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OSCILLATING MULTIPLIERS ON THE HEISENBERG GROUP

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Abstract. Let $\mathcal{L}$ be the sublaplacian on the Heisenberg group H^n. A recent result of Müller and Stein shows that the operator $\mathcal{L}^{-1/2}\sin\sqrt{\mathcal{L}}$ is bounded on $L^p(H^n)$ for all p satisfying $|1/p-1/2|<1/(2n)$. In this paper we show that the same operator is bounded on L^p in the bigger range $|1/p-1/2|<1/(2n-1)$ if we consider only functions which are band limited in the central variable.

1. Introduction and main results. Consider the Heisenberg group $H^n=\mathbb{C}^n\times\mathbb{R}$ with the group law

$$(z,t)(w,s)=\big(z+w,t+s+\tfrac{1}{2}\operatorname{Im} z.\overline{w}\big).$$

The vector fields

$$T=\frac{\partial}{\partial t},\quad X_j=\frac{\partial}{\partial x_j}-\frac{1}{2}y_j\frac{\partial}{\partial t},\quad Y_j=\frac{\partial}{\partial y_j}+\frac{1}{2}x_j\frac{\partial}{\partial t},\quad j=1,\ldots,n,$$

form a basis for the Lie algebra of left invariant vector fields on the Heisenberg group. The operator

$$\mathcal{L}=-\sum_{j=1}^{n}(X_j^2+Y_j^2)$$

is called the *sublaplacian*; it plays the same role as the ordinary Laplacian does on $\mathbb{R}^n$. It is well known that $\mathcal{L}$ is hypoelliptic and represents the simplest example of the subelliptic realm.

The sublaplacian $\mathcal{L}$ is self-adjoint and nonnegative and hence admits the spectral decomposition

$$\mathcal{L}=\int_0^\infty \lambda\,dE_\lambda.$$

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Given a bounded function m defined on $(0, \infty)$ one can define the operator $m(\mathcal{L})$ formally by setting

$$m(\mathcal{L})f = \int_0^\infty m(\lambda)\, dE_\lambda f.$$

This operator is clearly bounded on $L^2(H^n)$ but need not be bounded on $L^p(H^n)$ for $p \neq 2$ unless some more conditions are imposed on the multiplier m. This problem has been studied by several authors and sufficient conditions on m have been found. See the works [1], [6], [7] and [19]. The optimal result has been proved in Müller–Stein [9] and Hebisch [5].

When $m(\lambda) = m_s(\lambda) = \lambda^{-1/2} \sin s\sqrt{\lambda}$, the function $u(z, t, s)$ defined by

$$u(z, t, s) = m_s(\mathcal{L})f(z, t)$$

solves the Cauchy problem for the wave equation associated with the sublaplacian. Namely, $u(z, t, s)$ solves the equation

$$\partial_s^2 u(z, t, s) = \mathcal{L}u(z, t, s)$$

with initial conditions

$$u(z, t, 0) = 0, \qquad \partial_s u(z, t, 0) = f(z, t).$$

The L^p boundedness of the operator $m_s(\mathcal{L})$ has been studied by Müller and Stein in [10], where they have established the following result.

Theorem 1.1 (Müller–Stein). *For $|1/p - 1/2| < 1/(2n)$, the operator $\mathcal{L}^{-1/2} \sin s\sqrt{\mathcal{L}}$ extends to a bounded operator on $L^p(H^n)$.*

The analogue of this theorem for the Euclidean Laplacian has been proved by Peral [15] and Miyachi [8]. Similar multipliers on noncompact symmetric spaces have been studied by Giulini and Meda [3]. Results for the sublaplacian on stratified groups have been obtained by Mauceri and Meda [7]. Recently we have studied the wave equation associated with Hermite and special Hermite expansions in [13]. For certain Schrödinger operators see the work of Zhong [21] .

Observe that the multiplier $m(\lambda) = \lambda^{-1/2} \sin \sqrt{\lambda}$ satisfies the conditions

$$|m^{(j)}(\lambda)| \leq C_j (1 + \lambda^2)^{-1/4 - j/4}, \qquad \lambda > 0,$$

for $j = 0, 1, \ldots$ Therefore, we are led to consider operators of the form $m(\mathcal{L})$ when $m \in S_\varrho^\alpha(\mathbb{R})$ where the symbol class S_ϱ^α consists of all C^∞ functions on $\mathbb{R}$ satisfying the estimates

$$|m^{(j)}(\lambda)| \leq C_j (1 + \lambda^2)^{\alpha/2 - \varrho j/2}$$

for $j = 0, 1, \ldots$ In [13] the L^p boundedness of operators of the form $m(P)$ for $m \in S_\varrho^\alpha(\mathbb{R})$ has been studied. More generally, the following theorem has been established.

