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RECURRENCE FOR COSINE SERIES WITH BOUNDED GAPS

BY

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Abstract. Ullrich, Grubb and Moore proved that a lacunary trigonometric series satisfying Hadamard's gap condition is recurrent a.e. We prove the existence of a recurrent trigonometric series with bounded gaps.

1. Introduction. If we regard the sequence $\{\cos 2\pi n_k x\}$ as a sequence of random variables on the unit interval equipped with the Lebesgue measure, it behaves like a sequence of independent random variables when n_k diverges rapidly. For example, by assuming Hadamard's gap condition

$$n_{k+1}/n_k > q > 1 \quad (k = 1, 2, \dots),$$

the central limit theorem for $\sum \cos 2\pi n_k x$ was proved by Salem and Zygmund [9], the law of the iterated logarithm by Erdős and Gál [4], and the almost sure invariance principles by Philipp and Stout [8].

As to recurrence, Hawkes [7] proved that $\{\sum_{k=1}^{N} \exp(2\pi i n_k x)\}_{N\in\mathbb{N}}$ is dense in the complex plane for a.e. x assuming the very strong gap condition $\sum n_k/n_{k+1} < \infty$. Anderson and Pitt [1] weakened the gap condition to $n_{k+1}/n_k \to \infty$ or $n_k = a^k$, where $a \ge 2$ is an integer. These results imply the recurrence of $\sum_{k=1}^{N} \cos 2\pi n_k x$. For this one-dimensional recurrence, Ullrich, Grubb and Moore [11, 5] succeeded in weakening the condition to Hadamard's gap condition.

It is very natural to ask if the gap condition can be replaced by a weaker one. For the central limit theorem, Erdős [3] relaxed the gap condition to $n_{k+1}/n_k > 1 + c_k/\sqrt{k}$ with $c_k \to \infty$. This condition is best possible. Actually Erdős [3] and Takahashi [10] constructed counterexamples to the central limit theorem satisfying $n_{k+1}/n_k > 1 + c/\sqrt{k}$ with $c > 0$. But there still remains the possibility that some series having smaller gaps may obey the central limit theorem. Indeed, for any $\phi(k) \uparrow \infty$, Berkes [2] proved the existence of $\sum \cos 2\pi n_k x$ with small gaps $n_{k+1} - n_k = O(\phi(k))$ which obeys the central limit theorem. And it was a longstanding problem whether some trigonometric series with bounded gaps $n_{k+1} - n_k = O(1)$ can obey the

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central limit theorem. Recently the existence of such series was proved in [6] and the problem was solved.

In this paper, we consider the same problem for recurrence, and prove the existence of recurrent series with bounded gaps.

THEOREM 1. *Suppose that $\{n_k\}$ satisfies Hadamard's gap condition and let $\{m_j\}$ be an increasing arrangement of $\mathbb{N} \setminus \{n_k\}$. Put*

$$S_N(x) = \sum_{j=1}^{N} \cos 2\pi m_j x.$$

Then $\{S_N(x)\}$ is recurrent for a.e. x.

The sequence $\{n_k\}$ satisfying Hadamard's gap condition has null density, $\lim_{k\to\infty} n_k/k = 0$, and its complement sequence $\{m_k\}$ defined above has full density, $\lim_{k\to\infty} m_k/k = 1$. Both of these define recurrent trigonometric series. We can also construct a sequence with bounded gaps and intermediate density defining recurrent trigonometric series.

THEOREM 2. *Let p/q $(p, q \in \mathbb{N})$ be an arbitrary rational number in $(0, 1)$. Put $I_{p,q} = \{lq + j \mid l = 0, 1, \dots; j = 1, \dots, p\}$ and suppose that $\{n_k\}$ is a sequence satisfying Hadamard's gap condition and $\{n_k\} \cap I_{p,q} = \emptyset$. Let $\{m_j\}$ be an increasing arrangement order of $\{n_k\} \cup I_{p,q}$. Then $\sum \cos 2\pi m_k x$ is recurrent for a.e. x, and $\{m_j\}$ has density $\lim_{k\to\infty} m_k/k = p/q$.*

The proofs are modifications of those in Grubb and Moore [5]. We use the properties of the Dirichlet kernel.

