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DISTINGUISHING DERIVED EQUIVALENCE CLASSES USING THE SECOND HOCHSCHILD COHOMOLOGY GROUP

BY

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Abstract. We study the second Hochschild cohomology group of the preprojective algebra of type D_4 over an algebraically closed field K of characteristic 2. We also calculate the second Hochschild cohomology group of a non-standard algebra which arises as a socle deformation of this preprojective algebra and so show that the two algebras are not derived equivalent. This answers a question raised by Holm and Skowroński.

Introduction. The main work in this paper goes into determining the second Hochschild cohomology group $\mathrm{HH}^2(\Lambda)$ for two finite-dimensional algebras Λ over a field of characteristic 2 in order to show that they are not derived equivalent. We let $\mathcal{A}_1$ denote the preprojective algebra of type D_4; this is a standard algebra. We introduce, in Section 1, an algebra $\mathcal{A}_2$ by quiver and relations; it is a non-standard algebra which is socle equivalent to $\mathcal{A}_1$. We note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic in the case where the underlying field has characteristic not 2. (We refer to [6] for more information about standard and non-standard selfinjective algebras.) The work in this paper is motivated by the question asked by Holm and Skowroński as to whether or not these two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are derived equivalent.

The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are selfinjective algebras of polynomial growth. The main result of this paper (Corollary 4.2) shows that they are not derived equivalent. This answer to the question of Holm and Skowroński enables one to complete their derived equivalence classification of all symmetric algebras of polynomial growth in [6, 5.20]. We recall that the complete derived equivalence classification of selfinjective algebras of finite representation type was given in [2]. Computation of the second Hochschild cohomology group was then used in [1] to give an alternative proof to distinguish between derived equivalence classes of standard and non-standard selfinjective algebras of finite representation type. Thus the second Hochschild cohomology group is a powerful tool for distinguishing algebras up to derived equivalence.

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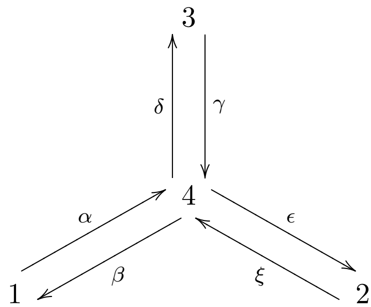
Throughout this paper, we let Λ denote a finite-dimensional algebra over an algebraically closed field K. We start, in Section 1, by giving the quiver and relations for $\mathcal{A}_1$ and $\mathcal{A}_2$. We are interested only in the case when $\operatorname{char} K = 2$, since if $\operatorname{char} K \neq 2$ then the algebras are isomorphic and the second Hochschild cohomology group is known by [3]. In Section 2, we give a short description of the projective resolution of [4] which we use to find $\mathrm{HH}^2(\Lambda)$. The remaining two sections determine $\mathrm{HH}^2(\Lambda)$ for $\Lambda = \mathcal{A}_1, \mathcal{A}_2$. As a consequence, we show in Corollary 4.2 that $\dim \mathrm{HH}^2(\mathcal{A}_1) \neq \dim \mathrm{HH}^2(\mathcal{A}_2)$ and hence these two algebras are not derived equivalent.

1. The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. In this section we describe the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ by quivers and relations. We assume that K is an algebraically closed field and $\operatorname{char} K = 2$. The standard algebra $\mathcal{A}_1$ is the preprojective algebra of type D_4, and it was shown in [3] that, in the case when $\operatorname{char} K \neq 2$, we have $\mathrm{HH}^2(\mathcal{A}_1) = 0$. We will see that this is in contrast to the $\operatorname{char} K = 2$ case.

The algebra $\mathcal{A}_1$ is given by the quiver $\mathcal{Q}$:



with relations

$$\beta\alpha + \delta\gamma + \epsilon\xi = 0, \quad \gamma\delta = 0, \quad \xi\epsilon = 0, \quad \alpha\beta = 0.$$

The algebra $\mathcal{A}_2$ is the non-standard algebra given by the same quiver $\mathcal{Q}$ with relations

$$\beta\alpha + \delta\gamma + \epsilon\xi = 0, \quad \gamma\delta = 0, \quad \xi\epsilon = 0, \quad \alpha\beta\alpha = 0, \quad \beta\alpha\beta = 0, \quad \alpha\beta = \alpha\delta\gamma\beta.$$

Note that we write our paths in a quiver from left to right.