THEOREM 1.2. *Let $m \in S_{\varrho}^{-\alpha}(\mathbb{R})$ be such that $m(\lambda) = 0$ for $|\lambda| \le 1/2$. Let P be a Rockland operator on H^n which is homogeneous of degree 2. Then $m(P)$ is bounded on $L^p(H^n)$ provided $\alpha > Q(1-\varrho)|1/p - 1/2|$, $1 < p < \infty$, where $Q = 2n+2$ is the homogeneous dimension of H^n.*

In particular, by taking $P = \mathcal{L}$ and $m(\lambda) = \lambda^{-1/2}\sin\sqrt{\lambda}$ we see that $\mathcal{L}^{-1/2}\sin\sqrt{\mathcal{L}}$ is bounded on $L^p(H^n)$ for $|1/p - 1/2| < 1/Q$. We see that the result of Müller and Stein is much stronger than this. The interesting thing to note is that in their result it is not the homogeneous dimension $2n+2$ but the Euclidean dimension $2n+1$ which restricts the range of L^p boundedness.

Our aim in this paper is to slightly improve the result of Müller and Stein on the wave equation in the case when f is band limited in the t-variable. Let $L_B^p(H^n)$ stand for those functions f in $L^p(H^n)$ for which the partial inverse Fourier transform $f^\lambda(z)$ in the t-variable is supported in $|\lambda| \le B$. On this space we have the following improvement of Theorem 1.1.

THEOREM 1.3. *Let $n \ge 2$. The operator $\mathcal{L}^{-1/2}\sin\sqrt{\mathcal{L}}$ is bounded on $L_B^p(H^n)$ for $|1/p - 1/2| < 1/(2n-1)$.*

More generally, we can consider operators of the form $\mathcal{L}^{-\alpha/2}J_\alpha(\sqrt{\mathcal{L}})$ where J_α is the Bessel function of order α.

THEOREM 1.4. *The operators $\mathcal{L}^{-\alpha/2}J_\alpha(\sqrt{\mathcal{L}})$ are bounded on $L_B^p(H^n)$ for $|1/p - 1/2| < (2\alpha+1)/(4n-2)$ provided $6\alpha \le 4n-5$. Otherwise, they are bounded on $L_B^p(H^n)$ in the smaller range $|1/p - 1/2| < (2\alpha+3)/(4n+4)$.*

Note that when $\alpha = 1/2$, we have $\lambda^{-\alpha/2}J_\alpha(\sqrt{\lambda}) = \sqrt{2/\pi}\,\lambda^{-1/2}\sin\sqrt{\lambda}$ and hence we only need to prove Theorem 1.4.

The operators $\mathcal{L}$ and T commute and so they admit a joint spectral decomposition which can be written down explicitly. Let

$$\varphi_k(z) = L_k^{n-1}(|z|^2/2)e^{-|z|^2/4}$$

be the Laguerre functions of type $n-1$. Define

$$e_k^\lambda(z,t) = e^{i\lambda t}\varphi_k^\lambda(z) = e^{i\lambda t}\varphi_k(\sqrt{|\lambda|}z)$$

for $\lambda \in \mathbb{R}$, $\lambda \ne 0$. Then $e_k^\lambda(z,t)$ are joint eigenfunctions of $\mathcal{L}$ and T:

$$\mathcal{L}e_k^\lambda(z,t) = (2k+n)|\lambda|e_k^\lambda(z,t), \qquad Te_k^\lambda(z,t) = i\lambda e_k^\lambda(z,t).$$

The explicit spectral decomposition of $\mathcal{L}$ and T studied in great detail by Strichartz [16] and [17] is then written as

$$f(z,t) = c_n \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} f * e_k^\lambda(z,t)\Big)|\lambda|^n\,d\lambda.$$

Given a bounded function $m(\xi,\eta)$ of two variables we can consider the operator

$$Mf(z,t) = c_n \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} m(k,\lambda) f * e_k^\lambda(z,t) \Big) |\lambda|^n \, d\lambda.$$

One can naturally ask for conditions on $m(k,\lambda)$ so that M extends to a bounded operator on $L^p(H^n)$.

Recently this problem has received considerable attention. In the papers [11] and [12] Müller, Ricci and Stein have obtained sufficient conditions on $m(\xi,\eta)$ so that M is bounded on $L^p(H^n)$. More precisely, if $m(\xi,\eta)$ satisfies the Marcinkiewicz type conditions

$$|(\xi\partial_\xi)^\alpha (\eta\partial_\eta)^\beta m(\xi,\eta)| \le C_{\alpha,\beta}$$

for sufficiently many derivatives, then M is bounded on $L^p(H^n)$, $1 < p < \infty$. In [12] the authors have obtained a sharp Marcinkiewicz multiplier theorem where the above conditions are required to hold only for an optimal number of derivatives.