2. Proof. We use a lemma which is a modification of that in Grubb and Moore [5].

LEMMA 3. *Let I be a non-empty open interval, $E_N, F_N \subset I$ $(N \in \mathbb{N})$, $c > 0$, and $0 < \delta_N \downarrow 0$. Assume that for any $x \in E_N$, there exists N_0 such that for $N \geq N_0$, there exists an interval J_N with $x \in J_N$, $|J_N| = \delta_N$ and $|F_N \cap J_N| \geq c|J_N|$. If $x \in E_N$ infinitely often for a.e. $x \in I$, then $x \in F_N$ infinitely often for a.e. $x \in I$.*

Proof of Theorem 1. Take $\rho > 0$ arbitrarily and take an open interval $I \subset [\,0, 1\,]$ such that $2 \sin \pi x > \rho$ on I. Since ρ is arbitrary, it is sufficient to prove the recurrence for a.e. $x \in I$.

Put $\varDelta = 2\pi(q/(q-1) + 4/\rho^2)$ and take an arbitrary $\varepsilon \in (0, \varDelta/2)$. We have

$$S_N(x) = D_{m_N}(x) - \frac{1}{2} - \sum_{j\,:\,n_j \le m_N} \cos 2\pi n_j x,$$

where D_n is the Dirichlet kernel given by

$$D_n(x) = \frac{1}{2} + \sum_{j=1}^{n} \cos 2\pi j x = \frac{\sin \pi(2n+1)x}{2\sin \pi x}.$$

It is easily verified that $|D_n'(x)| \le 2\pi(2n+2)/\rho^2 \le 8\pi n/\rho^2$ on I and $|T_j'(x)| \le 2\pi(n_1 + \cdots + n_j) \le 2\pi n_j q/(q-1)$ where $T_j(x) = \cos 2\pi n_1 x + \cdots + \cos 2\pi n_j x$. Hence $|S_N'(x)| \le \Delta m_N$ on I. Take an arbitrary $a \in \mathbb{R}$ and put

$$E_N = \{x \in I : S_N(x) \ge a,\ S_{N+1}(x) < a\},$$
$$F_N = \{x \in I : |S_N(x) - a| < \varepsilon \text{ or } |S_{N+1}(x) - a| < \varepsilon\}.$$

By noting $|D_n(x)| \le 1/\rho$ and the properties $\sup_j T_j(x) = \infty$ and $\inf_j T_j(x) = -\infty$ a.e. of lacunary trigonometric series (p. 205 of Zygmund [12]), we have $\sup_N S_N(x) = \infty$ and $\inf_N S_N(x) = -\infty$ for a.e. $x \in I$. Hence $x \in E_N$ infinitely often for a.e. $x \in I$.

Pick an arbitrary $x \in E_N$. Put $\delta_N = 1/m_{N+1}$ and $J_N = (x - \delta_N/2, x + \delta_N/2)$. We have $J_N \subset I$ for large N. We divide the proof into two cases:

Case I: *there exists an $x_0 \in J_N$ such that $S_N(x_0) = a$.* Then we have $|S_N(x) - a| < \varepsilon$ on $(x_0 - |J_N|\varepsilon/\Delta, x_0 + |J_N|\varepsilon/\Delta)$. Since $|J_N|\varepsilon/\Delta \le |J_N|/2$, either $(x_0 - |J_N|\varepsilon/\Delta)$ or $(x_0, x_0 + |J_N|\varepsilon/\Delta)$ is contained in J_N and hence in $F_N \cap J_N$. Therefore $|F_N \cap J_N| \ge |J_N|\varepsilon/\Delta$.

Case II: *$S_N(x) > a$ on J_N.* As $x \in E_N$, we have $S_N(x) \ge a$ and $S_{N+1}(x) < a$. Since $|J_N| = 1/m_{N+1}$, there exists an $x_1 \in J_N$ such that $\cos 2\pi m_{N+1} x_1 = 0$. Hence $S_{N+1}(x_1) = S_N(x_1) \ge a$, and therefore we can find $x_2 \in J_N$ such that $S_{N+1}(x_2) = a$. In the same way as in the previous case, we can see that $|F_N \cap J_N| \ge |J_N|\varepsilon/\Delta$.

Applying the lemma, we see that $x \in F_N$ infinitely often for a.e. $x \in I$. ∎

Theorem 2 can be proved in the same way by noting that

$$\sum_{l=1}^{n} \cos 2\pi(lq + j)x = \frac{\sin \pi((2n+1)q + 2j)x - \sin \pi(q+2j)x}{2\sin \pi q x}.$$

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