We need to find a minimal set of relations for each algebra. We start with $\mathcal{A}_2$. The set $\{\alpha\beta - \alpha\delta\gamma\beta, \xi\epsilon, \gamma\delta, \beta\alpha + \delta\gamma + \epsilon\xi, \alpha\beta\alpha, \beta\alpha\beta\}$ is not a minimal set of generators for I where $\mathcal{A}_2 = K\mathcal{Q}/I$. Let $x = \beta\alpha + \delta\gamma + \epsilon\xi$ and let $y = \alpha\beta - \alpha\delta\gamma\beta$. We will show that $\alpha\beta\alpha$ is in the ideal generated by $x, y, \gamma\delta, \xi\epsilon$. Using that $\operatorname{char} K = 2$, we have

$$\begin{aligned}\alpha\beta\alpha &= y\alpha + \alpha\delta\gamma\beta\alpha \\ &= y\alpha + \alpha x\beta\alpha + \alpha(\beta\alpha + \epsilon\xi)\beta\alpha\end{aligned}$$

$$= y\alpha + \alpha x\beta\alpha + \alpha\beta\alpha\beta\alpha + \alpha\epsilon\xi x + \alpha\epsilon\xi(\delta\gamma + \epsilon\xi)$$
$$= y\alpha + \alpha x\beta\alpha + \alpha\epsilon\xi x + \alpha\beta\alpha\beta\alpha + \alpha x\delta\gamma + \alpha(\beta\alpha + \delta\gamma)\delta\gamma + \alpha\epsilon\xi\epsilon\xi$$
$$= y\alpha + \alpha x\beta\alpha + \alpha\epsilon\xi x + \alpha x\delta\gamma + \alpha\epsilon\xi\epsilon\xi + \alpha\beta\alpha\beta\alpha + \alpha\beta\alpha x + \alpha\beta\alpha(\beta\alpha + \epsilon\xi) + \alpha\delta\gamma\delta\gamma$$
$$= y\alpha + \alpha x\beta\alpha + \alpha\epsilon\xi x + \alpha x\delta\gamma + \alpha\epsilon\xi\epsilon\xi + \alpha\beta\alpha x + \alpha\beta\alpha\epsilon\xi + \alpha\delta\gamma\delta\gamma.$$

However, $\alpha\beta\alpha\epsilon\xi = y\alpha\epsilon\xi + \alpha\delta\gamma\beta\alpha\epsilon\xi = y\alpha\epsilon\xi + \alpha\delta\gamma x\epsilon\xi + \alpha\delta\gamma(\delta\gamma + \epsilon\xi)\epsilon\xi$. Thus $\alpha\beta\alpha$ is in the ideal generated by $x, y, \gamma\delta, \xi\epsilon$. Using a similar argument for $\beta\alpha\beta$, we see that I is generated by the set $\{\alpha\beta - \alpha\delta\gamma\beta, \xi\epsilon, \gamma\delta, \beta\alpha + \delta\gamma + \epsilon\xi\}$. This gives the following result.

Proposition 1.1. *For $\mathcal{A}_2$ let*

$$f_1^2 = \alpha\beta - \alpha\delta\gamma\beta, \quad f_2^2 = \xi\epsilon,$$
$$f_3^2 = \gamma\delta, \quad f_4^2 = \beta\alpha + \delta\gamma + \epsilon\xi.$$

Then $f^2 = \{f_1^2, f_2^2, f_3^2, f_4^2\}$ is a minimal set of generators of I where $\mathcal{A}_2 = K\mathcal{Q}/I$.

We now consider the algebra $\mathcal{A}_1$.

Proposition 1.2. *For $\mathcal{A}_1$ let*

$$f_1^2 = \alpha\beta, \quad f_2^2 = \xi\epsilon,$$
$$f_3^2 = \gamma\delta, \quad f_4^2 = \beta\alpha + \delta\gamma + \epsilon\xi.$$

Then $f^2 = \{f_1^2, f_2^2, f_3^2, f_4^2\}$ is a minimal set of generators for I' where $\mathcal{A}_1 = K\mathcal{Q}/I'$.

2. The projective resolution. To find the Hochschild cohomology groups for any finite-dimensional algebra Λ, a projective resolution of Λ as a Λ,Λ-bimodule is needed. In this section we look at the projective resolutions of [4] and [5] in order to describe the second Hochschild cohomology group. Let K be a field and let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional algebra where $\mathcal{Q}$ is a quiver, and I is an admissible ideal of $K\mathcal{Q}$. Fix a minimal set f^2 of generators for the ideal I. For any $x \in f^2$, we may write $x = \sum_{j=1}^{r} c_j a_{1j} \cdots a_{kj} \cdots a_{s_j j}$, where the a_{ij} are arrows in $\mathcal{Q}$ and $c_j \in K$, that is, x is a linear combination of paths $a_{1j} \cdots a_{kj} \cdots a_{s_j j}$ for $j = 1, \ldots, r$. We may assume that there are (unique) vertices v and w such that each path $a_{1j} \cdots a_{kj} \cdots a_{s_j j}$ starts at v and ends at w for all j, so that $x = vxw$. We write $\mathfrak{o}(x) = v$ and $\mathfrak{t}(x) = w$. Similarly $\mathfrak{o}(a)$ is the origin of the arrow a and $\mathfrak{t}(a)$ is the terminus of a.