When $m(k,\lambda) = m((2k+n)|\lambda|)$ the operator M is nothing but $m(\mathcal{L})$ and the Marcinkiewicz conditions hold when $m \in S_1^0(\mathbb{R})$. In the general case, when $m \in S_1^0(\mathbb{R}^2)$, the corresponding operator M is bounded on $L^p(H^n)$, $1 < p < \infty$, as proved in [12]. It is an interesting problem to study the L^p boundedness of M when $m \in S_\varrho^\alpha(\mathbb{R}^2)$. We plan to return to this problem in the near future.

We now describe how we plan to prove Theorem 1.4. The proof of Theorem 1.2 given in [13] can be modified to show that the multipliers $m((2k+n)|\lambda|)$ and $m((2k+\beta)|\lambda|)$ have the same L^p boundedness properties when $m \in S_\varrho^{-\alpha}(\mathbb{R})$. In view of this, in order to prove Theorem 1.4 it is enough to consider the multipliers

$$m_r^\alpha(k,\lambda) = b_\alpha((2k+\alpha+1)|\lambda|r^2)^{-\alpha/2} J_\alpha(\sqrt{(2k+\alpha+1)|\lambda|r^2})$$

where $b_\alpha = 2^\alpha \Gamma(\alpha+1)$ and $r > 0$ is fixed. Let M_r^α be the operator defined by

$$M_r^\alpha f = c_n \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} m_r^\alpha(k,\lambda) f * e_k^\lambda(z,t) \Big) |\lambda|^n \, d\lambda.$$

We plan to study these operators by first studying the family of operators T_r^α defined by

$$T_r^\alpha f = c_n \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} \psi_k^\alpha(r\sqrt{|\lambda|}) f * e_k^\lambda(z,t) \Big) |\lambda|^n \, d\lambda$$

where

$$\psi_k^\alpha(r) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha\Big(\frac{1}{2} r^2\Big) e^{-r^2/4}$$

are the Laguerre functions of type α.

The operators T_r^α can be defined even for complex α as long as $\operatorname{Re}\alpha \geq -1/2$. When $\alpha = n-1$ we note that $T_r^{n-1}f = f * \mu_r$ where μ_r is the normalised surface measure on the sphere $S_r = \{(z,0) : |z| = r\}$. Using this and analytic interpolation we obtain

THEOREM 1.5. (i) *If* $\alpha > (2n-1)|1/p-1/2|-1/2$, *then* T_r^α *are uniformly bounded on* $L_B^p(H^n)$ *for* $0 < r \leq 1$.

(ii) *If* $\alpha > (2n-4/3)|1/p-1/2|-1/3$, *then* T_r^α *are uniformly bounded on* $L^p(H^n)$ *for all* $r > 0$.

Once we have Theorem 1.5, Theorem 1.4 and hence Theorem 1.3 are proved by comparing the multiplier $m_r^\alpha(k,\lambda)$ with $\psi_k^\alpha(\sqrt{|\lambda|}r)$. To this end we make use of a Hilb type asymptotic expansion [18] of the Laguerre polynomials. In the course of the proof we will make use of Theorem 1.2 in dealing with the error terms.

We closely follow the notations employed in [20]. For various results concerning the Heisenberg group we refer the reader to the monographs [2] and [20].

2. Proof of Theorem 1.5. As indicated in the introduction we prove Theorem 1.5 by using analytic interpolation. Let μ_r be the normalised surface measure on the sphere S_r. Then it is well known (see [14]) that

$$f * \mu_r = c_n \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} \psi_k^{n-1}(\sqrt{|\lambda|}r) f * e_k^\lambda\Big)|\lambda|^n \, d\lambda. \tag{2.1}$$

Now Laguerre functions of different type are related by the formula (see [18])

$$L_k^{\alpha+\beta}(r) = \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(\beta)\Gamma(k+\alpha+1)} \int_0^1 s^\alpha (1-s)^{\beta-1} L_k^\alpha(sr)\, ds,$$

which is valid for $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > 0$. Using this we can write, when $\alpha = n-1+\delta+i\sigma$,

$$\begin{aligned}\psi_k^\alpha(r) = {} & \frac{\Gamma(k+n+\delta+i\sigma)}{\Gamma(\delta+i\sigma)\Gamma(k+n)} \\ & \times \int_0^1 s^{n-1}(1-s)^{\delta+i\sigma-1} e^{-(1-s^2)r^2/4} \psi_k^{n-1}(sr)\, ds.\end{aligned} \tag{2.2}$$

Let us define an operator $A_r f$ by

$$(A_r f)^\lambda(z) = e^{-r|\lambda|/4} f^\lambda(z)$$

where $f^\lambda(z)$ is the partial inverse Fourier transform of $f(z,t)$ in the

t-variable. We then have the formula

$$T_r^\alpha f = \frac{\Gamma(n+\delta+i\sigma)}{\Gamma(\delta+i\sigma)\Gamma(n)} \int_0^1 s^{n-1}(1-s)^{\delta+i\sigma-1} T_{rs}^{n-1} A_{(1-s^2)r^2} f\, ds. \tag{2.3}$$

