COLLOQUIUM MATHEMATICUM

VOL. 90 2001 NO. 1

TWO-GENERATED IDEMPOTENT GROUPOIDS WITH SMALL CLONES

BY

J. GAŁUSZKA (Gliwice)

Abstract. A characterization of all classes of idempotent groupoids having no more than two essentially binary term operations with respect to small finite models is given.

1. Introduction. In [6] J. Dudek described all varieties of idempotent groupoids having no more than two essentially binary term operations. In this paper we characterize these varieties with respect to small finite models. To investigate the varietes described in [6] we use a technique analogous to the methods used for example in [4], [7] and [9]. The notations and notions used in this paper are standard (see [10] and [11]).

Let $\mathfrak{G} = (G, \cdot)$ be a groupoid. We denote by $p_n(\mathfrak{G})$ the number of essentially n-ary term operations over $\mathfrak{G}$ and by $p_0(\mathfrak{G})$ the number of unary constant term operations over $\mathfrak{G}$. Moreover, $\boldsymbol{p}(\mathfrak{G})$ denotes the sequence $(p_0(\mathfrak{G}), p_1(\mathfrak{G}), p_2(\mathfrak{G}), \ldots)$.

For the class $\mathcal{G}$ of groupoids we use the following notations: we write xy instead of $x \cdot y$, xy^n instead of $(\ldots(xy)\ldots)y$ and ${}^n yx$ instead of $y(\ldots(yx)\ldots)$ where y appears n times. Recall that a groupoid $\mathfrak{G}$ is *nontrivial* if $\mathrm{card}(G) \geq 2$, and *proper* if the operation xy in $\mathfrak{G}$ depends on both its variables. The *dual groupoid* $\mathfrak{G}^{\mathrm{d}} = (G, \circ)$ is defined by $x \circ y = yx$. If $\mathcal{C}$ is a class of groupoids, then $\mathcal{C}^{\mathrm{d}}$ denotes the class of all groupoids $\mathfrak{G}^{\mathrm{d}}$ for $\mathfrak{G} \in \mathcal{C}$.

We say that a groupoid $\mathfrak{G}$ is *idempotent* if it satisfies $x^2 = x$. In the whole paper we are dealing with idempotent groupoids only. We say that $\mathfrak{G}$ is *medial* (or *entropic*) if it satisfies $(xy)(zt) = (xz)(yt)$. An idempotent commutative groupoid satisfying $xy^2 = x$ is called a *Steiner quasigroup*; an idempotent commutative groupoid satisfying $xy^2 = xy$ is called a *near-semilattice*, and an idempotent associative groupoid satisfying $(xy)z = xz$ is called a *diagonal semigroup* (for details see [12]). We also use the following notation:

- $\mathcal{G}_{\mathrm{I}}$ denotes the class of all idempotent groupoids,
- $\mathcal{G}_{\mathrm{C}}$ denotes the class of all commutative groupoids,
- $\mathcal{G}_{\mathrm{IM}}$ denotes the class of all idempotent medial groupoids.

2000 *Mathematics Subject Classification*: 08A40, 20N02.

In general $\mathcal{C}_{(r(x_1,\dots,x_n))}$ denotes the subclass of the class $\mathcal{C}$ ($\subseteq \mathcal{G}$) satisfying the condition r.

Let us recall the results of J. Dudek summarized in [6] and [3].

THEOREM 1.1 ([6]). *$\mathfrak{G} \in \mathcal{G}_{\mathrm{I}(p_2(\mathfrak{G})\le 1)}$ if and only if $\mathfrak{G}$ belongs to one of the following varieties*:

$$\mathcal{G}_1^1: \quad xy = yx,\ xy^2 = x \qquad (\textit{the variety of Steiner quasigroups});$$

$$\mathcal{G}_2^1: \quad xy = yx,\ xy^2 = xy \qquad (\textit{the variety of near-semilattices}).$$

THEOREM 1.2 ([6]). *$\mathfrak{G} \in \mathcal{G}_{\mathrm{I}(p_2(\mathfrak{G})\le 2)}$ if and only if $\mathfrak{G}$ belongs to one of the following varieties*:

$$\mathcal{G}_1^2: \quad xy^2 = x,\ xy = (xy)x = x(yx),\ {}^2xy = (xy)(yx) = x;$$

$$\mathcal{G}_2^2: \quad xy^2 = y,\ (xy)(yx) = (xy)x = x,\ xy = {}^2xy = y(xy);$$

$$\mathcal{G}_3^2: \quad xy^2 = y,\ (xy)x = x,\ xy = {}^2xy = y(xy) = (yx)(xy);$$

$$\mathcal{G}_4^2: \quad xy^2 = y,\ xy = (yx)y = y(xy) = {}^2xy = (yx)(xy);$$

$$\mathcal{G}_5^2: \quad xy^2 = xy,\ (xy)x = x(yx) = (xy)(yx) = x;$$

$$\mathcal{G}_6^2: \quad xy^2 = xy = (xy)x = x(yx) = {}^2xy = (xy)(yx);$$

$$\mathcal{G}_7^2: \quad xy^2 = yx,\ (xy)x = x(yx) = y,\ {}^2xy = yx,\ (xy)(yx) = x;$$

$$\mathcal{G}_8^2: \quad xy^2 = x,\ xy = yx \quad (\textit{the variety of Steiner quasigroups});$$

$$\mathcal{G}_9^2: \quad xy^2 = yx^2,\ xy = yx,\ xy^2 = xy^3 \quad (\textit{the variety } \mathcal{N}_2)$$

or to one of the varieties $\mathcal{G}_i^{2\mathrm{d}}$ ($i = 1,\dots,9$).

