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ON THE NEUMANN PROBLEM FOR AN ELLIPTIC SYSTEM OF EQUATIONS INVOLVING THE CRITICAL SOBOLEV EXPONENT

BY

J. CHABROWSKI (St. Lucia) and JIANFU YANG (Campinas)

Abstract. We consider the Neumann problem for an elliptic system of two equations involving the critical Sobolev nonlinearity. Our main objective is to study the effect of the coefficient of the critical Sobolev nonlinearity on the existence and nonexistence of least energy solutions. As a by-product we obtain a new weighted Sobolev inequality.

1. Introduction. The main purpose of this work is to study the existence of a solution to the following problem:

$$(1_\Lambda)\quad \begin{cases} -\Delta u + \lambda_1 u = \dfrac{\alpha}{2^\star} Q(x)|u|^{\alpha-2}u|v|^\beta, \\ -\Delta v + \lambda_2 v = \dfrac{\beta}{2^\star} Q(x)|u|^{\alpha}|v|^{\beta-2}v \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad u, v > 0 \quad \text{on } \Omega, \end{cases}$$

where $\lambda_1, \lambda_2 > 0$ are parameters, $\alpha, \beta > 1$ and $\alpha + \beta = 2^\star$, where $2^\star$ denotes the critical Sobolev exponent, that is, $2^\star = 2N/(N-2)$, $N \ge 3$. ν is the unit outward normal at the boundary $\partial\Omega$. We assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. The coefficient Q is Hölder continuous on $\overline{\Omega}$ and $Q(x) > 0$ for $x \in \overline{\Omega}$. Further conditions guaranteeing the solvability of problem (1_Λ) will be formulated later. Systems (1_Λ) appear in biological pattern formation theory (see [13], [15], [12]).

In this paper we establish the existence of least energy solutions. We also examine the concentration phenomena of these solutions when $\lambda_1 \to \infty$ and $\lambda_2 \to \infty$. We use a variational approach to problem (1_Λ) based on a version of P.-L. Lions' concentration-compactness principle [14] which is suitable for the Neumann problem. To study the concentration phenomena of the least energy solutions we adopt the technique from the paper [4].

2. Concentration-compactness principle. We commence by extending P.-L. Lions' concentration-compactness principle to $H^1(\Omega)\times H^1(\Omega)$,

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where by $H^1(\Omega)$ we denote the usual Sobolev space equipped with the norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2)\,dx.$$

Let

$$S_{\alpha,\beta} = \inf_{u,v\in H_0^1(\Omega),\, u,v\neq 0} \frac{\int_\Omega (|\nabla u|^2 + |\nabla v|^2)\,dx}{(\int_\Omega |u|^\alpha |v|^\beta\,dx)^{2/2^\star}}. \tag{1}$$

It is known that

$$S_{\alpha,\beta} = \left[(\alpha/\beta)^{\beta/2^\star} + (\alpha/\beta)^{-\alpha/2^\star}\right] S, \tag{2}$$

where S is the best Sobolev constant (see Theorem 5 in [7]). For the future use we set

$$A_{\alpha,\beta} = (\alpha/\beta)^{\beta/2^\star} + (\alpha/\beta)^{-\alpha/2^\star}.$$

We recall that the best Sobolev constant is defined by

$$S = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^2\,dx;\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{2^\star}\,dx = 1\Big\},$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the space obtained as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$

The best Sobolev constant is achieved by

$$U(x) = \left[\frac{N(N-2)}{N(N-2)+|x|^2}\right]^{(N-2)/2}.$$

The function U, called an *instanton*, satisfies the equation

$$-\Delta U = U^{2^\star-1} \quad \text{in } \mathbb{R}^N.$$

We also have

$$\int_{\mathbb{R}^N} |\nabla U|^2\,dx = \int_{\mathbb{R}^N} U^{2^\star}\,dx = S^{N/2}.$$

We set

$$U_{\varepsilon,y}(x) = \varepsilon^{-(N-2)/2} U\left(\frac{x-y}{\varepsilon}\right)$$

for $y \in \mathbb{R}^N$, $\varepsilon > 0$. If $y = 0$, we write $U_\varepsilon = U_{\varepsilon,0}$.

We denote strong convergence in $H^1(\Omega)$ by "$\to$" and weak convergence by "$\rightharpoonup$".

We need the following lemmas:

LEMMA 2.1. *Let $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $H^1(\Omega)$. Then*

$$\lim_{n\to\infty} \int_\Omega |u_n|^\alpha |v_n|^\beta\,dx = \lim_{n\to\infty} \int_\Omega |u_n - u|^\alpha |v_n - v|^\beta\,dx + \int_\Omega |u|^\alpha |v|^\beta\,dx.$$

Proof. By Sobolev's embedding theorem we may assume that $u_n \to u$ and $v_n \to v$ in $L^p(\Omega)$ for $1 \le p < 2^\star$. We write

$$\begin{aligned}
&\int_\Omega |u_n|^\alpha |v_n|^\beta\,dx - \int_\Omega |u_n - u|^\alpha |v_n - v|^\beta\,dx \\
&\quad= \int_\Omega (|u_n|^\alpha(|v_n|^\beta - |v_n - v|^\beta) + |v_n - v|^\beta(|u_n|^\alpha - |u_n - u|^\alpha))\,dx \\
&\quad= -\int_\Omega |u_n|^\alpha \int_0^1 \frac{d}{dt}|v_n - tv|^\beta\,dt\,dx - \int_\Omega |v_n - v|^\beta \int_0^1 \frac{d}{dt}|u_n - tu|^\alpha\,dt\,dx \\
&\quad= \beta \int_\Omega \int_0^1 |u_n|^\alpha |v_n - tv|^{\beta-2}(v_n - tv)v\,dt\,dx \\
&\qquad+ \alpha \int_\Omega \int_0^1 |v_n - v|^\beta |u_n - tu|^{\alpha-2}(u_n - tu)u\,dt\,dx.
\end{aligned}$$

Since

$$\beta \lim_{n\to\infty} \int_\Omega \int_0^1 |u_n|^\alpha |v_n - tv|^{\beta-2}(v_n - tv)v\,dt\,dx = \int_\Omega |u|^\alpha |v|^\beta\,dx$$

and

$$\lim_{n\to\infty} \int_\Omega \int_0^1 |v_n - v|^\beta |u_n - tu|^{\alpha-2}(u_n - tu)u\,dt\,dx = 0,$$

the result readily follows. ∎

PROPOSITION 2.2. *Let $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $H^1(\Omega)$. Suppose that*

(i) $|\nabla u_n|^2 + |\nabla v_n|^2 \rightharpoonup \mu$ *weakly in the sense of measures,*
(ii) $|u_n|^\alpha |v_n|^\beta \rightharpoonup \nu$ *weakly in the sense of measures.*

Then there exists an at most countable index set J and sequences $\{x_j\} \subset \mathbb{R}^N$, $\{\mu_j\}, \{\nu_j\} \subset (0,\infty)$ such that

$$\nu = |u|^\alpha |v|^\beta + \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \mu \ge |\nabla u|^2 + |\nabla v|^2 + \sum_{j\in J} \mu_j \delta_{x_j}$$

and

(iii) $S_{\alpha,\beta}\nu_j^{2/2^\star} \le \mu_j$ *if* $x_j \in \Omega$,
(iv) $(S_{\alpha,\beta}/2^{2/N})\nu_j^{2/2^\star} \le \mu_j$ *if* $x_j \in \partial\Omega$.