In [4, Theorem 2.9], the first four terms of a minimal projective resolution of Λ as a Λ,Λ-bimodule are described:

$$\cdots \to Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \to 0.$$

The projective Λ,Λ-bimodules Q^0, Q^1, Q^2 are given by

$$Q^0 = \bigoplus_{v,\,\text{vertex}} \Lambda v \otimes v\Lambda,$$
$$Q^1 = \bigoplus_{a,\,\text{arrow}} \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda,$$
$$Q^2 = \bigoplus_{x\in f^2} \Lambda\mathfrak{o}(x) \otimes \mathfrak{t}(x)\Lambda.$$

Throughout, all tensor products are over K, and we write $\otimes$ for $\otimes_K$. The maps g, A_1, A_2 and A_3 are all Λ,Λ-bimodule homomorphisms. The map $g : Q^0 \to \Lambda$ is the multiplication map given by $v\otimes v \mapsto v$. The map $A_1 : Q^1 \to Q^0$ is given by $\mathfrak{o}(a)\otimes\mathfrak{t}(a) \mapsto \mathfrak{o}(a)\otimes\mathfrak{o}(a)a - a\mathfrak{t}(a)\otimes\mathfrak{t}(a)$ for each arrow a. With the notation for $x \in f^2$ given above, the map $A_2 : Q^2 \to Q^1$ is given by

$$\mathfrak{o}(x)\otimes\mathfrak{t}(x) \mapsto \sum_{j=1}^{r} c_j\Big(\sum_{k=1}^{s_j} a_{1j}\cdots a_{(k-1)j}\otimes a_{(k+1)j}\cdots a_{s_j j}\Big),$$

where $a_{1j}\cdots a_{(k-1)j}\otimes a_{(k+1)j}\cdots a_{s_j j} \in \Lambda\mathfrak{o}(a_{kj})\otimes\mathfrak{t}(a_{kj})\Lambda$.

In order to describe the projective bimodule Q^3 and the map A_3 in the Λ,Λ-bimodule resolution of Λ in [4], we need to introduce some notation from [5]. Recall that an element $y \in K\mathcal{Q}$ is *uniform* if there are vertices v, w such that $y = vy = yw$. We write $\mathfrak{o}(y) = v$ and $\mathfrak{t}(y) = w$. In [5], Green, Solberg and Zacharia show that there are sets f^n in $K\mathcal{Q}$, for $n \geq 3$, consisting of uniform elements y such that $y = \sum_{x\in f^{n-1}} xr_x = \sum_{z\in f^{n-2}} zs_z$ for unique $r_x, s_z \in K\mathcal{Q}$ such that $s_z \in I$. These sets have special properties relative to a minimal projective Λ-resolution of $\Lambda/\mathfrak{r}$, where $\mathfrak{r}$ is the Jacobson radical of Λ. Specifically, the nth projective in the minimal projective Λ-resolution of $\Lambda/\mathfrak{r}$ is $\bigoplus_{y\in f^n}\mathfrak{t}(y)\Lambda$.

In particular, to determine the set f^3, we follow explicitly the construction given in [5, §1]. Let f^1 denote the set of arrows of $\mathcal{Q}$. Suppose the intersection $(\bigoplus_i f_i^2 K\mathcal{Q}) \cap (\bigoplus_j f_j^1 I)$ is equal to some $(\bigoplus_l f_l^{3*} K\mathcal{Q})$. We then discard all elements of the form f^{3*} that are in $\bigoplus_i f_i^2 I$; the remaining ones form precisely the set f^3.

Thus, for $y \in f^3$ we have $y \in (\bigoplus_i f_i^2 K\mathcal{Q}) \cap (\bigoplus_j f_j^1 I)$. So we may write $y = \sum f_i^2 p_i = \sum q_i f_i^2 r_i$ with $p_i, q_i, r_i \in K\mathcal{Q}$ such that p_i, q_i are in the ideal generated by the arrows of $K\mathcal{Q}$, and the p_i are unique. Then [4] gives $Q^3 = \bigoplus_{y\in f^3}\Lambda\mathfrak{o}(y)\otimes\mathfrak{t}(y)\Lambda$ and, for $y \in f^3$ in the notation above, the component of $A_3(\mathfrak{o}(y)\otimes\mathfrak{t}(y))$ in the summand $\Lambda\mathfrak{o}(f_i^2)\otimes\mathfrak{t}(f_i^2)\Lambda$ of Q^2 is $\mathfrak{o}(y)\otimes p_i - q_i\otimes r_i$.

Given this part of the minimal projective Λ,Λ-bimodule resolution of Λ:

$$Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \to 0$$

we apply $\mathrm{Hom}(-,\Lambda)$ to get the complex

$$0 \to \mathrm{Hom}(Q^0,\Lambda) \xrightarrow{d_1} \mathrm{Hom}(Q^1,\Lambda) \xrightarrow{d_2} \mathrm{Hom}(Q^2,\Lambda) \xrightarrow{d_3} \mathrm{Hom}(Q^3,\Lambda)$$

where d_i is the map induced from A_i for $i = 1,2,3$. Then $\mathrm{HH}^2(\Lambda) = \mathrm{Ker}\, d_3/\mathrm{Im}\, d_2$.

When considering an element of the projective bimodule

$$Q^1 = \bigoplus_{a \text{ arrow}} \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda$$

it is important to keep track of the individual summands of Q^1. So to avoid confusion we usually denote an element in the summand $\Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda$ by $\lambda \otimes_a \lambda'$ using the subscript 'a' to remind us in which summand this element lies. Similarly, an element $\lambda \otimes_{f_i^2} \lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2)\Lambda$ of Q^2 and an element $\lambda \otimes_{f_i^3} \lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^3) \otimes \mathfrak{t}(f_i^3)\Lambda$ of Q^3. We keep this notation for the rest of the paper.

