RECURRENCE FOR COSINE SERIES WITH BOUNDED GAPSS

BY

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Abstract. Ullrich, Grubb and Moore proved that a lacunary trigonometric series satisfying Hadamard’s gap condition is recurrent a.e. We prove the existence of a recurrent trigonometric series with bounded gaps.

1. Introduction. If we regard the sequence \( \{ \cos 2\pi n_k x \} \) as a sequence of random variables on the unit interval equipped with the Lebesgue measure, it behaves like a sequence of independent random variables when \( n_k \) diverges rapidly. For example, by assuming Hadamard’s gap condition

\[ \frac{n_{k+1}}{n_k} > q > 1 \quad (k = 1, 2, \ldots), \]

the central limit theorem for \( \sum \cos 2\pi n_k x \) was proved by Salem and Zygmund [9], the law of the iterated logarithm by Erdős and Gál [4], and the almost sure invariance principles by Philipp and Stout [8].

As to recurrence, Hawkes [7] proved that \( \{ \sum_{k=1}^{N} \exp(2\pi i n_k x) \}_{N \in \mathbb{N}} \) is dense in the complex plane for a.e. \( x \) assuming the very strong gap condition \( \sum n_k/n_{k+1} < \infty \). Anderson and Pitt [1] weakened the gap condition to \( n_{k+1}/n_k \to \infty \) or \( n_k = a^k \), where \( a \geq 2 \) is an integer. These results imply the recurrence of \( \sum_{k=1}^{N} \cos 2\pi n_k x \). For this one-dimensional recurrence, Ullrich, Grubb and Moore [11, 5] succeeded in weakening the condition to Hadamard’s gap condition.

It is very natural to ask if the gap condition can be replaced by a weaker one. For the central limit theorem, Erdős [3] relaxed the gap condition to \( n_{k+1}/n_k > 1 + c_k/\sqrt{k} \) with \( c_k \to \infty \). This condition is best possible. Actually Erdős [3] and Takahashi [10] constructed counterexamples to the central limit theorem satisfying \( n_{k+1}/n_k > 1 + c/\sqrt{k} \) with \( c > 0 \). But there still remains the possibility that some series having smaller gaps may obey the central limit theorem. Indeed, for any \( \phi(k) \uparrow \infty \), Berkes [2] proved the existence of \( \sum \cos 2\pi n_k x \) with small gaps \( n_{k+1} - n_k = O(\phi(k)) \) which obeys the central limit theorem. And it was a longstanding problem whether some trigonometric series with bounded gaps \( n_{k+1} - n_k = O(1) \) can obey the
central limit theorem. Recently the existence of such series was proved in [6] and the problem was solved.

In this paper, we consider the same problem for recurrence, and prove the existence of recurrent series with bounded gaps.

**Theorem 1.** Suppose that \( \{n_k\} \) satisfies Hadamard’s gap condition and let \( \{m_j\} \) be an increasing arrangement of \( \mathbb{N} \setminus \{n_k\} \). Put

\[
S_N(x) = \sum_{j=1}^{N} \cos 2\pi m_j x.
\]

Then \( \{S_N(x)\} \) is recurrent for a.e. \( x \).

The sequence \( \{n_k\} \) satisfying Hadamard’s gap condition has null density, \( \lim_{k \to \infty} n_k/k = 0 \), and its complement sequence \( \{m_k\} \) defined above has full density, \( \lim_{k \to \infty} m_k/k = 1 \). Both of these define recurrent trigonometric series. We can also construct a sequence with bounded gaps and intermediate density defining recurrent trigonometric series.

**Theorem 2.** Let \( p/q \) \((p, q \in \mathbb{N})\) be an arbitrary rational number in \((0, 1)\). Put \( I_{p,q} = \{ql + j \mid l = 0, 1, \ldots; j = 1, \ldots, p\} \) and suppose that \( \{n_k\} \) is a sequence satisfying Hadamard’s gap condition and \( \{n_k\} \cap I_{p,q} = \emptyset \). Let \( \{m_j\} \) be an increasing arrangement order of \( \{n_k\} \cup I_{p,q} \). Then \( \sum \cos 2\pi m_k x \) is recurrent for a.e. \( x \), and \( \{m_j\} \) has density \( \lim_{k \to \infty} m_k/k = p/q \).

The proofs are modifications of those in Grubb and Moore [5]. We use the properties of the Dirichlet kernel.