Proof. This is a modification of P.-L. Lions' [14] concentration-compactness principle. We only sketch the proof. First, we prove the result assuming that $u = v = 0$ on $\partial\Omega$. Then (iii) is a consequence of (1). To obtain (iv) we need the following modification of the result due to X. J. Wang [16]:

Let $\widetilde{B} = B(0,1) \cap \{x_N > h(x')\}$, where $B(0,1)$ is the unit ball in $\mathbb{R}^N$, $h(x')$ is a C^1-function defined on $\{x' \in \mathbb{R}^{N-1};\ |x'| < 1\}$ with h, Dh vanishing at 0. Then for every $u, v \in H^1(B(0,1))$ with $\operatorname{supp} u, \operatorname{supp} v \subset B$ we have:

(A) if $h \equiv 0$, then (see [7])

$$\int_{\widetilde{B}} (|\nabla u|^2 + |\nabla v|^2)\,dx \ge 2^{-2/N} S_{\alpha,\beta} \Big(\int_{\widetilde{B}} |u|^\alpha |v|^\beta\,dx \Big)^{2/2^\star},$$

(B) for every $\varepsilon > 0$ there exists a $\delta > 0$ depending only on ε such that if $|\nabla h| \le \delta$, then

$$\int_{\widetilde{B}} (|\nabla u|^2 + |\nabla v|^2)\,dx \ge \left(\frac{S_{\alpha,\beta}}{2^{2/N}} - \varepsilon \right) \Big(\int_{\widetilde{B}} |u|^\alpha |v|^\beta\,dx \Big)^{2/2^\star}.$$

Using this result we deduce (iv). The general case $u \not\equiv 0$ and $v \not\equiv 0$ can be reduced to the above case through the substitution $u_n^1 = u_n - u$, $v_n^1 = v_n - v$ and Lemma 2.1 (see [18]). ■

3. Existence results. We formulate the existence results for a slightly more general system

$$(1_A) \quad \begin{cases} -\Delta u + au + bv = \dfrac{\alpha}{2^\star} Q(x)|u|^{\alpha-2}u|v|^\beta, \\ -\Delta v + bu + cv = \dfrac{\beta}{2^\star} Q(x)|u|^\alpha |v|^{\beta-2} v \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We assume that the matrix of coefficients $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ is positive definite.

We write

$$Q_{\mathrm{m}} = \max_{x \in \partial\Omega} Q(x), \quad Q_{\mathrm{M}} = \max_{x \in \overline{\Omega}} Q(x).$$

For $u, v \in H^1(\Omega)$ we set

$$J_A(u,v) = \int_\Omega (|\nabla u|^2 + |\nabla v|^2 + (AV, V))\,dx,$$

where $U = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$.

Solutions to problem (1_A) will be obtained as minimizers of the constrained variational problem

$$\begin{aligned} S_A &= \inf\Big\{ J_A(u,v);\ (u,v) \in H^1(\Omega) \times H^1(\Omega),\ \int_\Omega Q(x)|u|^\alpha |v^\beta\,dx = 1 \Big\} \\ &= \inf\left\{ \frac{J_A(u,v)}{(\int_\Omega Q(x)|u|^\alpha |v|^\beta\,dx)^{2/2^\star}};\ (u,v) \in H^1(\Omega) \times H^1(\Omega),\ u, v \not\equiv 0 \right\}. \end{aligned}$$

A minimizer (u,v) for S_A satisfies the system

$$-\Delta u + au + bv = \frac{\alpha S_A}{2^\star} Q(x)|u|^{\alpha-2}u|v|^{\beta},$$
$$-\Delta v + bu + cv = \frac{\beta S_A}{2^\star} Q(x)|u|^{\alpha}|v|^{\beta-2}v.$$

Hence a rescaled minimizer $(u/S_A^{1/(2^\star-2)}, v/S_A^{1/(2^\star-2)})$ is a solution of the system (1_A).

THEOREM 3.1. *If*

$$Q_{\mathrm{M}} \le 2^{2/(N-2)} Q_{\mathrm{m}} \tag{3}$$

and

$$S_A < \frac{S_{\alpha,\beta}}{2^{2/N} Q_{\mathrm{m}}^{(N-2)/N}}, \tag{4}$$

then there exists a minimizer for S_A.

Proof. Let $\{u_m, v_m\}$ be a minimizing sequence for S_A. Since $(AU,U) \ge \mu_1(u^2+v^2)$ for some constant $\mu_1 > 0$, we may assume that $\{u_m, v_m\}$ is bounded in $H^1(\Omega)\times H^1(\Omega)$. Therefore, up to a subsequence we can assume that $u_m \rightharpoonup u$ and $v_m \rightharpoonup v$ in $H^1(\Omega)$. By Proposition 2.2 we have

$$1 = \int_\Omega Q(x)|u|^\alpha |v|^\beta + \sum_{j\in J} \nu_j Q(x_j) \tag{5}$$

and

$$\begin{aligned}
S_A &\ge \int_\Omega (|\nabla u|^2 + |\nabla v|^2 + (AU,U))\,dx + \sum_{j\in J}\mu_j \\
&\ge S_A\Big(\int_\Omega Q(x)|u|^\alpha|v|^\beta\,dx\Big)^{2/2^\star} + \sum_{x_j\in\partial\Omega}\mu_j + \sum_{x_j\in\Omega}\mu_j \\
&\ge S_A\Big(\int_\Omega Q(x)|u|^\alpha|v|^\beta\,dx\Big)^{2/2^\star} + \sum_{x_j\in\partial\Omega}\frac{S_{\alpha,\beta}}{2^{2/N}Q(x_j)^{(N-2)/N}}(Q(x_j)\nu_j)^{(N-2)/N} \\
&\quad + \sum_{x_j\in\Omega}\frac{S_{\alpha,\beta}}{Q(x_j)^{(N-2)/N}}(Q(x_j)\nu_j)^{(N-2)/N} \\
&\ge S_A\Big(\int_\Omega Q(x)|u|^\alpha|v|^\beta\,dx\Big)^{2/2^\star} + \sum_{x_j\in\partial\Omega}\frac{S_{\alpha,\beta}}{2^{2/N}Q_{\mathrm{m}}^{(N-2)/N}}(Q(x_j)\nu_j)^{(N-2)/N} \\
&\quad + \sum_{x_j\in\Omega}\frac{S_{\alpha,\beta}}{Q_{\mathrm{M}}^{(N-2)/N}}(Q(x_j)\nu_j)^{(N-2)/N} \\
&\ge S_A\Big(\int_\Omega Q(x)|u|^\alpha|v|^\beta\,dx\Big)^{2/2^\star} + \sum_{j\in J}\frac{S_{\alpha,\beta}}{2^{2/N}Q_{\mathrm{m}}^{(N-2)/N}}(Q(x_j)\nu_j)^{(N-2)/N}.
\end{aligned}$$