Similarly when $\alpha = -1/2+\delta+i\sigma$ we have

$$T_r^\alpha f = \frac{\Gamma(-1/2+\delta+i\sigma)}{\Gamma(\delta+i\sigma)\Gamma(-1/2)} \int_0^1 s^{-1/2}(1-s)^{\delta+i\sigma-1} T_{rs}^{-1/2} A_{(1-s^2)r^2} f\, ds. \tag{2.4}$$

The operators $A_r f$ are nothing but the Poisson integrals in the t-variable and so they are uniformly bounded on $L^p(H^n)$ for all $1 \le p \le \infty$. Therefore, from (2.3) we see that

$$\|T_r f\|_p \le C(\sigma)\|f\|_p, \quad 1 \le p \le \infty,$$

when $\alpha = n-1+\delta+i\sigma$. When $\alpha = -1/2$, the Laguerre functions $\psi_k^{-1/2}(r)$ are uniformly bounded in k as long as r remains bounded. Let $\chi \in C_0^\infty(|\lambda| \le B+1)$ be such that $\chi(\lambda) = 1$ for $|\lambda| \le B$ and define $\chi(i\partial_t)$ to be the operator

$$(\chi(i\partial_t)f)^\lambda(z) = \chi(\lambda) f^\lambda(z).$$

Then the multiplier corresponding to $T_r^\alpha \chi(i\partial_t)$ is $\psi_k^\alpha(\sqrt{|\lambda|}r)\chi(\lambda)$, which is uniformly bounded; that is,

$$|\psi_k^\alpha(\sqrt{|\lambda|}r)\chi(\lambda)| \le C$$

for all $\lambda \in \mathbb{R}$, $k = 0, 1, \ldots$ and $0 \le r \le 1$. Therefore, by Plancherel's theorem,

$$\|T_r^\alpha \chi(i\partial_t) f\|_2 \le C_B(\sigma)\|f\|_2$$

when $\alpha = -1/2+\delta+i\sigma$. Using Stirling's formula for the gamma function we can check that $C(\sigma)$ and $C_B(\sigma)$ are of admissible growth.

By appealing to Stein's analytic interpolation theorem we obtain

$$\|T_r^\alpha \chi(i\partial_t) f\|_p \le C\|f\|_p$$

for $\alpha > (2n-1)(1/p-1/2)-1/2$. This proves part (i) of Theorem 1.5. To prove the other part we use the uniform estimate $|\psi_k^{-1/3}(t)| \le C$, which is valid for all $r > 0$ and $k = 0, 1, \ldots$ (see Szegő [18]). As before, analytic interpolation will prove part (ii).

3. A variant of Theorem 1.2. In the next section we will use Theorem 1.5 to study multipliers of the form $m((2k+\alpha+1)|\lambda|)$. However, in order to prove Theorem 1.3 we need to treat multipliers of the form $m((2k+n)|\lambda|)$. This can be achieved by comparing these two multipliers.

Taking $m(t) = t^{-\alpha/2} J_\alpha(t)$ we have the equation

$$m((2k+n)|\lambda|) - m((2k+\alpha+1)|\lambda|) = |\lambda| \int\limits_{\alpha+1}^{n} m'((2k+t)|\lambda|)\,dt.$$

Since $m'(t) = -\frac{1}{2} t^{-(\alpha+1)/2} J_{\alpha+1}(\sqrt{t})$ we have

$$m((2k+n)|\lambda|) - m((2k+\alpha+1)|\lambda|) = c|\lambda| \int\limits_{\alpha+1}^{n} \frac{J_{\alpha+1}(\sqrt{(2k+t)|\lambda|})}{(\sqrt{(2k+t)|\lambda|})^{\alpha+1}}\,dt. \tag{3.1}$$

Note that $\lambda^{-(\alpha+1)/2} J_{\alpha+1}(\sqrt{\lambda})$ belongs to the symbol class $S_{1/2}^{-\alpha/2-3/4}(\mathbb{R})$ whereas $m(\lambda) = \lambda^{-\alpha/2} J_\alpha(\sqrt{\lambda})$ belongs to $S_{1/2}^{-\alpha/2-1/4}(\mathbb{R})$.

Therefore, if we can show that the operators $J_r^\alpha f$ defined by

$$J_r^\alpha f = \int\limits_{-\infty}^{\infty} \bigg(\sum_{k=0}^{\infty} \frac{J_{\alpha+1}(\sqrt{(2k+r)|\lambda|})}{(\sqrt{(2k+r)|\lambda|})^{\alpha+1}} f * e_k^\lambda \bigg) |\lambda|^n \,d\lambda$$

are uniformly bounded on $L^p(H^n)$ for $2\alpha+3 > 2Q(1/p-1/2)$, $\alpha+1 \le r \le n$, then from (3.1) it will follow that $m(\mathcal{L})$ is bounded on $L_B^p(H^n)$ when the multiplier $m((2k+\alpha+1)|\lambda|)$ defines a bounded operator on $L_B^p(H^n)$. Thus we require the following variant of Theorem 1.2.