Let us recall that the variety $\mathcal{N}_2$ was described by J. Dudek in [4]. By Theorem 1.2, $\mathcal{G}_{\mathrm{I}(p_2(\mathfrak{G})\le 2)} = \mathcal{G}_1^2 \cup \mathcal{G}_1^{2\mathrm{d}} \cup \dots \cup \mathcal{G}_9^2 \cup \mathcal{G}_9^{2\mathrm{d}}$.

THEOREM 1.3 ([3]). *Let $\mathfrak{G} \in \mathcal{G}_{\mathrm{IM}}$. Then*

(i) *$p_2(\mathfrak{G}) = 1$ if and only if $\mathfrak{G}$ is either a semilattice or an affine space over $GF(3)$.*

(ii) *$p_2(\mathfrak{G}) = 2$ if and only if either $\mathfrak{G}$ is a diagonal semigroup, or $\mathfrak{G}$ represents the sequence ω, or $\mathfrak{G}$ is an affine space over $GF(4)$.*

(The definition of a groupoid representing a sequence is recalled in the next section.)

2. Theorems. Using Cayley's tables we define groupoids needed in the next theorems. In what follows, $\mathfrak{Sl}_2$ denotes a two-element semilattice, and $\mathfrak{Sl}_3$ a three-element semilattice which is not a chain. Some of the groupoids $\mathfrak{G}_i^j$ and $\widehat{\mathfrak{G}}_i^j$ defined below are described in [1] and [2].

$\mathfrak{Sl}_2$	0	1
0	0	0
1	0	1

$\mathfrak{P}_0$	0	1
0	0	0
1	1	1

$\mathfrak{P}_1$	0	1
0	0	1
1	0	1

$\mathfrak{G}_1^2$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

$\widehat{\mathfrak{G}}_1^2$	0	1	2
0	0	2	0
1	1	1	1
2	2	0	2

$\mathfrak{G}_2^2$	0	1	2	3
0	0	2	2	3
1	3	1	2	3
2	0	1	2	0
3	0	1	1	3

$\mathfrak{G}_3^2$	0	1	2	3
0	0	2	2	3
1	3	1	2	3
2	0	1	2	3
3	0	1	2	3

$\widehat{\mathfrak{G}}_3^2$	0	1	2
0	0	2	2
1	2	1	2
2	0	1	2

$\mathfrak{G}_4^2$	0	1	2	3
0	0	2	2	3
1	3	1	2	3
2	3	1	2	3
3	0	2	2	3

$\widehat{\mathfrak{G}}_4^2$	0	1	2
0	0	1	2
1	2	1	2
2	0	1	2

$\mathfrak{G}_5^2$	0	1	2	3
0	0	2	2	0
1	3	1	1	3
2	0	2	2	0
3	3	1	1	3

$\mathfrak{G}_6^2$	0	1	2	3
0	0	2	2	2
1	3	1	3	3
2	2	2	2	2
3	3	3	3	3

$\widehat{\mathfrak{G}}_6^2$	0	1	2
0	0	2	2
1	1	1	1
2	2	2	2

$\mathfrak{Sl}_3$	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

$\mathfrak{G}_7^2$	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

$\mathfrak{G}_8^2$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

$\mathfrak{G}_9^2$	0	1	2	3
0	0	2	3	3
1	2	1	3	3
2	3	3	2	3
3	3	3	3	3

Let $\mathcal{C} \subseteq \mathcal{G}$. We denote by $\mathcal{S}_n(\mathcal{C})$ the following class of groupoids: $\mathfrak{G} \in \mathcal{S}_n(\mathcal{C})$ if and only if $\mathfrak{G}$ is isomorphic to an n-generated subgroupoid of some $\mathfrak{H} \in \mathcal{C}$.

We use the following conventions:

• any two isomorphic groupoids are treated as identical,

• "n-generated" means that the groupoid is generated by a set of cardinality n and it is not generated by any set of cardinality less than n.

For $\mathcal{C} = \{\mathfrak{G}\}$ we write $\mathcal{S}_n(\mathfrak{G})$ instead of $\mathcal{S}_n(\{\mathfrak{G}\})$.

THEOREM 2.1. *For the class* $\mathcal{G}_{\mathrm{I}(p_2(\mathfrak{G})\leq 2)} = \mathcal{G}_1^2 \cup \mathcal{G}_1^{2\mathrm{d}} \cup \ldots \cup \mathcal{G}_9^2 \cup \mathcal{G}_9^{2\mathrm{d}}$ *we have*:

(1) $\mathcal{S}_2(\mathcal{G}_1^2) = \{\mathfrak{P}_0, \widehat{\mathfrak{G}}_1^2, \mathfrak{G}_1^2\}$.
(2) $\mathcal{S}_2(\mathcal{G}_2^2) = \{\mathfrak{P}_1, \mathfrak{G}_2^2\}$.
(3) $\mathcal{S}_2(\mathcal{G}_3^2) = \{\mathfrak{P}_1, \widehat{\mathfrak{G}}_3^2, \mathfrak{G}_3^2\}$.