This combined with (4) implies that $\nu_j = 0$ for all $j \in J$. Hence

$$\int_\Omega Q(x)|u|^\alpha |v|^\beta \, dx = 1$$

and by the lower semicontinuity of J_A with respect to the weak convergence we have

$$\int_\Omega (|\nabla u|^2 + |\nabla v|^2 + (AU, U)) \, dx \le S_A.$$

This means that (u, v) is a minimizer for S_A. Since (u_n, v_n) can be replaced by $(|u_n|, |v_n|)$ we may assume that $u, v \ge 0$ on Ω. By the strong maximum principle we have $u, v > 0$ on $\overline{\Omega}$. ■

In a similar manner we can prove

THEOREM 3.2. *Let*

$$Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}. \tag{6}$$

If

$$S_A < \frac{S_{\alpha,\beta}}{Q_{\mathrm{M}}^{(N-2)/N}}, \tag{7}$$

then there exists a minimizer for S_A.

We now formulate conditions guaranteeing that (4) holds. We need an additional assumption:

(H) there exists a point $y \in \partial\Omega$ such that $Q_{\mathrm{m}} = Q(y)$ and $H(y) > 0$ and moreover

$$|Q(x) - Q(y)| = o(|x - y|) \quad \text{for } x \text{ near } y.$$

Here $H(y)$ denotes the mean curvature of $\partial\Omega$ at $y \in \partial\Omega$ with respect to the inner normal to $\partial\Omega$ at y. It is also known that (see [1], [2], [17])

$$\frac{\int_\Omega |\nabla U_{\varepsilon,y}|^2 \, dx}{(\int_\Omega U_{\varepsilon,y}^{2^\star} \, dx)^{2/2^\star}} = 2^{-2/N} S - \begin{cases} A_N H(y) \varepsilon \log(1/\varepsilon) + O(\varepsilon), & N = 3, \\ A_N H(y) \varepsilon + O(\varepsilon^2 \log(1/\varepsilon)), & N = 4, \\ A_N H(y) \varepsilon + O(\varepsilon^2), & N \ge 5, \end{cases}$$

where $A_N > 0$ is a constant depending on N. Let $s, t > 0$. Then

$$J_A\bigg(\frac{sU_{\varepsilon,y}, tU_{\varepsilon,y}}{(\int_\Omega s^\alpha t^\beta Q U_{\varepsilon,y}^{2^\star} \, dx)^{1/2^\star}}\bigg) = \frac{s^2 + t^2}{(s^\alpha t^\beta)^{2/2^\star}} \cdot \frac{\int_\Omega |\nabla U_{\varepsilon,y}|^2 \, dx}{(\int_\Omega Q U_{\varepsilon,y}^{2^\star} \, dx)^{2/2^\star}} + \varrho(\varepsilon),$$

where

$$\varrho(\varepsilon) = \begin{cases} O(\varepsilon), & N = 3, \\ \varepsilon^2 \log(1/\varepsilon), & N = 4, \\ \varepsilon^2, & N \ge 5. \end{cases}$$

We now set $s/t = \sqrt{\beta/\alpha}$. Using (H) we see that

$$J_A\left(\frac{sU_{\varepsilon,y}, tU_{\varepsilon,y}}{(s^\alpha t^\beta \int_\Omega QU_{\varepsilon,y}^{2^\star})^{1/2^\star}}\right) < \frac{SA_{\alpha,\beta}}{2^{2/N} Q_{\mathrm{m}}^{N-2/N}}$$

for sufficiently small $\varepsilon > 0$. The condition (7), under assumption (6), certainly holds for any positive definite matrix A with sufficiently small coefficients a, b and c.

4. System (1_Λ)**.** First we rescale a solution (u,v) of (1_Λ) in the following way. We set $u_1 = su$, $v_1 = tv$ to get

$$-\Delta u_1 + \lambda_1 u_1 = \frac{\alpha}{2^\star} s^{-(\alpha-2)} t^{-\beta} Q u_1^{\alpha-1} v_1^\beta,$$
$$-\Delta v_1 + \lambda_2 v_1 = \frac{\beta}{2^\star} s^{-\alpha} t^{-(\beta-2)} Q u_1^\alpha v_1^{\beta-1}.$$

Choosing s and t so that

$$(8) \qquad \frac{\alpha}{2^\star} s^{-(\alpha-2)} t^{-\beta} = 1 \quad \text{and} \quad \frac{\beta}{2^\star} s^{-\alpha} t^{-(\beta-2)} = 1,$$

that is, $\beta/\alpha = s^2/t^2$, we see that (1_Λ) is reduced to the system

$$(1_{\Lambda^*}) \quad \begin{cases} -\Delta u_1 + \lambda_1 u_1 = Q u_1^{\alpha-1} v_1^\beta, \\ -\Delta v_1 + \lambda_2 v_1 = Q u_1^\alpha v_1^{\beta-1} \quad \text{in} \Omega, \\ \partial u_1/\partial\nu = \partial v_1/\partial\nu = 0 \quad \text{on } \partial\Omega. \end{cases}$$

For the future use we note the formula

$$(9) \qquad \frac{t^2+s^2}{t^2 s^2} = A_{\alpha,\beta}^{N/2}.$$

Indeed, solving the equations (8) we get

$$s = \left[\frac{2^\star}{\alpha}\left(\frac{\beta}{\alpha}\right)^{\alpha/2-1}\right]^{1/(2-(\alpha+\beta))} \left(\frac{\beta}{\alpha}\right)^{1/2}, \qquad t = \left[\frac{2^\star}{\alpha}\left(\frac{\beta}{\alpha}\right)^{\alpha/2-1}\right]^{1/(2-(\alpha+\beta))}.$$