Now we are ready to compute $\mathrm{HH}^2(\Lambda)$ for the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$.

3. $\mathrm{HH}^2(\mathcal{A}_2)$. In this section we determine $\mathrm{HH}^2(\mathcal{A}_2)$ for the non-standard algebra $\mathcal{A}_2$.

THEOREM 3.1. *For the non-standard algebra $\mathcal{A}_2$ with $\mathrm{char}\, K = 2$, we have $\dim \mathrm{HH}^2(\mathcal{A}_2) = 4$.*

Proof. The set f^2 of minimal relations was given in Proposition 1.1.

Following [5] as described above, we may choose the set f^3 to be $\{f_1^3, f_2^3, f_3^3, f_4^3\}$, where

$$\begin{aligned}
f_1^3 &= f_1^2\alpha\delta\gamma\beta + f_1^2\alpha\beta \\
&= \alpha\delta\gamma\beta f_1^2 + \alpha\beta f_1^2 \in e_1 K\mathcal{Q}e_1,\\
f_2^3 &= f_2^2\xi\delta\gamma\epsilon + f_2^2\xi\beta\alpha\epsilon \\
&= \xi f_4^2\beta\alpha\epsilon + \xi f_4^2\delta\gamma\epsilon + \xi\delta\gamma f_4^2\epsilon + \xi\beta\alpha f_4^2\epsilon + \xi\delta\gamma\epsilon f_2^2 + \xi\beta\alpha\epsilon f_2^2 \in e_2 K\mathcal{Q}e_2,\\
f_3^3 &= f_3^2\gamma\beta\alpha\delta + f_3^2\gamma\epsilon\xi\delta \\
&= \gamma f_4^2\epsilon\xi\delta + \gamma f_4^2\beta\alpha\delta + \gamma\beta\alpha f_4^2\delta + \gamma\epsilon\xi f_4^2\delta + \gamma\beta\alpha\delta f_3^2 + \gamma\epsilon\xi\delta f_3^2 \in e_3 K\mathcal{Q}e_3,\\
f_4^3 &= f_4^2\beta\alpha\delta\gamma + f_4^2\epsilon\xi\delta\gamma \\
&= \epsilon f_2^2\xi\delta\gamma + \delta f_3^2\gamma\beta\alpha + \delta f_3^2\gamma\epsilon\xi + \delta\gamma f_4^2\beta\alpha + \delta\gamma f_4^2\epsilon\xi \\
&\quad + \beta\alpha f_4^2\delta\gamma + \beta\alpha\delta f_3^2\gamma + \delta\gamma\epsilon\xi f_4^2 + \delta\gamma\beta\alpha f_4^2 \in e_4 K\mathcal{Q}e_4.
\end{aligned}$$

We remark that in line with [5, Theorem 2.4], the semisimple module $\mathcal{A}_2/\mathfrak{r}$ has a minimal projective resolution as a right $\mathcal{A}_2$-module which begins:

$$\cdots \to \bigoplus_{y\in f^3} \mathfrak{t}(y)\mathcal{A}_2 \xrightarrow{\partial_3} \bigoplus_{x\in f^2} \mathfrak{t}(x)\mathcal{A}_2 \xrightarrow{\partial_2} \bigoplus_{a\in f^1} \mathfrak{t}(a)\mathcal{A}_2 \xrightarrow{\partial_1} \bigoplus_{i=1}^{4} v_i\mathcal{A}_2 \to \mathcal{A}_2/\mathfrak{r} \to 0$$

where the maps are given by

$$\begin{aligned}
\partial_3 : \quad & \mathfrak{t}(f_1^3) \mapsto \mathfrak{t}(f_1^2)(\alpha\delta\gamma\beta + \alpha\beta), \\
& \mathfrak{t}(f_2^3) \mapsto \mathfrak{t}(f_2^2)(\xi\delta\gamma\epsilon + \xi\beta\alpha\epsilon), \\
& \mathfrak{t}(f_3^3) \mapsto \mathfrak{t}(f_3^2)(\gamma\beta\alpha\delta + \gamma\epsilon\xi\delta), \\
& \mathfrak{t}(f_4^3) \mapsto \mathfrak{t}(f_4^2)(\beta\alpha\delta\gamma + \epsilon\xi\delta\gamma), \\
\partial_2 : \quad & \mathfrak{t}(f_1^2) \mapsto \mathfrak{t}(\alpha)(\beta - \delta\gamma\beta), \\
& \mathfrak{t}(f_2^2) \mapsto \mathfrak{t}(\xi)\epsilon, \\
& \mathfrak{t}(f_3^2) \mapsto \mathfrak{t}(\gamma)\delta, \\
& \mathfrak{t}(f_4^2) \mapsto \mathfrak{t}(\beta)\alpha + \mathfrak{t}(\delta)\gamma + \mathfrak{t}(\epsilon)\xi, \\
\partial_1 : \quad & \mathfrak{t}(\alpha) \mapsto v_4, \quad \mathfrak{t}(\delta) \mapsto v_3, \\
& \mathfrak{t}(\beta) \mapsto v_1, \quad \mathfrak{t}(\epsilon) \mapsto v_2, \\
& \mathfrak{t}(\gamma) \mapsto v_4, \quad \mathfrak{t}(\xi) \mapsto v_4,
\end{aligned}$$

with each term being in the obvious summand of the appropriate projective module.