(4) $\mathcal{S}_2(\mathcal{G}_4^2) = \{\mathfrak{P}_1, \widehat{\mathfrak{G}}_4^2, \mathfrak{G}_4^2\}$.
(5) $\mathcal{S}_2(\mathcal{G}_5^2) = \{\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{G}_5^2\}$.
(6) $\mathcal{S}_2(\mathcal{G}_6^2) = \{\mathfrak{P}_0, \mathfrak{Sl}_2, \mathfrak{Sl}_3, \widehat{\mathfrak{G}}_6^2, \mathfrak{G}_6^2\}$.
(7) $\mathcal{S}_2(\mathcal{G}_7^2) = \{\mathfrak{G}_7^2\}$.
(8) $\mathcal{S}_2(\mathcal{G}_8^2) = \{\mathfrak{G}_8^2\}$.
(9) $\mathcal{S}_2(\mathcal{G}_9^2) = \{\mathfrak{Sl}_2, \mathfrak{Sl}_3, \mathfrak{G}_9^2\}$.

Here $\mathfrak{G}_i^2$ *is a free 2-generated groupoid in the variety* $\mathcal{G}_i^2$ *for* $i = 1, \ldots, 9$. *The dual versions of* (1)–(9) *for the classes* $\mathcal{G}_1^{2\mathrm{d}}, \ldots, \mathcal{G}_9^{2\mathrm{d}}$ *are analogous.*

THEOREM 2.2. *Let* $\mathfrak{G} \in \mathcal{G}_{\mathrm{I}(p_2(\mathfrak{G})\le 2)}$. *Then*:

(1) $\mathfrak{G} \in \mathcal{G}^2_{1(p_2(\mathfrak{G})=2)}$ *if and only if* $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_0, \widehat{\mathfrak{G}}_1^2, \mathfrak{G}_1^2\}$ *and either* $\widehat{\mathfrak{G}}_1^2$ *or* $\mathfrak{G}_1^2$ *can be embedded in* $\mathfrak{G}$.

(2) $\mathfrak{G} \in \mathcal{G}^2_{2(p_2(\mathfrak{G})=2)}$ *if and only if* $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_1, \mathfrak{G}_2^2\}$ *and* $\mathfrak{G}_2^2$ *can be embedded in* $\mathfrak{G}$.

(3) $\mathfrak{G} \in \mathcal{G}^2_{3(p_2(\mathfrak{G})=2)}$ *if and only if* $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_1, \widehat{\mathfrak{G}}_3^2, \mathfrak{G}_3^2\}$ *and either* $\widehat{\mathfrak{G}}_3^2$ *or* $\mathfrak{G}_3^2$ *can be embedded in* $\mathfrak{G}$.

(4) $\mathfrak{G} \in \mathcal{G}^2_{4(p_2(\mathfrak{G})=2)}$ *if and only if* $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_1, \widehat{\mathfrak{G}}_4^2, \mathfrak{G}_4^2\}$ *and* $\widehat{\mathfrak{G}}_4^2$ *can be embedded in* $\mathfrak{G}$.

(5) $\mathfrak{G} \in \mathcal{G}^2_{5(p_2(\mathfrak{G})=2)}$ *if and only if* $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{G}_5^2\}$ *and both* $\mathfrak{P}_0$ *and* $\mathfrak{P}_1$ *can be embedded in* $\mathfrak{G}$.

(5) $\mathfrak{G} \in \mathcal{G}^2_{6(p_2(\mathfrak{G})=2)}$ *if and only if* $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_0, \mathfrak{Sl}_2, \mathfrak{Sl}_3, \widehat{\mathfrak{G}}_6^2, \mathfrak{G}_6^2\}$ *and both* $\mathfrak{P}_0$ *and* $\mathfrak{Sl}_2$ *can be embedded in* $\mathfrak{G}$.

(6) $\mathfrak{G} \in \mathcal{G}^2_{7(p_2(\mathfrak{G})=2)}$ *if and only if* $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{G}_7^2\}$ *and* $\mathfrak{G}_7^2$ *can be embedded in* $\mathfrak{G}$.

(7) *If* $p_2(\mathfrak{G}) = 2$ *then* $\mathfrak{G} \notin \mathcal{G}_8^2$.

(8) $\mathfrak{G} \in \mathcal{G}^2_{9(p_2(\mathfrak{G})=2)}$ *if and only if* $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{Sl}_2, \mathfrak{Sl}_3, \mathfrak{G}_9^2\}$ *and* $\mathfrak{G}_9^2$ *can be embedded in* $\mathfrak{G}$.