Then

$$\frac{t^2+s^2}{t^2s^2} = \frac{1+\frac{\beta}{\alpha}}{\left[\frac{2^\star}{\alpha}\left(\frac{\beta}{\alpha}\right)^{\alpha/2-1}\right]^{2/(2-(\alpha+\beta))}\frac{\beta}{\alpha}} = \frac{\alpha+\beta}{\beta}\left[\left(\frac{\beta}{\alpha}\right)^{(\alpha-2)/2} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2}\right]^{N/2-1}$$
$$= \frac{\alpha+\beta}{\beta}\left[\left(\frac{\beta}{\alpha}\right)^{\alpha/2}\left(\frac{\alpha}{\beta}+1\right)\right]^{N/2-1} = \left(\frac{\beta}{\alpha}\right)^{\alpha(N/2-1)/2}\left(\frac{\alpha+\beta}{\beta}\right)^{N/2}.$$

By easy computations we get

$$A_{\alpha,\beta}^{N/2} = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\alpha}{\beta}\right)^{-\alpha/(\alpha+\beta)}\right]^{N/2} = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)}\left(1+\frac{\beta}{\alpha}\right)\right]^{N/2}$$

$$= \left(\frac{\beta}{\alpha}\right)^{N\alpha/(2(\alpha+\beta))}\left(\frac{\alpha+\beta}{\beta}\right)^{N/2} = \left(\frac{\beta}{\alpha}\right)^{\alpha(N/2-1)/2}\left(\frac{\alpha+\beta}{\beta}\right)^{N/2}$$

and the formula (9) follows.

PROPOSITION 4.1. (i) *Suppose that* $\lambda_1 = \lambda_2$. *If* (u, v) *is a positive solution of* (1_Λ), *then* $u = s^{-1}w$ *and* $v = t^{-1}w$, *where* s, t *are positive constants satisfying* (8) *and* w *is a positive solution of the problem*

$$\begin{cases} -\Delta w + \lambda_1 w = Q(x) w^{2^\star - 1} & \text{in } \Omega, \\ \partial w/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

(ii) *Suppose that* $\lambda_1 > \lambda_2$. *If* (u, v) *is a positive solution of* (1_Λ), *then*

$$su \le tv \quad \text{on } \Omega$$

where $s, t > 0$ *are constants satisfying* (8).

Proof. (i) The rescaled functions $u_1 = su$ and $v_1 = tv$ satisfy (1_{Λ^*}). From this we deduce that

$$\begin{cases} -\Delta(u_1 - v_1) + [\lambda_1 + Q u_1^{\alpha-1} v_1^{\beta-1}](u_1 - v_1) = 0 & \text{in } \Omega, \\ \partial(u_1 - v_1)/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle we have $u_1 = v_1$ on Ω. Setting $u_1 = v_1 = w$, we see that w satisfies (10) and the result follows.

(ii) The difference $u_1 - v_1$ satisfies

$$-\Delta(u_1 - v_1) + \lambda_1(u_1 - v_1) + (\lambda_1 - \lambda_2)v_1 + Q u_1^{\alpha-1} v_1^{\beta-1}(u_1 - v_1) = 0 \quad \text{in } \Omega.$$

Hence

$$-\Delta(u_1 - v_1) + [\lambda_1 + Q u_1^{\alpha-1} v_1^{\beta-1}](u_1 - v_1) \le 0 \quad \text{in } \Omega,$$
$$\frac{\partial(u_1 - v_1)}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

By the maximum principle we get $u_1 \le v_1$ on Ω. ∎

In what follows, we study the behaviour of the least energy solutions of (1_Λ) as $\lambda_1, \lambda_2 \to \infty$. According to the previous section, these solutions are minimizers of the problem

$$S_\Lambda = \inf\Big\{ \int_\Omega (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2)\, dx; \\ u, v \in H^1(\Omega),\ \int_\Omega Q|u|^\alpha |v|^\beta\, dx = 1\Big\}.$$

THEOREM 4.2. *Let* $Q_{\mathrm{M}} < 2^{2/(N-2)} Q_{\mathrm{m}}$. *Suppose that* $\lambda_1 = \lambda_2 + m$, $m > 0$. *Let* $u_\lambda^1 = s u_\lambda$ *and* $v_\lambda^1 = t v_\lambda$ *be a rescaled least energy solution for* (1_Λ), *where* s *and* t *satisfy the equations* (8). *If* $M_\lambda = \max_{x\in\overline{\Omega}} v_\lambda^1(x) =$

$v^1_\lambda(x_\lambda)$, $x_\lambda \in \overline{\Omega}$, *then* $u^1_\lambda \le M_\lambda$, $M_\lambda \to \infty$ *and* $x_\lambda \to x_\circ \in \partial\Omega$ *with* $Q(x_\circ) = Q_{\mathrm{m}}$ *as* $\lambda_2 \to \infty$. *Moreover,*

$$\lim_{\lambda_2\to\infty} \int_\Omega |\nabla(su_\lambda - U_{\varepsilon_\lambda,x_\lambda}(Bx))|^2\,dx = 0,$$

$$\lim_{\lambda_2\to\infty} \int_\Omega |\nabla(tv_\lambda - U_{\varepsilon_\lambda,x_\lambda}(Bx))|^2\,dx = 0,$$

where $B = Q_{\mathrm{M}}^{1/N} A_{\alpha,\beta}^{1/2} S^{1/2}$.

Proof. We commence by showing that $M_\lambda \to \infty$ as $\lambda_2 \to \infty$. Indeed, we have

$$\int_\Omega (|\nabla v^1_\lambda|^2 + \lambda_2 (v^1_\lambda)^2)\,dx = \int_\Omega Q(u^1_\lambda)^\alpha (v^1_\lambda)^\beta\,dx$$

and by Proposition 4.1(ii),

$$\int_\Omega Q(v^1_\lambda)^2((v^1_\lambda)^{2^\star-2} - \lambda_2)\,dx \ge 0.$$

Hence $Q(v^1_\lambda)^{2^\star-2} \ge \lambda_2$ somewhere in Ω. This means that $M_\lambda^{4/(N-2)} \ge \lambda_2/Q_{\mathrm{M}}$ and our claim follows. We now define

$$\widetilde{u}^1_\lambda(x) = \varepsilon_\lambda^{(N-2)/2} u^1_\lambda(\varepsilon_\lambda x + x_\lambda) \quad\text{and}\quad \widetilde{v}^1_\lambda(x) = \varepsilon_\lambda^{(N-2)/2} v^1_\lambda(\varepsilon_\lambda x + x_\lambda)$$

for $x \in \Omega_{\varepsilon_\lambda} = (\Omega - x_\lambda)/\varepsilon_\lambda$, where $\varepsilon_\lambda = 1/M_\lambda^{2/(N-2)}$. The functions $\widetilde{u}^1_\lambda$ and $\widetilde{v}^1_\lambda$ are solutions of the problem

$$\begin{cases} -\Delta\widetilde{u}^1_\lambda + \lambda_1\varepsilon_\lambda^2\widetilde{u}^1_\lambda = S_\Lambda Q(\varepsilon_\lambda x + x_\lambda)(\widetilde{u}^1_\lambda)^{\alpha-1}(\widetilde{v}^1_\lambda)^\beta, \\ -\Delta\widetilde{v}^1_\lambda + \lambda_2\varepsilon_\lambda^2\widetilde{v}^1_\lambda = S_\Lambda Q(\varepsilon_\lambda x + x_\lambda)(\widetilde{u}^1_\lambda)^{\alpha}(\widetilde{v}^1_\lambda)^{\beta-1} \quad \text{in } \Omega_{\varepsilon_\lambda}, \\ \partial\widetilde{v}^1_\lambda/\partial\nu = \partial\widetilde{u}^1_\lambda/\partial\nu = 0 \quad \text{on } \partial\Omega_{\varepsilon_\lambda}. \end{cases} \tag{11}$$