THEOREM 3.1. *Let $m \in S_\varrho^{-\alpha}(\mathbb{R})$ and let M_r be the operator with the multiplier $m((2k+r)|\lambda|)$ where $0 < \varepsilon < r < 2n-\varepsilon$. Then M_r are uniformly bounded on $L^p(H^n)$ when $\alpha > Q(1-\varrho)|1/p-1/2|$.*

Proof. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi(\lambda) = 0$ for $|\lambda| \le 1/2$ and $\varphi(\lambda) = 1$ for $|\lambda| \ge 1$. Then the multiplier

$$m_1(\xi,\eta) = m((2\xi+r)\eta)(1-\varphi((2\xi+r)\eta))$$

satisfies the conditions

$$\sup_{\xi>0,\,\eta\in\mathbb{R}} |(\xi\partial_\xi)^j (\eta\partial_\eta)^l m_1(\xi,\eta)| \le C_{jl}$$

for all j and l uniformly in r. Therefore, by a theorem of Müller, Ricci and Stein (Theorem 2.2 in [12]) the operators with multipliers $m_1(k,\lambda)$ are uniformly bounded on $L^p(H^n)$, $1 < p < \infty$. So, it is enough to consider the operator $\widetilde{M}_r$ with the multiplier $\widetilde{m}((2k+r)|\lambda|)$ where $\widetilde{m}(\lambda) = m(\lambda)\varphi(\lambda)$.

Let Hf be the Hilbert transform of f in the t-variable defined by

$$(Hf)^\lambda(z) = -i \operatorname{sgn} \lambda f^\lambda(z).$$

Write $g = \frac{1}{2}(f+iHf)$ and $h = \frac{1}{2}(f-iHf)$ so that $f = g+h$ and $\|g\|_p \le C\|f\|_p$, $\|h\|_p \le C\|f\|_p$. Note that $g^\lambda(z)$ vanishes for $\lambda < 0$ and $h^\lambda(z)$ for $\lambda > 0$. We have

$$\widetilde{M}_r g = c_n \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} \widetilde{m}((2k+n)|\lambda| + (r-n)\lambda) g * e_k^\lambda\Big) |\lambda|^n \, d\lambda,$$

which is nothing but $\widetilde{m}(\mathcal{L}+i(n-r)T)g$. Similarly, $\widetilde{M}_r h = \widetilde{m}(\mathcal{L}-i(n-r)T)h$.

Note that the operators $\mathcal{L}+i(n-r)T$ and $\mathcal{L}-i(n-r)T$ are homogeneous of degree 2 and since $0 < r < 2n$ it is easily verified that they are Rockland operators. Therefore, by appealing to Theorem 1.2 we can conclude that $\widetilde{m}(\mathcal{L} \pm i(n-r)T)$ are bounded on $L^p(H^n)$ for $\alpha > Q(1-\varrho)|1/p - 1/2|$. As $\widetilde{M}_r f = \widetilde{M}_r g + \widetilde{M}_r h$ we see that $\widetilde{M}_r$ is bounded on $L^p(H^n)$.

It remains to be shown that the operator norms of $\widetilde{M}_r$ are uniform in r as long as $\varepsilon \le r \le 2n - \varepsilon$. To this end we have to recall the main ideas involved in the proof of Theorem 1.2. In [13] we have treated multipliers for a wide class of operators. If P is a nonnegative self-adjoint operator on $\mathbb{R}^n$ for which the kernel $S_R^\delta(x,y)$ of the Bochner–Riesz mean $(1 - P/R)_+^\delta$ satisfies an estimate of the form

$$|S_R^\delta(x,y)| \le C R^{n/2} (1 + R^{1/2}|x-y|)^{-\delta+\beta} \tag{3.2}$$

then an analogue of Theorem 1.2 holds for $m(P)$, $m \in S_\varrho^\alpha(\mathbb{R})$. Therefore, if we can show that the Bochner–Riesz kernels associated with the operators $\mathcal{L} \pm i(n-r)T$ satisfy the above estimates with C independent of r, then the operators $\widetilde{M}_r$ will be uniformly bounded.

For $a \in \mathbb{R}$ consider the operator $P_a = \mathcal{L} + iaT$, which is a Rockland operator as long as a is admissible. We will show that if $|a| \le n - \varepsilon$, $\varepsilon > 0$, then the Bochner–Riesz kernel associated with P_a satisfies uniform estimates of the form (3.2). To do this we make use of a method developed by Hebisch [3] which only requires uniform estimates on the heat kernel associated with P_a. In the present case we can easily obtain estimates on the heat kernel.

PROPOSITION 3.2. *Let $p_{s,a}(z,t)$ be the kernel of the operator e^{-sP_a}, $s > 0$. Then*

$$|p_{s,a}(z,t)| \le C s^{-Q/2} e^{-(A/s)(|z|^2+|t|)}$$

where A and C are independent of a for $|a| \le n - \varepsilon$.