The dual versions of (1)–(9) *are also true.*

We say that a groupoid $\mathfrak{G}$ *represents* a sequence $\boldsymbol{a} = (a_0, a_1, a_2, \ldots)$ (finite or not) if $\boldsymbol{a}$ is a subsequence of $\boldsymbol{p}(\mathfrak{G})$ (written $\boldsymbol{a} \subseteq \boldsymbol{p}(\mathfrak{G})$). A sequence $\boldsymbol{a}$ is *representable* (resp. *representable in a class* $\mathcal{C}$) if there exists a groupoid $\mathfrak{G}$ (resp. $\mathfrak{G} \in \mathcal{C}$) such that $\mathfrak{G}$ represents $\boldsymbol{a}$. If $\boldsymbol{a}$ is a finite sequence representable by a given groupoid $\mathfrak{G}$ (resp. $\mathfrak{G} \in \mathcal{C}$) then $\boldsymbol{p}(\mathfrak{G})$ is called an *extension* of $\boldsymbol{a}$ (resp. *extension of* $\boldsymbol{a}$ *in* $\mathcal{C}$). On the class of sequences of cardinal numbers we have a natural partial order: $\boldsymbol{a} \le \boldsymbol{b} \overset{\mathrm{def}}{\iff} \forall i \in \mathbb{N},\ a_i \le b_i$. Take a (finite) sequence $\boldsymbol{a}$ and consider the set $\{\boldsymbol{p}(\mathfrak{G}) \mid \boldsymbol{a} \subseteq \boldsymbol{p}(\mathfrak{G})\}$ (resp. $\{\boldsymbol{p}(\mathfrak{G}) \mid \mathfrak{G} \in \mathcal{C},\ \boldsymbol{a} \subseteq \boldsymbol{p}(\mathfrak{G})\}$) of all extensions of $\boldsymbol{a}$ ordered by $\le$. A least element in this set is called the *minimal extension* of $\boldsymbol{a}$ (resp. *minimal extension* of $\boldsymbol{a}$ *in* $\mathcal{C}$). Combining the results of [1], [5] and [8] with Theorem 2.2 we obtain the simple but interesting observations presented below. We use the standard notations

Σ_1, Σ_2, Σ_3 for the varieties of groupoids representing the sequence $\omega = (0,1,2,\ldots)$ described by J. Płonka. Recall that these varieties are defined by the following identities:

$$\begin{aligned}
\Sigma_1 :&\quad x^2 = x,\ (xy)z = x(yz),\ x(yz) = x(zy);\\
\Sigma_2 :&\quad x^2 = x,\ (xy)z = (xz)y,\ x(yz) = xy,\ xy^2 = xy;\\
\Sigma_3 :&\quad x^2 = x,\ (xy)z = (xz)y,\ x(yz) = xy,\ xy^2 = x
\end{aligned}$$

(for details see [10], pp. 394–395 and [13]).

(1) $\widehat{\mathfrak{G}}_1^2$ and $\mathfrak{G}_1^2$ are both medial proper groupoids in Σ_3 having exactly two essentially binary term operations; $\boldsymbol{p}(\widehat{\mathfrak{G}}_1^2)$ is a minimal extension of the sequence $(0,1,2)$ in $\mathcal{G}_1^2$; $\boldsymbol{p}(\widehat{\mathfrak{G}}_1^2) = \omega$.

(2) $\mathfrak{G}_2^2$ is a nonmedial groupoid having exactly two essentially binary term operations; it is neither a diagonal semigroup nor a member of $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ nor an affine space over $GF(4)$; $\boldsymbol{p}(\mathfrak{G}_2^2)$ is a minimal extension of the sequence $(0,1,2)$ in $\mathcal{G}_2^2$; $p_3(\mathfrak{G}_2^2) \geq 6$.

(3) $\widehat{\mathfrak{G}}_3^2$ and $\mathfrak{G}_3^2$ are both nonmedial groupoids having exactly two essentially binary term operations; they are neither diagonal semigroups nor members of $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ nor affine spaces over $GF(4)$; $\boldsymbol{p}(\widehat{\mathfrak{G}}_3^2)$ is a minimal extension of the sequence $(0,1,2)$ in $\mathcal{G}_3^2$; $p_3(\widehat{\mathfrak{G}}_3^2) = 21$.

(4) $\widehat{\mathfrak{G}}_4^2$ and $\mathfrak{G}_4^2$ are both medial proper groupoids in Σ_2^{d} having exactly two essentially binary term operations; $\boldsymbol{p}(\widehat{\mathfrak{G}}_4^2) = \omega$ is a minimal extension of the sequence $(0,1,2)$ in $\mathcal{G}_4^2$.

(5) $\mathfrak{G}_5^2$ is a diagonal semigroup having exactly two essentially binary term operations; $\boldsymbol{p}(\mathfrak{G}_5^2) = (0,1,2,0,0,\ldots)$ is a minimal extension of the sequence $(0,1,2)$ in $\mathcal{G}_5^2$.

(6) $\widehat{\mathfrak{G}}_6^2$ and $\mathfrak{G}_6^2$ are both medial proper two-generated groupoids in Σ_1; $\boldsymbol{p}(\widehat{\mathfrak{G}}_6^2) = \omega$ is a minimal extension of the sequence $(0,1,2)$ in $\mathcal{G}_6^2$.

