$$|K \setminus R| < 2a_p^2 + (3a_q - a_p)a_p < \frac{9}{4}a_p^2 + \frac{9}{4}a_q^2 = \frac{9}{4}\big(|S_p| + |S_q|\big)$$

(see Fig. 3 (left), where $v < 3a_q - a_p$).

Now consider the case where K is a hexagon. Denote by b and c the length of the sides of K parallel to the x-axis and let $t = a_p - \frac{1}{2}|b - c|$ (see Fig. 4).

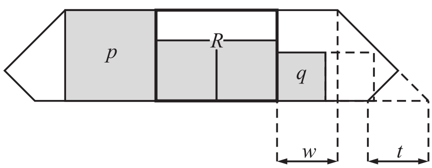

Fig. 4

If $t \leq a_q$, then we argue as in the case where K is a trapezoid (see Fig. 3, right).

If $t > a_q$, then

$$|K \setminus R| < 2a_p^2 - \left(\frac{t}{\sqrt{2}}\right)^2 + a_p(2a_q + t - a_p) = a_p^2 + 2a_p a_q - \frac{1}{2}t^2 + a_p t$$

(see Fig. 4, where $w < 2a_q + t - a_p$). Consequently,

$$|K \setminus R| < a_p^2 + 2a_pa_q - \frac{1}{2}a_p^2 + a_p^2 = \frac{3}{2}a_p^2 + 2a_pa_q.$$

Since

$$\frac{9}{4}a_p^2 - \frac{3}{2}a_p^2 - 2a_pa_q + \frac{9}{4}a_q^2 > \frac{3}{4}a_p^2 - \frac{3\sqrt{3}}{2}a_pa_q + \frac{9}{4}a_q^2 = \left(\frac{\sqrt{3}}{2}a_p - \frac{3}{2}a_q\right)^2 \geq 0$$

it follows that

$$|K \setminus R| < \frac{9}{4}a_p^2 + \frac{9}{4}a_q^2 = \frac{9}{4}(|S_p| + |S_q|).$$

We conclude from (1) and (2) that

$$\sum_{i=p}^{q} |S_i| > \frac{4}{9}|K|.$$

CASE 2: $a_q < \frac{1}{2}a_p$. Let m be an integer such that $2^{-m-1}a_p < a_q \leq 2^{-m}a_p$. Denote by $K_q(1), \ldots, K_q(2^m)$ the q-open containers of height $2^{-m}a_p$ obtained by partitioning K. For each $i \in \{1, \ldots, 2^m\}$ denote by $s_q(i)$ the maximum value of the x-coordinate on $(\sigma_1 S_1 \cup \cdots \cup \sigma_q S_q) \cap \operatorname{Int} K_q(i)$. Let $R_q(i)$ be the set of points of $K_q(i)$ lying between the right side of $\sigma_p S_p$ and the straight line $x = s_q(i)$ and let $R_q = \bigcup_{i=1}^{2^m} R_q(i)$. By the description of the packing method we deduce that

$$\sum_{i=p+1}^{q} |S_i| \geq \frac{1}{2}|R_q|$$

(see Fig. 5). Moreover,

$$|K \setminus R_q| < \frac{3}{2}a_p^2 + 2^m \cdot \frac{3}{2}(2^{-m}a_p)^2 = \frac{3}{2}a_p^2 + \frac{3}{2} \cdot 2^{-m}a_p^2 \leq a_p^2\left(\frac{3}{2} + \frac{3}{4}\right) = \frac{9}{4}|S_p|.$$

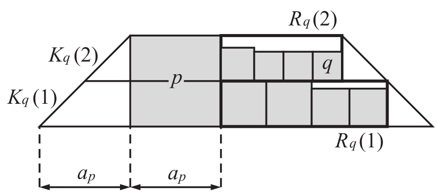

Fig. 5

Consequently,

$$\sum_{i=p}^{q} |S_i| > \frac{4}{9}|K|. \blacksquare$$

4. The main result

THEOREM. *Assume that S is a square and that S' is a square with a diagonal parallel to a side of S. Any (finite or infinite) sequence of homothetic copies of S can be packed translatively into S' provided the total area of the copies does not exceed $\frac{4}{9}|S'|$.*

Proof. Due to the affine invariant nature of the problem we can assume that the vertices of S' are $(0,-1)$, $(1,0)$, $(0,1)$, $(-1,0)$. Let (S_i) be a sequence of homothetic copies of S and let $\sum |S_i| \le \frac{4}{9}|S'|$. Denote by a_i the side length of S_i for $i = 1, 2, \ldots$. Without loss of generality we can assume that $a_1 \ge a_2 \ge \cdots$.

We show that $S_1, S_2, \ldots$ can be packed translatively into S'.

Suppose that it is impossible to pack $S_1, S_2, \ldots$ into S' by the method described in Section 2. Let S_z be the first square which cannot be packed into S'.