Proof. By homogeneity it is enough to consider $s = 1$. Let us write $p_{1,a}(z,t) = K_a(z,t)$. It is well known that the kernel is given by the formula

$$K_a(z,t) = c_n \int k_a(z,t,\lambda) \, d\lambda$$

where

$$k_a(z,t,\lambda) = e^{-a\lambda} \Big(\frac{\lambda}{\sinh \lambda}\Big)^n e^{-\lambda(\coth\lambda)|z|^2/4} e^{i\lambda t}.$$

Note that $k_a(z,t,\lambda)$ extends to a holomorphic function of λ in the strip $|\mathrm{Im}\,\lambda| < \pi/2$. Hence by Cauchy's theorem

$$K_a(z,t) = \lim_{R\to\infty} \left\{ \int_0^{\pi/4} k_a(z,t,-R+i\sigma)\,d\sigma + \int_{-R}^{R} k_a\left(z,t,\lambda+i\frac{\pi}{4}\right)d\lambda - \int_0^{\pi/4} k_a(z,t,R+i\sigma)\,d\sigma \right\}.$$

In the above the first and last integrals go to zero uniformly in a as $R\to\infty$, provided $|a| \le n-\varepsilon$. Then we get

$$K_a(z,t) = c_n \int k_a\left(z,t,\lambda+i\frac{\pi}{4}\right)d\lambda$$

and from this we obtain

$$|K_a(z,t)| \le Ce^{-\pi|t|/4}, \qquad t>0, \tag{3.3}$$

where C is independent of a. The same estimate holds for $t<0$ as well. As $\coth\lambda$ behaves like λ for λ small we easily get the estimate

$$|K_a(z,t)| \le Ce^{-|z|^2/4}. \tag{3.4}$$

The estimates (3.3) and (3.4) put together prove the proposition.

Using the heat kernel estimate proved above and following a method of Hebisch [4] we can obtain uniform estimates on the Bochner–Riesz kernels associated with P_a. Write $w=(z,t)$ and let $|w|$ be the homogeneous norm defined by $|w|^4 = |z|^4 + |t|^2$.

Proposition 3.3. *Let $S^\delta_{R,a}(w)$ be the kernel of the Bochner–Riesz means associated with P_a. Then for $|a| \le n-\varepsilon$ and δ large,*

$$|S^\delta_{R,a}(w)| \le CR^{Q/2}(1+R^{1/2}|w|)^{-\delta+\beta}$$

where C is independent of a and R, and β is a fixed constant.

Proof. Due to homogeneity of the operators P_a it is enough to consider $R=1$. Following Hebisch we let $E^a_n(w)$ be the kernel of the operator $e^{inK}K$ with $K=e^{-P_a}$. By appealing to Theorem 3.1 of [4] we get the estimate

$$\int_{H^n} |E^a_n(w)|(1+|w|)^\gamma\,dw \le C(1+|n|)^{\gamma+Q/2}$$

for every $\gamma \ge 0$ and C independent of a. Defining e^a_n to be the kernel of $e^{inK}K^2$ we have

$$e^a_n(w) = E^a_n * p_{1,a}(w).$$

Using the L^1 estimate of E_n^a and the heat kernel estimate of P_a we easily get the estimate

$$|e_n^a(w)| \leq C(1+|w|)^{-\gamma}(1+|n|)^{\gamma+Q/2} \tag{3.5}$$

for all $\gamma \geq 0$ with C independent of a.

We can now make use of the functional calculus developed in [4] to get estimates of the Bochner–Riesz kernel. For the sake of completeness we briefly indicate the method. Let $F(\lambda) = (1-\lambda)_+^\delta \psi(\lambda)$ where $\psi \in C^\infty$ is such that $\psi(\lambda) = 1$ for $\lambda \geq 0$ and $\psi(\lambda) = 0$ for $\lambda \leq -e^{-1}$. Let $G(\lambda) = \lambda^{-2}F(-\log\lambda)$ for $\lambda > 0$ and $G(\lambda) = 0$ otherwise. Then $G(\lambda)$ is supported in $[0, e]$ and $F(P_a) = G(e^{-P_a})e^{-2P_a}$. Expanding $G(\lambda)$ into Fourier series as $G(\lambda) = \sum \widehat{G}(n)e^{in\lambda}$ we get

$$F(P_a) = \sum \widehat{G}(n)e^{inK}K^2$$

where, as before, $K = e^{-P_a}$.