(7) $\mathfrak{G}_7^2$ is an affine space over $GF(4)$ having exactly two essentially binary term operations; $\boldsymbol{p}(\mathfrak{G}_7^2)$ is a minimal extension of the sequence $(0,1)$ in $\mathcal{G}_7^2$; $\mathfrak{G}_7^2$ represents the sequence $(0,1,2,7)$ (cf. [8]).

(8) $\mathfrak{G}_8^2$ is an affine space over $GF(3)$ and a medial Steiner quasigroup; it can be embedded in every nontrivial groupoid $\mathfrak{G}$ from $\mathcal{G}_8^2$; $\boldsymbol{p}(\mathfrak{G}_8^2)$ is a minimal extension of the sequence $(0,1)$ in $\mathcal{G}_8^2$; $\mathfrak{G}_8^2$ represents the sequence $(0,1,1,3)$.

(9) $\mathcal{G}_9^2 = \mathcal{N}_2$ so evidently $\mathfrak{G}_9^2$ is Dudek's groupoid and hence it is a free two-generated groupoid in $\mathcal{N}_2$; $\boldsymbol{p}(\mathfrak{G}_9^2)$ is a minimal extension of the sequence $(0,1,2)$ in $\mathcal{G}_9^2$ (and in $\mathcal{G}_{\mathrm{C}}$); $\mathfrak{G}_9^2$ represents the sequence $(0,1,2,10)$.

Analogously we can formulate the dual versions of (1)–(9).

Theorem 2.2 also yields the following remark:

REMARK. The assumption $\mathfrak{G} \in \mathcal{G}_{\mathrm{IM}}$ in Theorem 1.3(ii) cannot be omitted. For example $p_2(\widehat{\mathfrak{G}}_3^2) = 2$ but $\widehat{\mathfrak{G}}_3^2$ is neither a diagonal semigroup nor a member of $\Sigma_1 \cup \Sigma_1^{\mathrm{d}} \cup \Sigma_2 \cup \Sigma_2^{\mathrm{d}} \cup \Sigma_3 \cup \Sigma_3^{\mathrm{d}}$ nor an affine space over $GF(4)$.

3. Proofs of theorems. We say that term operations $p(x_{i_1}, \ldots, x_{i_m})$ and $q(x_{j_1}, \ldots, x_{j_n})$ (built from the binary operation "$\cdot$") are *left-uniform* if the variables x_{i_1} and x_{j_1} really occur in p and q respectively and $x_{i_1} = x_{j_1}$. Analogously p and q are *right-uniform* if $x_{i_m} = x_{j_n}$. We say that the identity $p = q$ is *left-uniform* (resp. *right-uniform*) if the term operations p and q are left-uniform (resp. right-uniform). We say that a variety $\mathcal{V}$ is *left-uniform* (resp. *right-uniform*) if the identities defining it are all left-uniform (resp. right-uniform). Evidently if a variety $\mathcal{V}$ in the class of groupoids is left-uniform (resp. right-uniform) then $\mathfrak{P}_0$ (resp. $\mathfrak{P}_1$) is a nontrivial groupoid in $\mathcal{V}$.

Proof of Theorem 2.1. We present the steps of the proof for the classes $\mathcal{G}_1^2, \ldots, \mathcal{G}_9^2$ only. The proof for the dual classes proceeds analogously and is omitted. We leave it to the reader to check that the models of groupoids presented are members of the given classes. The author has checked it using an unpublished program written by Marek Żabka. We present the details of the proof of the first item only. The parts of the proofs of (2)–(9) which are analogous to the proof of (1) are omitted.

(1) Consider the class $\mathcal{G}_1^2$ and take a free groupoid $\mathfrak{F}$ generated by two free variables x, y. The Cayley table of this groupoid is

$\mathfrak{F}$	x	y	xy	yx
x	x	xy	x	xy
y	yx	y	yx	y
xy	xy	x	xy	x
yx	y	yx	y	yx

The function f defined on the set $\{x, y, xy, yx\}$ as follows: $f(x) = 0$, $f(y) = 1$, $f(xy) = 2$, $f(yx) = 3$ is an isomorphism from $\mathfrak{F}$ onto $\mathfrak{G}_1^2$. The variety $\mathcal{G}_1^2$ is left-uniform so clearly $\mathfrak{P}_0$ is a member of $\mathcal{G}_1^2$. Assume that $\mathfrak{G}$ is a nontrivial member of $\mathcal{G}_1^2$. So there exist $a, b \in G$ such that $a \neq b$. Consider the subgroupoid $\mathfrak{G}(a, b) = (G(a, b), \cdot)$ of $\mathfrak{G}$ generated by $\{a, b\}$. Then $G(a, b) = \{a, b, ab, ba\}$ and $\mathrm{card}(G(a, b)) \leq 4$. Evidently $a \neq ba$. Indeed, suppose that $a = ba$. Then $a = a^2 = ba^2 = b$, a contradiction. Assume that $a = ab$ and $b = ba$. Then $\mathrm{card}(G(a, b)) = 2$ and the Cayley table of $\mathfrak{G}(a, b)$ is