Since $M_\lambda^{4/(N-2)} \ge \lambda_2/Q_{\mathrm{M}}$ and $\lambda_1 = \lambda_2 + m$, we see that $\varepsilon_\lambda^2\lambda_1$ and $\varepsilon_\lambda^2\lambda_2$ are bounded as $\lambda_2 \to \infty$. The elliptic regularity theory implies that $\widetilde{u}^1_\lambda \to \widetilde{u}$ and $\widetilde{v}^1_\lambda \to \widetilde{v}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$. We may also assume that $x_\lambda \to x_\circ$, $\varepsilon_\lambda^2\lambda_1 \to a_1$ and $\varepsilon_\lambda^2\lambda_2 \to a_2$ as $\lambda_2 \to \infty$, where $0 \le a_1, a_2 < \infty$. We also observe that

$$\lim_{\lambda_2\to\infty} S_\Lambda = \frac{A_{\alpha,\beta}S}{2^{2/N}Q_{\mathrm{m}}^{(N-2)/N}} =: S_{\Lambda_\infty}.$$

This can be easily established using the concentration-compactness principle. Hence $(\widetilde{u}, \widetilde{v})$ is a solution of the problem

$$\begin{cases} -\Delta\widetilde{u} + a_1\widetilde{u} = S_{\Lambda_\infty} Q(x_\circ)\widetilde{u}^{\alpha-1}\widetilde{v}^\beta, \\ -\Delta\widetilde{v} + a_2\widetilde{v} = S_{\Lambda_\infty} Q(x_\circ)\widetilde{u}^{\alpha}\widetilde{v}^{\beta-1} \quad \text{in } \Omega_\infty, \\ \partial\widetilde{u}/\partial\nu = \partial\widetilde{v}/\partial\nu = 0 \quad \text{on } \partial\Omega_\infty, \end{cases} \tag{12}$$

where $\Omega_{\varepsilon_\lambda} \to \Omega_\infty$ as $\lambda_2 \to \infty$. Since $\widetilde{v}^1_\lambda(0) = 1$ we see that $\widetilde{v} \not\equiv 0$. We now show that $\widetilde{u} \not\equiv 0$. (As we will see later, Ω_∞ is either $\mathbb{R}^N$ or a half-space in $\mathbb{R}^N_+$.) By the Fatou lemma

$$\int_{\Omega_\infty} (|\nabla \widetilde{u}|^2 + m\widetilde{u}^2 + |\nabla \widetilde{v}|^2 + m\widetilde{v}^2)\,dx \le \lim_{\lambda_2\to\infty} S_\Lambda < \infty,$$

therefore $\widetilde{u}, \widetilde{v} \in H^1(\Omega_\infty)$. If $\widetilde{u} \equiv 0$ on Ω_∞, then

$$-\Delta\widetilde{v} + a_2\widetilde{v} = 0 \quad \text{in } \Omega_\infty, \qquad \frac{\partial\widetilde{v}}{\partial n} = 0 \quad \text{on } \partial\Omega_\infty.$$

Hence $\widetilde{v} \equiv 0$ on Ω_∞ in both cases $\Omega_\infty = \mathbb{R}^N$ and $\Omega_\infty = \mathbb{R}^N_+$, which is a contradiction. By Pokhozhaev's identity (see Appendix) $a_1 = a_2 = 0$. Therefore the system (12) is reduced to

$$\begin{cases} -\Delta\widetilde{u} = S_{\Lambda_\infty} Q(x_\circ)\widetilde{u}^{\alpha-1}\widetilde{v}^{\beta}, \\ -\Delta\widetilde{v} = S_{\lambda_\infty} Q(x_\circ)\widetilde{u}^{\alpha}\widetilde{v}^{\beta-1} \quad \text{in } \Omega_\infty, \\ \partial\widetilde{u}/\partial\nu = \partial\widetilde{v}/\partial\nu = 0 \quad \text{on } \partial\Omega_\infty. \end{cases} \tag{13}$$

By Proposition 4.1(i), we see that $\widetilde{u} = \widetilde{v}$ on Ω_∞. We now distinguish two cases:

(a) $\operatorname{dist}(x_\lambda, \partial\Omega)/\varepsilon_\lambda \to \infty$ or
(b) $\operatorname{dist}(x_\lambda, \partial\Omega)/\varepsilon_\lambda$ is bounded as $\lambda_2 \to \infty$.

In the first case we have

$$\widetilde{u} = \widetilde{v} = U(S^{1/2}_{\Lambda_\infty} Q(x_\circ)^{1/2} x) \quad \text{and} \quad \Omega_\infty = \mathbb{R}^N.$$

Let $b = S^{1/2}_{\Lambda_\infty} Q(x_\circ)^{1/2}$. Then by Fatou's lemma

$$\begin{aligned} \frac{s^2+t^2}{s^2t^2} b^{2-N} \int_{\mathbb{R}^N} |\nabla U|^2\,dx &= \frac{s^2+t^2}{s^2t^2} b^{2-N} S^{N/2} \\ &\le \lim_{\lambda_2\to\infty} \int_\Omega (|\nabla u_\lambda|^2 + |\nabla v_\lambda|^2)\,dx \le \frac{S_{\alpha,\beta}}{2^{2/N} Q_{\mathrm{m}}^{(N-2)/2}}. \end{aligned} \tag{14}$$