**2. Proof.** We use a lemma which is a modification of that in Grubb and Moore [5].

**Lemma 3.** Let \( I \) be a non-empty open interval, \( E_N, F_N \subset I \) \((N \in \mathbb{N})\), \( c > 0 \), and \( 0 < \delta_N \downarrow 0 \). Assume that for any \( x \in E_N \), there exists \( N_0 \) such that for \( N \geq N_0 \), there exists an interval \( J_N \) with \( x \in J_N \), \( |J_N| = \delta_N \) and \( |F_N \cap J_N| \geq c|J_N| \). If \( x \in E_N \) infinitely often for a.e. \( x \in I \), then \( x \in F_N \) infinitely often for a.e. \( x \in I \).

*Proof of Theorem 1.* Take \( \rho > 0 \) arbitrarily and take an open interval \( I \subset [0, 1] \) such that \( 2 \sin \pi x > \rho \) on \( I \). Since \( \rho \) is arbitrary, it is sufficient to prove the recurrence for a.e. \( x \in I \).

Put \( \Delta = 2\pi(q/(q-1) + 4/\rho^2) \) and take an arbitrary \( \varepsilon \in (0, \Delta/2) \). We have
where $D_n$ is the Dirichlet kernel given by

$$D_n(x) = \frac{1}{2} + \sum_{j=1}^{n} \cos 2\pi jx = \frac{\sin \pi(2n+1)x}{2\sin \pi x}.$$ 

It is easily verified that $|D_n'(x)| \leq 2\pi (2n+2)/\rho^2 \leq 8\pi n/\rho^2$ on $I$ and $|T_j'(x)| \leq 2\pi (n_1 + \cdots + n_j) \leq 2\pi nj/(q-1)$ where $T_j(x) = \cos 2\pi n_1x + \cdots + \cos 2\pi njx$. Hence $|S_n'(x)| \leq \Delta m_N$ on $I$. Take an arbitrary $a \in \mathbb{R}$ and put

$$E_N = \{x \in I : S_N(x) \geq a, S_{N+1}(x) < a\},$$

$$F_N = \{x \in I : |S_N(x) - a| < \varepsilon \text{ or } |S_{N+1}(x) - a| < \varepsilon\}.$$

By noting $|D_n(x)| \leq 1/\rho$ and the properties $\sup_j T_j(x) = \infty$ and $\inf_j T_j(x) = -\infty$ a.e. of lacunary trigonometric series (p. 205 of Zygmund [12]), we have $\sup_N S_N(x) = \infty$ and $\inf_N S_N(x) = -\infty$ for a.e. $x \in I$. Hence $x \in E_N$ infinitely often for a.e. $x \in I$.

Pick an arbitrary $x \in E_N$. Put $\delta_N = 1/m_{N+1}$ and $J_N = (x - \delta_N/2, x + \delta_N/2)$. We have $J_N \subset I$ for large $N$. We divide the proof into two cases:

**Case I:** there exists an $x_0 \in J_N$ such that $S_N(x_0) = a$. Then we have $|S_N(x) - a| < \varepsilon$ on $(x_0 - |J_N|\varepsilon/\Delta, x_0 + |J_N|\varepsilon/\Delta)$. Since $|J_N|\varepsilon/\Delta \leq |J_N|/2$, either $(x_0 - |J_N|\varepsilon/\Delta)$ or $(x_0, x_0 + |J_N|\varepsilon/\Delta)$ is contained in $J_N$ and hence in $F_N \cap J_N$. Therefore $|F_N \cap J_N| \geq |J_N|\varepsilon/\Delta$.

**Case II:** $S_N(x) > a$ on $J_N$. As $x \in E_N$, we have $S_N(x) \geq a$ and $S_{N+1}(x) < a$. Since $|J_N| = 1/m_{N+1}$, there exists an $x_1 \in J_N$ such that $\cos 2\pi m_{N+1}x_1 = 0$. Hence $S_{N+1}(x_1) = S_N(x_1) \geq a$, and therefore we can find $x_2 \in J_N$ such that $S_{N+1}(x_2) = a$. In the same way as in the previous case, we can see that $|F_N \cap J_N| \geq |J_N|\varepsilon/\Delta$.

Applying the lemma, we see that $x \in F_N$ infinitely often for a.e. $x \in I$. ■

Theorem 2 can be proved in the same way by noting that

$$\sum_{l=1}^{n} \cos 2\pi (lq + j)x = \frac{\sin \pi((2n+1)q + 2j)x - \sin \pi(q + 2j)x}{2\sin \pi qx}.$$

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**REFERENCES**


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