Denote by K_1^+ the set of the points of S' with y-coordinate not greater than $-1 + \frac{3}{2}a_1$. All basic containers are denoted by $K_2, \ldots, K_{l+1}$ in such a way that K_i is higher than K_j provided $i > j$ ($l = 3$ and $z = 8$ in Fig. 2). Moreover, let K_l^+ be the set of points of S' lying above the base of K_l. Into $K_{l+1} = K(z)$ no square has been packed—this container is z-open, but it is impossible to pack translatively S_z into K_{l+1}.

First we show that $l \ge 2$. Since $|S_1| \le \frac{4}{9}|S'| < \frac{1}{2}|S'|$ it follows that $l \ge 1$ ($|S_1| = \frac{1}{2}|S|$ in Fig. 1, right). If $l = 1$, then $\frac{3}{2}a_1 + \frac{3}{2}a_2 > 2$ (see Fig. 1 (left), where $\frac{3}{2}a_1 + \frac{3}{2}a_2 = 2$). Consequently,

$$a_1^2 + a_2^2 > a_1^2 + \left(\frac{4}{3} - a_1\right)^2 \ge \frac{8}{9} = \frac{4}{9}|S'|,$$

which is a contradiction.

Obviously,

$$S' = K_1^+ \cup K_2 \cup \cdots \cup K_{l-1} \cup K_l^+. \tag{3}$$

Observe that

$$|S_1| \ge \frac{4}{9}|K_1^+| \tag{4}$$

(see Fig. 6, left).

Denote by $\sigma_r S_r$ the first square packed into K_l. We show that

$$\sum_{i=r}^{z} |S_i| > \frac{4}{9}|K_l^+|. \tag{5}$$

Let T_l be the smallest right-angled isosceles triangle containing K_l^+. Obviously, if K_l is a trapezoid, then $T_l = K_l^+$. Denote by b_l the length of the hypotenuse of T_l and denote by b_{l+1} the length of the base of K_{l+1}.

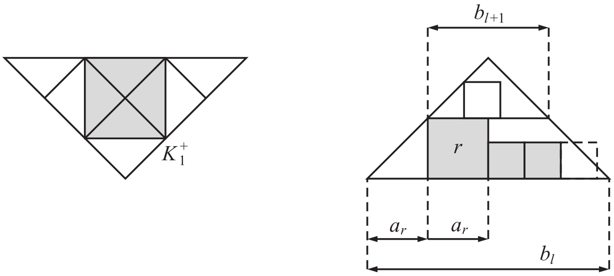

Fig. 6

Observe that $z \le r+2$. If $z \ge r+3$, then $2a_r + a_{r+1} + 2a_{r+2} \le b_l$ (see Fig. 6, right). Since $2a_r + 3a_{r+2} \le b_l$ and $b_{l+1} = b_l - 2a_r$ it follows that $3a_{r+2} \le b_{l+1}$, i.e., S_z can be packed into K_{l+1}, which is a contradiction.

There are two possibilities: either $z = r+1$ or $z = r+2$.

If $z = r+1$, then $2a_r + 2a_z > b_l$ (see Fig. 7, left). Consequently,

$$|S_r| + |S_z| > a_r^2 + \left(\frac{1}{2}b_l - a_r\right)^2 = 2a_r^2 - a_r b_l + \frac{1}{4}b_l^2 \ge \frac{1}{8}b_l^2 \ge \frac{1}{2}|K_l^+| > \frac{4}{9}|K_l^+|.$$

If $z = r+2$, then $2a_r + a_{r+1} + 2a_z > b_l$ (see Fig. 7, right). Consequently,

$$|S_r| + |S_{r+1}| + |S_z| = a_r^2 + a_{r+1}^2 + a_z^2 > a_r^2 + a_{r+1}^2 + \left(\frac{1}{2}b_l - a_r - \frac{1}{2}a_{r+1}\right)^2.$$

By using the standard method of finding the minimum of a function of two variables it is easy to check that this value is not less than $\frac{1}{9}b_l^2 \ge \frac{4}{9}|K_l^+|$.

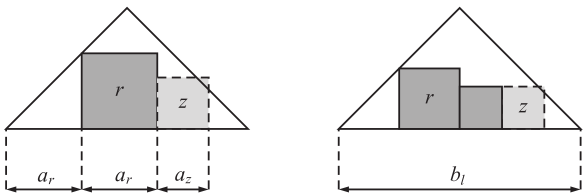

Fig. 7

It is easy to see that if $j \in \{2, \ldots, l\}$ and if only one square is packed into K_j, then $j = l$ (as in Fig. 7, left). Consequently, at least two squares are packed into K_j for $j = 2, \ldots, l-1$. By (3)–(5) and by the Lemma we deduce that

$$\sum_{i=1}^{z} |S_i| > \frac{4}{9}(|K_1^+| + |K_2| + \cdots + |K_{l-1}| + |K_l^+|) = \frac{4}{9}|S'|,$$

which is a contradiction. ∎

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