$\mathfrak{G}(a, b)$	a	b
a	a	a
b	b	b

so $\mathfrak{G}(a,b) = \mathfrak{P}_0$. Assume now that $\mathfrak{G}$ is a proper groupoid. Then the term operation xy is not a projection. Thus there exist $a, b \in G$ such that $a \neq ab$. Then evidently $a \neq b$. Suppose that $a = ba$. Then $a = a^2 = ba^2 = b$, a contradiction. Thus $a \neq ba$. Analogously $b \neq ab$. Suppose now that $ab = ba$. Then $a = ab^2 = (ba)b = ba$, a contradiction. So we have two possibilities only: $b = ba$ or $b \neq ba$. Assume that $b = ba$. Then $\mathrm{card}(G(a,b)) = 3$ and the Cayley table of $\mathfrak{G}(a,b)$ is

$\mathfrak{G}(a,b)$	a	b	ab
a	a	ab	a
b	b	b	b
ab	ab	a	ab

so $\mathfrak{G}(a,b) = \widehat{\mathfrak{G}}_1^2$. Assume now that $b \neq ba$. Then $\mathrm{card}(G(a,b)) = 4$ and the Cayley table of $\mathfrak{G}(a,b)$ is

$\mathfrak{G}(a,b)$	a	b	ab	ba
a	b	ab	a	ab
b	ba	b	ba	b
ab	ab	a	ab	a
ba	b	ba	b	ba

so $\mathfrak{G}(a,b) = \mathfrak{G}_1^2$. Evidently $\widehat{\mathfrak{G}}_1^2$ is a homomorphic image of $\mathfrak{G}_1^2$.

(2) The variety $\mathcal{G}_2^2$ is right-uniform so $\mathfrak{P}_1 \in \mathcal{G}_2^2$. Let $\mathfrak{G} \in \mathcal{G}_2^2$ be nontrivial. Then there exist $a, b \in G$ such that $a \neq b$. Evidently $ab \neq ba$. Suppose that $a = ab$. Then $a = ab = ab^2 = b$, a contradiction. If $a = ba$, then $ab = (ba)b = b$ and $\mathfrak{G}(a,b) = \mathfrak{P}_1$. Assume now that $\mathfrak{G}$ is a proper groupoid. Then there exist $a, b \in G$ such that $a \neq ba$. Then $G(a,b) = \{a, b, ab, ba\}$ is a four-element set and $\mathfrak{G}(a,b) = \mathfrak{G}_2^2$.

(3) The variety $\mathcal{G}_3^2$ is right-uniform so $\mathfrak{P}_1 \in \mathcal{G}_3^2$. Let $\mathfrak{G} \in \mathcal{G}_3^2$ be nontrivial. Then there exist $a, b \in G$ such that $a \neq b$. Hence $G(a,b) = \{a, b, ab, ba\}$ where $a \neq ab$. If $a = ba$ then $ab = b$ and $\mathfrak{G}(a,b) = \mathfrak{P}_1$. Assume that $\mathfrak{G}$ is proper. Then there exist $a, b \in G$ such that $a \neq ba$. Hence $a \neq b$, $a \neq ab$, $a \neq ba$, $b \neq ba$, $b \neq ab$. If $ab = ba$ then $\mathfrak{G}(a,b) = \widehat{\mathfrak{G}}_3^2$. If $ab \neq ba$ then $\mathfrak{G}(a,b) = \mathfrak{G}_3^2$.

(4) The variety $\mathcal{G}_4^2$ is right-uniform so $\mathfrak{P}_1 \in \mathcal{G}_4^2$. Let $\mathfrak{G} \in \mathcal{G}_4^2$ be nontrivial. Let $a, b \in G$ be such that $a \neq b$. If $a = ba$ and $b = ab$ then $\mathfrak{G}(a,b) = \mathfrak{P}_1$. Assume that $\mathfrak{G}$ is proper. Take $a, b \in G$ such that $a \neq ba$. Then a, b, ba are all distinct. Moreover $a \neq ab$. Suppose that $ab = ba$. Then $ba = (ab)a = ba^2 = a$, a contradiction. Assume that $ab = b$. Then $\mathfrak{G}(a,b) = \widehat{\mathfrak{G}}_4^2$. If $ab \neq b$ then $\mathfrak{G}(a,b) = \mathfrak{G}_4^2$.

(5) The variety $\mathcal{G}^2_5$ is left-uniform and right-uniform simultaneously. Thus $\mathfrak{P}_0, \mathfrak{P}_1 \in \mathcal{G}^2_5$. Let $\mathfrak{G} \in \mathcal{G}^2_5$ be nontrivial. Let $a, b \in G$, $a \neq b$. Consider the groupoid $\mathfrak{G}(a,b)$. We have $G(a,b) = \{a, b, ab, ba\}$. Evidently $ab \neq ba$. If $a = ab$, $b = ba$ then $\mathfrak{G}(a,b) = \mathfrak{P}_0$. If $a = ba$, $b = ab$ then $\mathfrak{G}(a,b) = \mathfrak{P}_1$. If $\mathfrak{G}$ is proper then (a) there exist $a, b \in G$ such that $a \neq ab$ and (b) there exist $a, b \in G$ such that $b \neq ab$. (a) Assume that $a \neq ab$. Then $b \neq ba$. Thus we have two possibilities only: either $b = ab$ and $\mathfrak{G}(a,b) = \mathfrak{P}_1$, or $b \neq ab$ and $\mathfrak{G}(a,b) = \mathfrak{G}^2_5$. (b) For the case $b \neq ab$ we analogously conclude that either $\mathfrak{G}(a,b) = \mathfrak{P}_0$ or $\mathfrak{G}(a,b) = \mathfrak{G}^2_5$.