Thus (writing Λ for $\mathcal{A}_2$) the projective bimodule $Q^3 = \bigoplus_{y\in f^3} \Lambda\mathfrak{o}(y) \otimes \mathfrak{t}(y)\Lambda = (\Lambda e_1 \otimes e_1\Lambda) \oplus (\Lambda e_2 \otimes e_2\Lambda) \oplus (\Lambda e_3 \otimes e_3\Lambda) \oplus (\Lambda e_4 \otimes e_4\Lambda)$. We know that $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3/\operatorname{Im} d_2$. First we will find $\operatorname{Im} d_2$. Let $f \in \operatorname{Hom}(Q^1, \Lambda)$ and so write

$$\begin{aligned}
f(e_1 \otimes_\alpha e_4) &= c_1\alpha + c_2\alpha\delta\gamma, & f(e_4 \otimes_\beta e_1) &= c_3\beta + c_4\delta\gamma\beta, \\
f(e_3 \otimes_\gamma e_4) &= c_5\gamma + c_6\gamma\beta\alpha, & f(e_4 \otimes_\delta e_3) &= c_7\delta + c_8\beta\alpha\delta, \\
f(e_4 \otimes_\epsilon e_2) &= c_9\epsilon + c_{10}\delta\gamma\epsilon, & f(e_2 \otimes_\xi e_4) &= c_{11}\xi + c_{12}\xi\delta\gamma,
\end{aligned}$$

where $c_1, \ldots, c_{12} \in K$. Now we find $fA_2 = d_2 f$. We have

$$\begin{aligned}
fA_2(e_1 \otimes_{f_1^2} e_1) &= f(e_1 \otimes_\alpha e_4)\beta + \alpha f(e_4 \otimes_\beta e_1) - f(e_1 \otimes_\alpha e_4)\delta\gamma\beta \\
&\quad - \alpha f(e_4 \otimes_\delta e_3)\gamma\beta - \alpha\delta f(e_3 \otimes_\gamma e_4)\beta - \alpha\delta\gamma f(e_4 \otimes_\beta e_1) \\
&= c_1\alpha\beta + c_2\alpha\delta\gamma\beta + c_3\alpha\beta + c_4\alpha\delta\gamma\beta - c_1\alpha\delta\gamma\beta - c_7\alpha\delta\gamma\beta \\
&\quad - c_5\alpha\delta\gamma\beta - c_3\alpha\delta\gamma\beta \\
&= (c_1 + c_2 + c_3 + c_4 - c_1 - c_7 - c_5 - c_3)\alpha\beta \\
&= (c_2 + c_4 + c_7 + c_5)\alpha\beta.
\end{aligned}$$

Also

$$\begin{aligned}
fA_2(e_2 \otimes_{f_2^2} e_2) &= f(e_2 \otimes_\xi e_4)\epsilon + \xi f(e_4 \otimes_\epsilon e_2) = (c_{12} + c_{10})\xi\delta\gamma\epsilon, \\
fA_2(e_3 \otimes_{f_3^2} e_3) &= f(e_3 \otimes_\gamma e_4)\delta + \gamma f(e_4 \otimes_\delta e_3) = (c_6 + c_8)\gamma\beta\alpha\delta,
\end{aligned}$$

and

$$\begin{aligned}
fA_2(e_4 \otimes_{f_4^2} e_4) &= f(e_4 \otimes_\beta e_1)\alpha + f(e_4 \otimes_\delta e_3)\gamma + f(e_2 \otimes_\epsilon e_4)\xi \\
&\quad + \beta f(e_1 \otimes_\alpha e_4) + \delta f(e_3 \otimes_\gamma e_4) + \epsilon f(e_2 \otimes_\xi e_4) \\
&= c_3\beta\alpha + c_4\delta\gamma\beta\alpha + c_7\delta\gamma + c_8\beta\alpha\delta\gamma + c_9\epsilon\xi + c_{10}\delta\gamma\epsilon\xi + c_1\beta\alpha \\
&\quad + c_2\beta\alpha\delta\gamma + c_5\delta\gamma + c_6\delta\gamma\beta\alpha + c_{11}\epsilon\xi + c_{12}\epsilon\xi\delta\gamma \\
&= (c_3 + c_1)\beta\alpha + (c_7 + c_5)\delta\gamma + (c_9 + c_{11})\epsilon\xi \\
&\quad + (c_4 + c_2 + c_7 + c_5 + c_{10} + c_{12})\delta\gamma\beta\alpha \\
&= (c_3 + c_1 + c_9 + c_{11})\beta\alpha + (c_7 + c_5 + c_9 + c_{11})\delta\gamma \\
&\quad + (c_4 + c_2 + c_7 + c_5 + c_{10} + c_{12})\delta\gamma\beta\alpha.
\end{aligned}$$