Using the estimate (3.5) we get

$$|S_{1,a}^\delta(x,y)| \leq C(1+|w|)^{-\gamma} \sum |\widehat{G}(n)|(1+|n|)^{\gamma+Q/2}.$$

The coefficients $\widehat{G}(n)$ are given by

$$\widehat{G}(n) = \frac{1}{2\pi}\int_0^e G(\lambda)e^{-in\lambda}\,d\lambda.$$

Making a change of variables we get

$$\widehat{G}(n) = \frac{1}{2\pi}\int_{-e^{-1}}^{1} F(t)e^t e^{-ine^{-t}}\,dt.$$

As $F(t) = (1-t)_+^\delta \psi(t)$ we easily get the estimate

$$|\widehat{G}(n)| \leq C(1+|n|)^{-l}$$

provided $\delta > l-1$. Taking $\delta = \gamma + Q/2 + 2$ we have

$$|\widehat{G}(n)| \leq C(1+|n|)^{-\gamma-Q/2-2}$$

and hence

$$|S_{1,a}^\delta(w)| \leq C(1+|w|)^{-\delta+Q/2+2}$$

where C is independent of a. This completes the proof of the proposition.

Once we have uniform estimates on the Bochner–Riesz kernels $S_{R,a}^\delta$ we can prove Theorem 3.1. See [13] for the details.

4. Proof of Theorem 1.4. In view of Theorem 3.1 and the remarks preceding it, it is enough to consider the operator M_r^α given by the multiplier $m_r^\alpha(k,\lambda)$. We now compare the multipliers $m_r^\alpha(k,\lambda)$ and $\psi_k^\alpha(\sqrt{|\lambda|}\,r)$ by using a Hilb type asymptotic formula for the Laguerre polynomials. Formula (8.64.3) on page 217 of Szegő [18] gives

$$\psi_k^\alpha(r) = m_r^\alpha(k,1) + e(k,\alpha,r) \tag{4.1}$$

where $e(k,\alpha,r)$ is given by the integral

$$\frac{\pi}{2^3}\,\frac{r^4}{\sin\alpha\pi}\int_0^1 (J_\alpha(r\sqrt{K})J_{-\alpha}(rs\sqrt{K}) - J_\alpha(rs\sqrt{K})J_\alpha(r\sqrt{K}))s^{\alpha+3}\psi_k^\alpha(rs)\,ds.$$

In the above formula $K = 2k+\alpha+1$. When α is an integer, $\sin\alpha\pi$ in the above formula has to be replaced by -1 and J_α by the modified Bessel function Y_α.

Define $a_\alpha(\lambda,r,s)$ for $\lambda > 0$ by

$$a_\alpha(\lambda,r,s) = (J_\alpha(r\sqrt{\lambda})J_{-\alpha}(rs\sqrt{\lambda}) - J_{-\alpha}(r\sqrt{\lambda})J_\alpha(rs\sqrt{\lambda}))s^{\alpha+3}r^4$$

and let $A_\alpha(r,s)$ be the operator whose multiplier is $a_\alpha((2k+n)|\lambda|,r,s)$. Let $\chi \in C_0^\infty(|\lambda| \le B+1)$ and $\chi(i\partial_t)$ be as before. From (4.1) it follows that

$$T_r^\alpha\chi(i\partial_t)f = M_r^\alpha\chi(i\partial_t)f + c_1\int_0^1 A_\alpha(r,s)T_{rs}^\alpha\chi_1(i\partial_t)f\,ds$$

where $\chi_1(\lambda) = \lambda^2\chi(\lambda)$ and c_1 is some constant. Another iteration produces the formula

$$\begin{aligned} M_r^\alpha\chi(i\partial_t)f = T_r^\alpha\chi(i\partial_t)f &+ c_1\int_0^1 A_\alpha(r,s)M_{rs}^\alpha\chi_1(i\partial_t)f\,ds \\ &+ c_2\int_0^1\int_0^1 A_\alpha(r,s)A_\alpha(rs,s')T_{rss'}^\alpha\chi_2(i\partial_t)f\,ds\,ds' \end{aligned} \tag{4.2}$$

where $\chi_2(\lambda) = \lambda^4\chi(\lambda)$ and c_1, c_2 are constants. For the symbols $a_\alpha(\lambda,r,s)$ we prove the following estimates.

LEMMA 4.1. *For $0 \le r,\ s \le 1$ we have the estimates*

$$|\partial_\lambda^k a_\alpha(\lambda,r,s)| \le C_k(1+\lambda)^{-k/2-1/2}$$

valid for all $\lambda > 0$, $k \ge 0$. More precisely,

$$\begin{aligned} &|\partial_\lambda^k a_\alpha(\lambda,r,s)| \le Cr^3s^{5/2}(1+\lambda)^{-(k+1)/2} \\ &\qquad\times\{(1+r^2\lambda)^{-\alpha/2}(1+r^2s^2\lambda)^{\alpha/2} + s^{2\alpha}(1+r^2\lambda)^{\alpha/2}(1+r^2s^2\lambda)^{-\alpha/2}\}. \end{aligned}$$