(6) The variety $\mathfrak{G}^2_6$ is left-uniform. Thus $\mathfrak{P}_0 \in \mathcal{G}^2_6$. Also $\mathfrak{Sl}_2 \in \mathcal{G}^2_6$. Let $\mathfrak{G} \in \mathcal{G}^2_6$ be nontrivial. Let $a, b \in G$, $a \neq b$. If $a = ab$ and $b = ba$ then $\mathfrak{G}(a,b) = \mathfrak{P}_0$. Assume that $\mathfrak{G}$ is proper. Let $a, b \in G$, $a \neq ab$. Suppose that $b = ab$. Then $ba = b$ and $\mathfrak{G}(a,b) = \mathfrak{Sl}_2$. Assume that $b \neq ab$. Then either $ba = b$ and $\mathfrak{G}(a,b) = \widehat{\mathfrak{G}}^2_6$, or $ba = ab$ and $\mathfrak{G}(a,b) = \mathfrak{Sl}_3$, or $ba \neq ab$ and $\mathfrak{G}(a,b) = \mathfrak{G}^2_6$.

(7) Consider a nontrivial groupoid $\mathfrak{G} \in \mathcal{G}^2_7$. Let $a, b \in G$, $a \neq b$. Then a, b, ab, ba are all distinct. Hence $\mathfrak{G}(a,b) = \mathfrak{G}^2_7$.

(8) Let $\mathfrak{G} \in \mathcal{G}^2_8$ be nontrivial. Then $\mathfrak{G}(a,b) = \mathfrak{G}^2_8$.

(9) This item is a consequence of the results presented in [4].

Proof of Theorem 2.2. (1) Let $\mathfrak{G} \in \mathcal{G}^2_1$ and $p_2(\mathfrak{G}) = 2$. Then by Theorem 2.1(1), $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_0, \widehat{\mathfrak{G}}^2_1, \mathfrak{G}^2_1\}$. Since $p_2(\mathfrak{G}) = 2$, $\mathfrak{G}$ is a proper groupoid. As in the proof of Theorem 2.1(1) we find that either $\widehat{\mathfrak{G}}^2_1$ or $\mathfrak{G}^2_1$ can be embedded in $\mathfrak{G}$. Conversely, if $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_0, \widehat{\mathfrak{G}}^2_1, \mathfrak{G}^2_1\}$ then by Theorem 2.1(1), $\mathfrak{G} \in \mathcal{G}^2_1$. Assume that either $\widehat{\mathfrak{G}}^2_1$ or $\mathfrak{G}^2_1$ can be embedded in $\mathfrak{G}$. Then $2 = p_2(\widehat{\mathfrak{G}}^2_1) = p_2(\mathfrak{G}^2_1) \leq p_2(\mathfrak{G}) \leq 2$. So $p_2(\mathfrak{G}) = 2$.

(2)–(3) and (7). The proofs are analogous to the proof of (1) and are omitted.

(4) The proof is similar to that of (1). Moreover $\widehat{\mathfrak{G}}^2_4$ can be embedded in $\mathfrak{G}^2_4$.

(5) Let $\mathfrak{G} \in \mathcal{G}^2_5$ and $p_2(\mathfrak{G}) = 2$. Then by Theorem 2.1(5), $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{G}^2_5\}$. As $\mathfrak{G}$ is a proper groupoid, as in Theorem 2.1(5) we conclude that either (a) $\mathfrak{P}_1$ or $\mathfrak{G}^2_5$ can be embedded in $\mathfrak{G}$, or (b) $\mathfrak{P}_0$ or $\mathfrak{G}^2_5$ can be embedded in $\mathfrak{G}$. Observe that both $\mathfrak{P}_0$ and $\mathfrak{P}_1$ can be embedded in $\mathfrak{G}^2_5$. Evidently $\mathcal{S}_2(\mathfrak{G}) \not\subseteq \{\mathfrak{P}_0\}$ and $\mathcal{S}_2(\mathfrak{G}) \not\subseteq \{\mathfrak{P}_1\}$. Thus in either case ((a) or (b)) both $\mathfrak{P}_0$, $\mathfrak{P}_1$ can be embedded in $\mathfrak{G}$. Conversely, assume now that $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{G}^2_5\}$ and both $\mathfrak{P}_0$, $\mathfrak{P}_1$ can be embedded in $\mathfrak{G}$. By Theorem 2.1(5), $\mathfrak{G} \in \mathcal{G}^2_5$. The term operation xy evidently depends on both variables x and y (xy depends on x in $\mathfrak{P}_0$ and on y in $\mathfrak{P}_1$). Moreover xy is noncommutative. Thus $p_2(\mathfrak{G}) \geq 2$. As $\mathfrak{G} \in \mathcal{G}^2_5$ we have $p_2(\mathfrak{G}) = 2$.