Hence, fA_2 is given by

$$\begin{aligned}
fA_2(e_1 \otimes_{f_1^2} e_1) &= d_1\alpha\beta, \\
fA_2(e_2 \otimes_{f_2^2} e_2) &= d_2\xi\delta\gamma\epsilon, \\
fA_2(e_3 \otimes_{f_3^2} e_3) &= d_3\gamma\beta\alpha\delta, \\
fA_2(e_4 \otimes_{f_4^2} e_4) &= d_4\beta\alpha + d_5\delta\gamma + (d_1 + d_2)\delta\gamma\beta\alpha,
\end{aligned}$$

for some $d_1, \ldots, d_5 \in K$. Since there are no further linear dependencies between $d_1, \ldots, d_5$, we have $\dim \operatorname{Im} d_2 = 5$.

Now we determine $\operatorname{Ker} d_3$. Let $h \in \operatorname{Ker} d_3$, so $h \in \operatorname{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Let $h : Q^2 \to \Lambda$ be given by

$$\begin{aligned}
h(e_1 \otimes_{f_1^2} e_1) &= c_1 e_1 + c_2\alpha\delta\gamma\beta, \\
h(e_2 \otimes_{f_2^2} e_2) &= c_3 e_2 + c_4\xi\delta\gamma\epsilon, \\
h(e_3 \otimes_{f_3^2} e_3) &= c_5 e_3 + c_6\gamma\beta\alpha\delta, \\
h(e_4 \otimes_{f_4^2} e_4) &= c_7 e_4 + c_8\beta\alpha + c_9\delta\gamma + c_{10}\beta\alpha\delta\gamma,
\end{aligned}$$

for some $c_1, \ldots, c_{10} \in K$. Then

$$\begin{aligned}
hA_3(e_1 \otimes_{f_1^3} e_1) &= h(e_1 \otimes_{f_1^2} e_1)\alpha\delta\gamma\beta + h(e_1 \otimes_{f_1^2} e_1)\alpha\beta \\
&\quad - \alpha\delta\gamma\beta h(e_1 \otimes_{f_1^2} e_1) - \alpha\beta h(e_1 \otimes_{f_1^2} e_1) \\
&= c_1\alpha\delta\gamma\beta + c_1\alpha\beta - c_1\alpha\delta\gamma\beta - c_1\alpha\beta = 0.
\end{aligned}$$

In a similar way and recalling that $\operatorname{char} K = 2$, we can show that $hA_3(e_2 \otimes_{f_2^3} e_2) = 0$ and $hA_3(e_3 \otimes_{f_3^3} e_3) = 0$. Finally,

$$\begin{aligned}
hA_3(e_4 \otimes_{f_4^3} e_4) &= h(e_4 \otimes_{f_4^2} e_4)\beta\alpha\delta\gamma + h(e_4 \otimes_{f_4^2} e_4)\epsilon\xi\delta\gamma - \epsilon h(e_2 \otimes_{f_2^2} e_2)\xi\delta\gamma \\
&\quad - \delta h(e_3 \otimes_{f_3^2} e_3)\gamma\beta\alpha - \delta h(e_3 \otimes_{f_3^2} e_3)\gamma\epsilon\xi - \delta\gamma h(e_4 \otimes_{f_4^2} e_4)\beta\alpha \\
&\quad - \delta\gamma h(e_4 \otimes_{f_4^2} e_4)\epsilon\xi - \beta\alpha h(e_4 \otimes_{f_4^2} e_4)\delta\gamma - \beta\alpha\delta h(e_3 \otimes_{f_3^2} e_3)\gamma \\
&\quad - \delta\gamma\epsilon\xi h(e_4 \otimes_{f_4^2} e_4) - \delta\gamma\beta\alpha h(e_4 \otimes_{f_4^2} e_4)
\end{aligned}$$

$$= c_7\beta\alpha\delta\gamma + c_7\epsilon\xi\delta\gamma - c_3\epsilon\xi\delta\gamma - c_5\delta\gamma\beta\alpha - c_5\delta\gamma\epsilon\xi - c_7\delta\gamma\beta\alpha - c_7\delta\gamma\epsilon\xi$$
$$- c_7\beta\alpha\delta\gamma - c_5\delta\gamma\beta\alpha - c_7\delta\gamma\epsilon\xi - c_7\delta\gamma\beta\alpha$$
$$= (c_7 - c_3 - c_5)\epsilon\xi\delta\gamma.$$

As $h \in \operatorname{Ker} d_3$ we have $c_7 = c_3 + c_5$.