Here we have used the fact that $\lim_{\lambda_2\to\infty} \lambda_1 \int_\Omega u_\lambda^2\,dx = \lim_{\lambda_2\to\infty} \lambda_2 \int_\Omega v_\lambda^2\,dx = 0$. From the above inequality and (9) we deduce that

$$A^{N/2}_{\alpha,\beta} Q(x_\circ)^{1-N/2} \frac{A^{1-N/2}_{\alpha,\beta} S^{1-N/2} S^{N/2}}{2^{(2/N)(1-N/2)} Q_{\mathrm{m}}^{((N-2)/N)(1-N/2)}} \le A_{\alpha\beta} \frac{S}{2^{2/N} Q_{\mathrm{m}}^{(N-2)/N}},$$

which is equivalent to $2Q_{\mathrm{M}}^{(N-2)/2} \le Q(x_\circ)^{(N-2)/2}$. Hence $2^{2/(N-2)} Q_{\mathrm{m}} \le Q_{\mathrm{M}}$, which is impossible. Therefore case (b) prevails and $x_\circ \in \partial\Omega$. In this case we may assume that $\Omega_\infty = \mathbb{R}^N_+$ and estimate (14) takes the form

$$\frac{s^2+t^2}{s^2t^2}b^{2-N}\int_{\mathbb{R}^N_+}|\nabla U|^2\,dx = \frac{s^2+t^2}{s^2t^2}b^{2-N}\frac{S^{N/2}}{2}$$
$$\le \lim_{\lambda_2\to\infty}\int_\Omega(|\nabla u_\lambda|^2+|\nabla v_\lambda|^2)\,dx = \frac{S_{\alpha,\beta}}{2^{2/N}Q_{\mathrm{m}}^{(N-2)/N}}.$$

From this we deduce that $Q(x_\circ)\ge Q_{\mathrm{m}}$ and hence $Q(x_\circ)=Q_{\mathrm{m}}$. Therefore, the above inequality becomes, in fact, equality. On the other hand, by the Fatou lemma and the fact that $\widetilde{u}^1_\lambda\to U(bx)$ and $\widetilde{v}^1_\lambda\to U(bx)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$, we get

$$b^{2-N}\int_{\mathbb{R}^N_+}|\nabla U|^2\,dx \le \lim_{\lambda_2\to\infty}\int_\Omega|\nabla u^1_\lambda|^2\,dx = \lim_{\lambda_2\to\infty}\int_{\Omega_{\varepsilon_\lambda}}|\nabla\widetilde{u}^1_\lambda|^2\,dx$$

and

$$b^{2-N}\int_{\mathbb{R}^N_+}|\nabla U|^2\,dx \le \lim_{\lambda_2\to\infty}\int_\Omega|\nabla v^1_\lambda|\,dx = \lim_{\lambda_2\to\infty}\int_{\Omega_{\varepsilon_\lambda}}|\nabla\widetilde{v}^1_\lambda|^2\,dx.$$

From this we deduce that

$$\lim_{\lambda_2\to\infty}\int_{\Omega_{\varepsilon_\lambda}}|\nabla\widetilde{u}^1_\lambda|^2\,dx = \lim_{\lambda_2\to\infty}\int_{\Omega_{\varepsilon_\lambda}}|\nabla\widetilde{v}^1_\lambda|^2\,dx = b^{2-N}\int_{\mathbb{R}^N_+}|\nabla U|^2\,dx$$

and the result readily follows. ■

5. The case $Q_{\mathrm{M}} > S_{\alpha,\beta}/Q_{\mathrm{m}}^{(N-2)/N}$. As in the previous section the parameters λ_1,λ_2 satisfy $\lambda_1=\lambda_2+m$, where $m>0$ is fixed. In Theorem 5.3 below we show that least energy solutions exist for $\lambda_2\in(0,\overline{\lambda})$ for some $\overline{\lambda}>0$ and there are no solutions for $\lambda_2>\overline{\lambda}$. This means that

$$S_\lambda = \frac{SA_{\alpha,\beta}}{Q_{\mathrm{M}}^{(N-2)/N}} \quad \text{for } \lambda_2>\overline{\lambda}.$$

We need the following lemmas:

LEMMA 5.1 ([17], Lemma 4.7). *Assume $N\ge 5$. Let $\lambda_n>0$ and $\lambda_n\to\infty$, $\sigma_n\to\infty$, $\sigma_n>0$, $P_n\in\Omega$, $P_n\to P_\circ$ with $P_\circ\in\Omega$ and $v_n\in H^1(\Omega)$, $v_n\ge 0$, $v_n\rightharpoonup v$ in $H^1(\Omega)$ be such that*

$$\lim_{n\to\infty}\left\|\nabla v_n-\nabla\left(\frac{U_{\sigma_n,P_n}}{\|U_{\sigma_n,P_n}\|_{L^{2^\star}(\Omega)}}\right)\right\|_{L^2(\Omega)}=0.$$

If $\int_\Omega(|\nabla v_n|^2+\lambda_n v_n^2)\,dx<S$ for large n, then there exist sequences $\{\delta_n\}$, $\delta_n>0$, and $\{y_n\}\subset\Omega$ such that, modulo a subsequence, $\delta_n/\sigma_n\to 1$, $y_n\to P_\circ$ and

$$\int_\Omega(|\nabla v_n|^2+\lambda_n v_n^2)\,dx \ge \frac{\int_\Omega(|\nabla U_{\delta_n,y_n}|^2+\lambda_n U^2_{\delta_n,y_n})\,dx}{\|U_{\delta_n,y_n}\|^2_{L^{2^\star}(\Omega)}}+O(\delta_n^2)+o(\lambda_n\delta_n^2)$$

and $o(\lambda_n\delta_n^2)=O(\delta_n^2)$.

LEMMA 5.2 (see [9]). *Suppose that $N \geq 5$. Let y be an interior point of Ω. Then there exists a constant $b_N > 0$, depending only on N, such that*

$$\frac{\int_\Omega (|\nabla U_{\delta,y}|^2 + \lambda U_{\delta,y}^2)\,dx}{(\int_\Omega U_{\delta,y}^{2^\star}\,dx)^{2/2^\star}} = S + b_n\lambda\delta^2 + O(\delta^2) + o(\lambda\delta^2),$$

$O(\cdot)$ and $o(\cdot)$ are uniform in λ and y as $\delta \to 0$ for $\lambda > 1$ and for y in compact subsets of Ω.

THEOREM 5.3. *Suppose that $N \geq 5$. Let $Q_{\mathrm{M}} > S_{\alpha,\beta}/Q_{\mathrm{m}}^{(N-2)/N}$ and $\lambda_1 = \lambda_2 + m$. Then there exists $\overline{\lambda} > 0$ such that least energy solutions exist only for $\lambda_2 \in (0, \overline{\lambda})$.*

Proof. We argue indirectly. Assume that there exists a least energy solution (u_λ, v_λ) for (1_Λ) for each $\lambda_2 > 0$. Set $u_\lambda^1 = su_\lambda$, $v_\lambda^1 = tv_\lambda$ with $s, t > 0$ satisfying (8) and $M_\lambda = \sup_{x\in\overline{\Omega}} v_\lambda(x) = v_\lambda(x_\lambda)$. As in the proof of Theorem 4.2 we show that $M_\lambda \to \infty$ as $\lambda_2 \to \infty$ and also $u_\lambda^1 \leq v_\lambda^1$ on Ω. We now apply the blow-up technique to the rescaled solutions

$$(\widetilde{u}_\lambda^1(x), \widetilde{v}_\lambda^1(x)) = (\varepsilon_\lambda^{(N-2)/2} u_\lambda^1(\varepsilon_\lambda x + x_\lambda), \varepsilon^{(N-2)/2} v_\lambda^1(\varepsilon_\lambda x + x_\lambda)).$$