(6) Assume that $\mathfrak{G} \in \mathcal{G}_6^2$ and $p_2(\mathfrak{G}) = 2$. Then by Theorem 2.1(6), $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_0, \mathfrak{Sl}_2, \mathfrak{Sl}_3, \widehat{\mathfrak{G}}_6^2, \mathfrak{G}_6^2\}$ and (a) there exist $a, b \in G$ such that $ab \neq ba$ and (b) there exist $a, b \in G$ such that $ab \neq a$.

(a) Assume that $a, b \in G$ and $ab \neq ba$. Suppose that $a = ab$. Then either $b = ba$ and $\mathfrak{G}(a, b) = \mathfrak{P}_0$, or $b \neq ba$ and $\mathfrak{G}(a, b) = \widehat{\mathfrak{G}}_6^2$. Assume that $a \neq ab$. Then $b \neq ab$ (if $b = ab$, then $ba = ab^2 = ab$, a contradiction). So we have two possibilities: either $b = ba$ and $\mathfrak{G}(a, b) = \widehat{\mathfrak{G}}_6^2$, or $b \neq ba$ and $\mathfrak{G}(a, b) = \mathfrak{G}_6^2$.

(b) Assume now that $a, b \in G$ and $ab \neq a$. If $ab = b$ then $ba = b(ab) = b$ and $\mathfrak{G}(a, b) = \mathfrak{Sl}_2$. Assume that $ab \neq b$. If $ab = ba$ then $\mathfrak{G}(a, b) = \mathfrak{Sl}_3$. If $ab \neq ba$ we have two possibilities only: either $ba = b$ and $\mathfrak{G}(a, b) = \widehat{\mathfrak{G}}_6^2$, or $ba \neq b$ and $\mathfrak{G}(a, b) = \mathfrak{G}_6^2$. Evidently $\widehat{\mathfrak{G}}_6^2$ can be embedded in $\mathfrak{G}_6^2$ and both $\mathfrak{P}_0$ and $\mathfrak{Sl}_2$ can be embedded in $\widehat{\mathfrak{G}}_6^2$. Thus in each case both $\mathfrak{P}_0$ and $\mathfrak{Sl}_2$ can be embedded in $\mathfrak{G}$.

Assume now that $\mathcal{S}_2(\mathfrak{G}) \subseteq \{\mathfrak{P}_0, \mathfrak{Sl}_2, \mathfrak{Sl}_3, \widehat{\mathfrak{G}}_6^2, \mathfrak{G}_6^2\}$ and both $\mathfrak{P}_0$ and $\mathfrak{Sl}_2$ can be embedded in $\mathfrak{G}$. Then by Theorem 2.1(6), $\mathfrak{G} \in \mathcal{G}_6^2$ and the term operation xy is noncommutative and depends on both variables. Thus $p_2(\mathfrak{G}) = 2$.

(8) If $\mathfrak{G} \in \mathcal{G}_8^2$ then $p_2(\mathfrak{G}) = 1$.

(9) This is a consequence of [4].

Acknowledgments. The author is greatly indebted to Professor Andrzej Kisielewicz for his valuable remarks and suggestions.

REFERENCES

[1] J. Berman, *Free spectra of 3-element algebras*, in: Universal Algebra and Lattice Theory (Puebla, 1982), Lecture Notes in Math. 1004, Springer, Berlin, 1983, 10–53.

[2] B. Csákány, *Three-element groupoids with minimal clones*, Acta Sci. Math. (Szeged) 45 (1983), 111–117.

[3] J. Dudek, *Medial idempotent groupoids I*, Czechoslovak Math. J. 41 (116) (1991), 249–259.

[4] —, *On minimal extension of sequences*, Algebra Universalis 23 (1986), 308–312.

[5] —, *Polynomial characterization of some idempotent algebras*, Acta Sci. Math. (Szeged) 50 (1986), 39–49.

[6] —, *Small idempotent clones I*, Czechoslovak Math. J. 48 (123) (1998), 105–118.

[7] —, *The minimal extension of sequences II. On problem 17 of Grätzer and Kisielewicz*, Period. Math. Hungar. 34 (1998), 177–183.

[8] J. Dudek and J. Tomasik, *Affine spaces over* $GF(4)$, Algebra Universalis 36 (1996), 279–285.

[9] J. Gałuszka, *A characterization of commutative and idempotent groupoids*, Discussiones Math. 15 (1995), 121–125.

[10] G. Grätzer, *Universal Algebra*, Springer, Berlin, 1979.

[11] G. Grätzer and A. Kisielewicz, *A survey of some open problems on* p_n*-sequences and free spectra of algebras and varieties*, in: Universal Algebra and Quasigroup Theory, A. Romanowska and J. D. H. Smith (eds.), Heldermann, Berlin, 1992, 57–88.

[12] J. Płonka, *Diagonal algebras*, Fund. Math. 58 (1966), 309–321.
[13] —, *On algebras with n distinct n-ary operations*, Algebra Universalis 1 (1971), 73–79.

Institute of Mathematics
Silesian University of Technology
44-100 Gliwice, Poland
E-mail: jagalusz@zeus.polsl.gliwice.pl

Received 19 July 2000;
revised 12 March 2001 (3956)