Proof. Let $B_\alpha(\lambda) = \lambda^{-\alpha/2}J_\alpha(\sqrt{\lambda})$ and when α is a negative integer replace J_α by Y_α. Then B_α satisfies the equation

$$\frac{d}{d\lambda}B_\alpha(\lambda) = -\frac{1}{2}B_{\alpha+1}(\lambda).$$

The asymptotic properties of the Bessel function give us the estimates

$$\left|\left(\frac{d}{d\lambda}\right)^k B_\alpha(\lambda)\right| \leq C(1+\lambda)^{-(\alpha+k+1/2)/2}.$$

Consider the first term, which is equal to $B_\alpha(r^2\lambda)B_{-\alpha}(r^2s^2\lambda)s^3r^4$. The kth derivative of that term is a linear combination of terms of the form

$$r^{2j+4}B_{\alpha+j}(r^2\lambda)(r^2s^2)^{k-j}B_{-\alpha+k-j}(r^2s^2\lambda)s^3,$$

which is bounded by a constant times

$$r^{2k+4}s^{2k-2j+3}(1+r^2\lambda)^{-(\alpha+j+1/2)/2}(1+r^2s^2\lambda)^{-(-\alpha+k-j+1/2)/2}.$$

As $0 \leq r, s \leq 1$, the above is bounded by a constant times

$$r^3s^{5/2}(1+\lambda)^{-(k+1)/2}(1+r^2\lambda)^{-\alpha/2}(1+r^2s^2\lambda)^{\alpha/2},$$

which in turn is bounded by $C(1+\lambda)^{-(k+1)/2}$. Similarly, the kth derivative of the second term is bounded by

$$Cr^3s^{2\alpha+5/2}(1+\lambda)^{-1/2-k/2}(1+r^2\lambda)^{\alpha/2}(1+r^2s^2\lambda)^{-\alpha/2},$$

which in turn is bounded by $C(1+\lambda)^{-(k+1)/2}$. This proves the lemma.

We are now in a position to prove Theorem 1.4. From Theorem 1.5 we know that $T_1^\alpha\chi(i\partial_t)$ is bounded on $L^p(H^n)$ for $|1/p-1/2|<(2\alpha+1)/(4n-2)$. If $6\alpha \leq 4n-5$, then $(2\alpha+1)/(4n-2) \leq (2\alpha+3)/(4n+4)$ and consequently $\alpha/2+3/4 > Q|1/p-1/2|/2$ whenever $|1/p-1/2| < (2\alpha+1)/(4n-2)$. The multiplier corresponding to the product $A_\alpha(1,s)M_s^\alpha$ is given by the symbol

$$m(\lambda, s) = a_\alpha(\lambda, 1, s)B_\alpha(s^2\lambda),$$

which belongs to the class $S_{1/2}^{-\alpha/2-3/4}(\mathbb{R})$. Using Lemma 4.1 we can show that

$$|\partial_\lambda^k m(\lambda, s)| \leq C(1+\lambda)^{-(\alpha+3/2+k)/2}$$

where C is uniform for $0 \leq s \leq 1$. Since $\alpha/2+3/4 > Q|1/p-1/2|/2$, from Theorem 3.1 we conclude that

$$\|A_\alpha(1,s)M_s^\alpha f\|_p \leq C\|f\|_p$$

where C is independent of s. Therefore, the operator

$$\int_0^1 A_\alpha(1,s)M_s^\alpha\chi(i\partial_t)f\,ds$$

is bounded on $L^p(H^n)$.

For the third term in (4.2), the symbol of the operator $A_\alpha(1,s)A_\alpha(s,s')$ comes from $S_{1/2}^{-1}(\mathbb{R})$ and the derivatives satisfy uniform estimates for $0 \leq$

$s, s' \leq 1$ in view of Lemma 4.1. If $0 \leq \alpha \leq 1/2$ we can conclude that the operator

$$\int_0^1 \int_0^1 A_\alpha(1,s) A_\alpha(s,s') T^\alpha_{ss'} \chi_2(i\partial_t) f \, ds \, ds'$$

is also bounded on $L^p(H^n)$. Therefore, from (4.2) we see that $M_1^\alpha \chi(i\partial_t)$ is bounded on $L^p(H^n)$. If $\alpha > 1/2$, we can perform further iterations and then the symbol of

$$A_\alpha(1, s_1) A_\alpha(s_1, s_2) \ldots A_\alpha(s_1 s_2 \ldots s_{l-1}, s_l)$$

will come from $S^{-l/2}_{1/2}(\mathbb{R})$ with estimates uniform in $s_1, \ldots, s_l$. We can choose l large enough so that $\alpha/2 + 3/4 \leq l/2$ and appealing to Theorem 3.1 we get the boundedness of M_1^α in the case when $6\alpha \leq 4n - 5$.

If $6\alpha > 4n - 5$ then we need to assume the condition $|1/p - 1/2| < (2\alpha + 3)/(4n + 4)$ so that $\alpha/2 + 3/4 > Q|1/p - 1/2|/2$. We then proceed as before to complete the proof.

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