Thus h is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1 e_1 + c_2\alpha\delta\gamma\beta,$$
$$h(e_2 \otimes_{f_2^2} e_2) = c_3 e_2 + c_4\xi\delta\gamma\epsilon,$$
$$h(e_3 \otimes_{f_3^2} e_3) = c_5 e_3 + c_6\gamma\beta\alpha\delta,$$
$$h(e_4 \otimes_{f_4^2} e_4) = (c_3 + c_5)e_4 + c_8\beta\alpha + c_9\delta\gamma + c_{10}\beta\alpha\delta\gamma.$$

Hence $\dim \operatorname{Ker} d_3 = 9$.

Therefore, $\dim \mathrm{HH}^2(\mathcal{A}_2) = \dim \operatorname{Ker} d_3 - \dim \operatorname{Im} d_2 = 9 - 5 = 4$. ■

4. $\mathrm{HH}^2(\mathcal{A}_1)$. In this section we determine $\mathrm{HH}^2(\mathcal{A}_1)$ for the standard algebra $\mathcal{A}_1$.

THEOREM 4.1. *For the standard algebra $\mathcal{A}_1$ with* $\operatorname{char} K = 2$, *we have* $\dim \mathrm{HH}^2(\mathcal{A}_1) = 3$.

Proof. The set f^2 of minimal relations was given in Proposition 1.2. Following [5], we may choose the set f^3 to be $\{f_1^3, f_2^3, f_3^3, f_4^3\}$, where

$$f_1^3 = f_1^2\alpha\epsilon\xi\beta$$
$$= \alpha f_4^2\epsilon\xi\beta + \alpha\delta\gamma f_4^2\beta + \alpha\delta\gamma\beta f_1^2 + \alpha\delta f_3^2\gamma\beta + \alpha\epsilon f_2^2\xi\beta \in e_1 K\mathcal{Q}e_1,$$
$$f_2^3 = f_2^2\xi\delta\gamma\epsilon = \xi f_4^2\delta\gamma\epsilon + \xi\beta\alpha f_4^2\epsilon + \xi\beta f_1^2\alpha\epsilon + \xi\beta\alpha\epsilon f_2^2 + \xi\delta f_3^2\gamma\epsilon \in e_2 K\mathcal{Q}e_2,$$
$$f_3^3 = f_3^2\gamma\epsilon\xi\delta$$
$$= \gamma f_4^2\epsilon\xi\delta + \gamma\beta\alpha f_4^2\delta + \gamma\beta f_1^2\alpha\delta + \gamma\beta\alpha\delta f_3^2 + \gamma\epsilon f_2^2\xi\delta \in e_3 K\mathcal{Q}e_3,$$
$$f_4^3 = f_4^2\beta\alpha\delta\gamma = \beta f_1^2\alpha\delta\gamma + \delta f_3^2\gamma\epsilon\xi + \epsilon f_2^2\xi\delta\gamma + \delta\gamma f_4^2\epsilon\xi + \epsilon\xi f_4^2\delta\gamma$$
$$+ \delta\gamma\beta f_1^2\alpha + \delta\gamma\epsilon f_2^2\xi + \epsilon\xi\delta f_3^2\gamma + \delta\gamma\beta\alpha f_4^2 \in e_4 K\mathcal{Q}e_4.$$

Thus (writing Λ for $\mathcal{A}_1$) the projective bimodule Q^3 equals $\bigoplus_{y\in f^3} \Lambda\mathfrak{o}(y) \otimes \mathfrak{t}(y)\Lambda = (\Lambda e_1 \otimes e_1\Lambda) \oplus (\Lambda e_2 \otimes e_2\Lambda) \oplus (\Lambda e_3 \otimes e_3\Lambda) \oplus (\Lambda e_4 \otimes e_4\Lambda)$.

Again, $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3/\operatorname{Im} d_2$. First we will find $\operatorname{Im} d_2$. Let $f \in \operatorname{Hom}(Q^1, \Lambda)$ and so write

$$f(e_1 \otimes_\alpha e_4) = c_1\alpha + c_2\alpha\delta\gamma, \qquad f(e_4 \otimes_\beta e_1) = c_3\beta + c_4\delta\gamma\beta,$$
$$f(e_3 \otimes_\gamma e_4) = c_5\gamma + c_6\gamma\beta\alpha, \qquad f(e_4 \otimes_\delta e_3) = c_7\delta + c_8\beta\alpha\delta,$$
$$f(e_4 \otimes_\epsilon e_2) = c_9\epsilon + c_{10}\delta\gamma\epsilon, \qquad f(e_2 \otimes_\xi e_4) = c_{11}\xi + c_{12}\xi\delta\gamma,$$

where $c_1, \dots, c_{12} \in K$. Now we find $fA_2 = d_2 f$. We have

$$\begin{aligned} fA_2(e_1 \otimes_{f_1^2} e_1) &= f(e_1 \otimes_\alpha e_4)\beta + \alpha f(e_4 \otimes_\beta e_1) \\ &= c_2\alpha\delta\gamma\beta + c_4\alpha\delta\gamma\beta = (c_2 + c_4)\alpha\delta\gamma\beta. \end{aligned}$$