Obviously we have $\widetilde{u}_\lambda^1 \to \widetilde{u}$ and $\widetilde{v}_\lambda^1 \to \widetilde{v}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\widetilde{u}$ and $\widetilde{v}$ satisfy the system (13) with S_{Λ_∞} replaced by $A_{\alpha,\beta}S/Q_{\mathrm{M}}^{(N-2)/N}$. We now consider the cases (a) and (b) from the proof of Theorem 4.2. Due to the assumption $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$, the case (b) does not hold. Therefore (a) occurs. In this case we have

$$\widetilde{u} = \widetilde{v} = U(S_{\widetilde{\Lambda}_\infty}^{1/2} Q(x_\circ)^{1/2} x),$$

where $S_{\widetilde{\Lambda}_\infty} = A_{\alpha,\beta}S/Q_{\mathrm{M}}^{(N-2)/N}$. By the Fatou lemma we have

$$\frac{t^2+s^2}{s^2t^2} Q(x_\circ)^{1-N/2} S_{\widetilde{\Lambda}_\infty}^{1-N/2} S^{N/2} \leq \frac{A_{\alpha,\beta}S}{Q_{\mathrm{M}}^{(N-2)/N}},$$

from which we deduce, using formula (9), that $Q(x_\circ) \geq Q_{\mathrm{M}}$ and necessarily $Q(x_\circ) = Q_{\mathrm{M}}$. We now observe that by the above argument we also have

$$\lim_{\lambda_2\to\infty} \lambda_1 \int_\Omega u_\lambda^2\,dx = \lim_{\lambda_2\to\infty} \lambda_2 \int_\Omega v_\lambda^2\,dx = 0.$$

Therefore

$$\lim_{\lambda_2\to\infty} \int_\Omega (|\nabla u_\lambda|^2 + |\nabla v_\lambda|^2)\,dx = \frac{A_{\alpha,\beta}S}{Q_{\mathrm{M}}^{(N-2)/N}}. \tag{15}$$

As in the proof of Theorem 4.2 we show that

$$\int_{\mathbb{R}^N} |\nabla U(Bx)|^2\,dx = \lim_{\lambda_2\to\infty} \int_\Omega |\nabla u_\lambda^1|^2\,dx = \lim_{\lambda_2\to\infty} \int_{\Omega_{\varepsilon_\lambda}} |\nabla \widetilde{u}_\lambda^1|^2\,dx, \tag{16}$$

$$\int_{\mathbb{R}^N} |\nabla U(Bx)|^2\,dx = \lim_{\lambda_2\to\infty}\int_\Omega |\nabla v_\lambda^1|^2\,dx = \lim_{\lambda_2\to\infty}\int_{\Omega_{\varepsilon_\lambda}} |\nabla \widetilde{v}_\lambda^1|^2\,dx, \tag{17}$$

where $B = S_{\widetilde{\Lambda}_\infty}^{1/2} Q_{\mathrm{M}}^{1/2}$. Since

$$\frac{s^2+t^2}{s^2t^2} B^{2-N} S^{N/2} = \frac{A_{\alpha,\beta}S}{Q_{\mathrm{M}}^{(N-2)/N}},$$

we may assume that along a subsequence $\lambda_2^n \to \infty$ we have

$$\int_{\Omega_{\varepsilon_{\lambda_n}}} |\nabla \widetilde{u}_{\lambda_n}^1|^2\,dx < \frac{A_{\alpha,\beta}S}{Q_{\mathrm{M}}^{(N-2)/N}}.$$

Then by (16),

$$\lim_{\lambda_2^n\to\infty}\int_\Omega |\nabla u_{\lambda_n}^1|^2\,dx = \lim_{\lambda_2^n\to\infty}\int_{\Omega_{\varepsilon_{\lambda_n}}} |\nabla \widetilde{u}_{\lambda_n}^1|^2\,dx = \frac{A_{\alpha,\beta}S}{Q_{\mathrm{M}}^{(N-2)/N}}. \tag{18}$$

We now set

$$w_{\lambda_n} = \frac{Q_{\mathrm{M}}^{(N-2)/(2N)}}{A_{\alpha,\beta}^{1/2}} \widetilde{u}_{\lambda_n}^1.$$

Then

$$\int_\Omega |\nabla w_{\lambda_n}|^2\,dx = \frac{s^2 Q_{\mathrm{M}}^{(N-2)/N}}{A_{\alpha,\beta}} \int_\Omega |\nabla u_{\lambda_n}|^2\,dx < S$$

and

$$\left\|\nabla w_{\lambda_n} - \overline{\varepsilon}^{-(N-2)/4} S^{-(N-2)/4} \nabla U\left(\frac{x-x_{\lambda_n}}{\overline{\varepsilon}_{\lambda_n}}\right)\right\|_{L^2(\Omega)} \to 0$$

as $n\to\infty$, where

$$\overline{\varepsilon}_{\lambda_n} = \frac{\varepsilon_{\lambda_n}}{S^{1/2}Q_{\mathrm{M}}^{1/2}A_{\alpha,\beta}^{1/2}}.$$

Using Lemma 5.1 with $P_n = x_{\lambda_n}$, we get sequences $\{y_n\}$ such that, modulo a subsequence, $\varepsilon_{\lambda_n}/\sigma_n \to 1$, $y_n \to x_\circ$ and moreover

$$\int_\Omega (|\nabla w_n|^2 + \lambda_1^n w_n^2)\,dx \ge \frac{\int_\Omega |\nabla U_{\sigma_n,y_n}|^2\,dx}{(\int_\Omega U_{\sigma_n,y_n}^{2^\star}\,dx)^{1/2^\star}} + O(\sigma_n^2) + o(\lambda_n\sigma_n^2).$$

Lemma 5.2 implies that $\int_\Omega (|\nabla w_n|^2 + \lambda_n w_n^2)\,dx > S$ for large n. This contradiction completes the proof. ∎

We now define $\overline{\lambda} = \inf\{\lambda_2;\ (1_\Lambda)$ has no least energy solution$\}$. It is clear that $S_\Lambda = A_{\alpha,\beta}S/Q_{\mathrm{M}}^{(N-2)/N}$ for $\lambda_2 \ge \overline{\lambda}$.

Remark 5.4. Theorem 4.2 remains true for $Q_{\mathrm{M}} = 2^{2/(N-2)}Q_{\mathrm{m}}$.

Indeed, in this case the concentration can only occur on the boundary or at an interior point of Ω. The concentration at an interior point can be excluded as in the proof of Theorem 4.2. Therefore, (u_λ, v_λ) can only concentrate at a boundary point and the assertion of Theorem 4.2 remains true in this case.

THEOREM 5.5. *Let $N \geq 5$ and suppose that $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$. Then there exists a least energy solution of (1_Λ) for $\lambda_2 = \overline{\lambda}$.*

Proof. Let $\lambda_2^n \in (0, \overline{\lambda})$ and $\lambda_2^n \to \overline{\lambda}$. By Theorem 5.3 for each λ_2^n there exists a least energy solution $(u_{\lambda_2^n}, v_{\lambda_2^n})$. Let $M_{\lambda_2^n} = \max_{x\in\overline{\Omega}} v_{\lambda_2^n}(x)$. We show that $\{M_{\lambda_2^n}\}$ is a bounded sequence. In the contrary case $M_{\lambda_2^n} \to \infty$. Then we can repeat the final part of the proof of Theorem 5.3, which gives a contradiction. Since $u_{\lambda_2^n} \leq v_{\lambda_2^n}$ on Ω we see that both sequences $u_{\lambda_2^n}$ and $v_{\lambda_2^n}$ are bounded. By the Sobolev embedding theorem we may assume that $u_{\lambda_2^n} \rightharpoonup u_{\bar{\lambda}}$ and $v_{\lambda_n^2} \rightharpoonup v_{\bar{\lambda}}$ in $H^1(\Omega)$ and also $u_{\lambda_2^n} \to u_{\overline{\lambda}}$ and $v_{\lambda_2^n} \to v_{\bar{\lambda}}$ a.e. on Ω. It then follows from the Lebesgue dominated convergence theorem that

$$1 = \lim_{n\to\infty} \int_\Omega Q(x) u_{\lambda_2^n}^\alpha v_{\lambda_2^n}^\beta \, dx = \int_\Omega Q(x) u_{\bar{\lambda}}^\alpha v_{\bar{\lambda}}^\beta \, dx$$

and on the other hand, by the lower semicontinuity of the norm with respect to the weak convergence, we get

$$\int_\Omega (|\nabla u_{\bar{\lambda}}|^2 + |\nabla v_{\bar{\lambda}}|^2 + (\overline{\lambda} + m)u_{\bar{\lambda}}^2 + \overline{\lambda} v_{\bar{\lambda}}^2)\, dx \leq \frac{A_{\alpha,\beta} S}{Q_{\mathrm{M}}^{(N-2)/N}}$$

and the result follows. ∎

6. Remark on a weighted Sobolev inequality. As a by-product of Theorem 5.5 we obtain the following inequality:

THEOREM 6.1. *Let $N \geq 5$ and suppose that $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$. Then there exists a constant $K = K(\Omega)$ such that*

$$\Big(\int_\Omega Q(x)|u|^\alpha |v|^\beta \, dx\Big)^{2/2^\star} \leq \frac{Q_{\mathrm{M}}^{(N-2)/N}}{A_{\alpha,\beta} S} \int_\Omega (|\nabla u|^2 + |\nabla v|^2)\, dx + K \int_\Omega (u^2 + v^2)\, dx$$

for all $u, v \in H^1(\Omega)$.

7. Appendix. We extend the Pokhozhaev identity to a system of two equations.

PROPOSITION 7.1. *Let $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ be a solution of the system*

$$\begin{cases} -\Delta u + \lambda_1 u = \dfrac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^\beta, \\ -\Delta v + \lambda_2 v = \dfrac{2\beta}{\alpha+\beta} |u|^\alpha |v|^{\beta-2} v, \end{cases} \tag{19}$$

in $\mathbb{R}^N$. *Then*

$$\frac{N-2}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)\,dx=\frac{2N}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,dx-\frac{N}{2}\int_{\mathbb{R}^N}(\lambda_1u^2+\lambda_2v^2)\,dx.$$

Proof. We follow the argument from Proposition 1 in [8] (p. 320). It follows from the first equation that

$$\int_{\mathbb{R}^N}(-\Delta u+\lambda_1u)(x\cdot\nabla u)\,dx=\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^{\alpha-2}u|v|^\beta(x\cdot\nabla u)\,dx.$$

We also have

$$\int_{\mathbb{R}^N}(-\Delta u)(x\cdot\nabla u)\,dx=\frac{2-N}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx.$$

Hence

$$\begin{aligned}&\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^{\alpha-2}u|v|^\beta(x\cdot\nabla u)\,dx\\&=\frac{2}{\alpha+\beta}\int_{\mathbb{R}^N}(|u|^\alpha)_{x_j}|v|^\beta x_j\,dx\\&=-\frac{2N}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,dx-\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^{\beta-2}vv_{x_j}x_j\,dx\\&=-\frac{2N}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,dx-\int_{\mathbb{R}^N}(-\Delta v+\lambda_2v)(x\cdot\nabla v)\,dx\\&=-\frac{2N}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,dx-\frac{2-N}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,dx-\lambda_2\int_{\mathbb{R}^N}v(x\cdot\nabla v)\,dx.\end{aligned}$$

We now observe that

$$\lambda_1\int_{\mathbb{R}^N}u(x\cdot\nabla u)\,dx=\lambda_1\int_{\mathbb{R}^N}ux_ju_{x_j}\,dx=\frac{\lambda_1}{2}\int_{\mathbb{R}^N}(u^2)_{x_j}x_j\,dx=-\frac{N\lambda_1}{2}\int_{\mathbb{R}^N}u^2\,dx.$$

Therefore

$$\begin{aligned}&\frac{2-N}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx-\frac{N\lambda_1}{2}\int_{\mathbb{R}^N}u^2\,dx\\&\qquad=-\frac{2N}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,dx-\frac{2-N}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,dx+\frac{N\lambda_2}{2}\int_{\mathbb{R}^N}v^2\,dx\end{aligned}$$

and this completes the proof. ∎

PROPOSITION 7.2. *The system of equations* (19) *has no positive solutions in* $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ *for* $\lambda_1,\lambda_2\ge 0$ *with* $\lambda_1+\lambda_2>0$.

Proof. Indeed, we have

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)\,dx = 2\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\,dx - \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)\,dx.$$

It then follows from the Pokhozhaev identity that

$$(N-2)\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\,dx - \frac{N-2}{2}\int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)\,dx$$
$$= (N-2)\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\,dx - \frac{N}{2}\int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)\,dx.$$

This yields

$$\int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)\,dx = 0$$

and consequently $u = v = 0$ on $\mathbb{R}^N$. ■

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Department of Mathematics
The University of Queensland
St. Lucia 4072, Qld, Australia
E-mail: jhc@maths.uq.edu.au

IMECC-UNICAMP
13083-970 Campinas, S.P. Brazil

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