Also

$$fA_2(e_2 \otimes_{f_2^2} e_2) = f(e_2 \otimes_\xi e_4)\epsilon + \xi f(e_4 \otimes_\epsilon e_2) = (c_{12} + c_{10})\xi\delta\gamma\epsilon,$$
$$fA_2(e_3 \otimes_{f_3^2} e_3) = f(e_3 \otimes_\gamma e_4)\delta + \gamma f(e_4 \otimes_\delta e_3) = (c_6 + c_8)\gamma\beta\alpha\delta.$$

Finally

$$\begin{aligned} fA_2(e_4 \otimes_{f_4^2} e_4) &= f(e_4 \otimes_\beta e_1)\alpha + f(e_4 \otimes_\delta e_3)\gamma + f(e_2 \otimes_\epsilon e_4)\xi \\ &\quad + \beta f(e_1 \otimes_\alpha e_4) + \delta f(e_3 \otimes_\gamma e_4) + \epsilon f(e_2 \otimes_\xi e_4) \\ &= (c_3 + c_9 + c_1 + c_{11})\beta\alpha + (c_7 + c_9 + c_5 + c_{11})\delta\gamma \\ &\quad + (c_4 + c_8 + c_{10} + c_2 + c_6 + c_{12})\delta\gamma\beta\alpha. \end{aligned}$$

Hence, fA_2 is given by

$$\begin{aligned} fA_2(e_1 \otimes_{f_1^2} e_1) &= d_1\alpha\delta\gamma\beta, \\ fA_2(e_2 \otimes_{f_2^2} e_2) &= d_2\xi\delta\gamma\epsilon, \\ fA_2(e_3 \otimes_{f_3^2} e_3) &= d_3\gamma\beta\alpha\delta, \\ fA_2(e_4 \otimes_{f_4^2} e_4) &= d_4\beta\alpha + d_5\delta\gamma + (d_1 + d_2 + d_3)\delta\gamma\beta\alpha, \end{aligned}$$

for some $d_1, \dots, d_5 \in K$. Since there are no further linear dependencies between $d_1, \dots, d_5$, we have $\dim \operatorname{Im} d_2 = 5$.

Now we determine $\operatorname{Ker} d_3$. Let $h \in \operatorname{Ker} d_3$, so $h \in \operatorname{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Let $h : Q^2 \to \Lambda$ be given by

$$\begin{aligned} h(e_1 \otimes_{f_1^2} e_1) &= c_1 e_1 + c_2\alpha\delta\gamma\beta, \\ h(e_2 \otimes_{f_2^2} e_2) &= c_3 e_2 + c_4\xi\delta\gamma\epsilon, \\ h(e_3 \otimes_{f_3^2} e_3) &= c_5 e_3 + c_6\gamma\beta\alpha\delta, \\ h(e_4 \otimes_{f_4^2} e_4) &= c_7 e_4 + c_8\beta\alpha + c_9\delta\gamma + c_{10}\beta\alpha\delta\gamma, \end{aligned}$$

for some $c_1, \dots, c_{10} \in K$.

It can be easily shown that $hA_3(e_1 \otimes_{f_1^3} e_1) = (-c_5 - c_3)\alpha\delta\gamma\beta$. As $h \in \operatorname{Ker} d_3$ and $\operatorname{char} K = 2$ we have $c_5 = c_3$, and $hA_3(e_2 \otimes_{f_2^3} e_2) = (-c_1 - c_5)\xi\delta\gamma\epsilon$, so that $c_1 = c_5$. Similarly, $hA_3(e_3 \otimes_{f_3^3} e_3) = (-c_1 - c_3)\gamma\beta\alpha\delta$ so that $c_1 = c_3$. Finally, we have $hA_3(e_2 \otimes_{f_4^3} e_2) = 0$.

Thus h is given by

$$\begin{aligned} h(e_1 \otimes_{f_1^2} e_1) &= c_1 e_1 + c_2\alpha\delta\gamma\beta, \\ h(e_2 \otimes_{f_2^2} e_2) &= c_1 e_2 + c_4\xi\delta\gamma\epsilon, \end{aligned}$$

$$h(e_3 \otimes_{f_3^2} e_3) = c_1 e_3 + c_6 \gamma\beta\alpha\delta,$$
$$h(e_4 \otimes_{f_4^2} e_4) = c_7 e_4 + c_8 \beta\alpha + c_9 \delta\gamma + c_{10} \beta\alpha\delta\gamma.$$

Hence $\dim \operatorname{Ker} d_3 = 8$.

Therefore $\dim \mathrm{HH}^2(\mathcal{A}_1) = \dim \operatorname{Ker} d_3 - \dim \operatorname{Im} d_2 = 8 - 5 = 3$. ∎

Thus we have shown that $\dim \mathrm{HH}^2(\mathcal{A}_1) \neq \dim \mathrm{HH}^2(\mathcal{A}_2)$. Since Hochschild cohomology is invariant under derived equivalence, it follows that these two algebras are not derived equivalent, which the main result of this paper:

Corollary 4.2. *For the finite-dimensional algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ over an algebraically closed field K with $\operatorname{char} K = 2$, we have $\dim \mathrm{HH}^2(\mathcal{A}_1) \neq \dim \mathrm{HH}^2(\mathcal{A}_2)$. Hence these two algebras are not derived equivalent.*

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