THE MEAN VALUE OF $|L(k, \chi)|^2$
AT POSITIVE RATIONAL INTEGERS $k \geq 1$

BY

STÉPHANE LOUBOUTIN (Marseille)

Abstract. Let $k \geq 1$ denote any positive rational integer. We give formulae for the sums

$$S_{\text{odd}}(k, f) = \sum_{\chi(-1) = -1} |L(k, \chi)|^2$$

(where $\chi$ ranges over the $\phi(f)/2$ odd Dirichlet characters modulo $f > 2$) whenever $k \geq 1$ is odd, and for the sums

$$S_{\text{even}}(k, f) = \sum_{\chi(-1) = +1} |L(k, \chi)|^2$$

(where $\chi$ ranges over the $\phi(f)/2$ even Dirichlet characters modulo $f > 2$) whenever $k \geq 1$ is even.

1. Introduction. The aim of this paper is to prove the following two results:

**Theorem 1.** Let $f > 2$, $k \geq 1$ and $l \geq 1$ denote rational integers. Set

$$\phi_l(f) = \prod_{p \mid f} (1 - 1/p^l) \quad \text{and} \quad \phi(f) = f\phi_1(f).$$

Then for any $k \geq 1$ there exists a polynomial $R_k(X) = \sum_{l=0}^{2k} r_{k,l} X^l$ of degree $2k$ with rational coefficients such that for all $f > 2$ we have

$$\frac{2}{\phi(f)} \sum_{\chi(-1) = (-1)^k} |L(k, \chi)|^2 = \frac{\pi^{2k}}{2((k - 1)!)^2} \sum_{l=1}^{2k} r_{k,l} \phi_l(f) f^{l-2k}$$

where $\chi$ ranges over the $\phi(f)/2$ Dirichlet characters modulo $f$ such that $\chi(-1) = (-1)^k$.

2000 Mathematics Subject Classification: Primary 11M06, 11M20, 11R18.

Key words and phrases: Dirichlet $L$-functions, characters.
THEOREM 2. Assume $f > 2$ and $k \geq 1$. Let

\[ M_{\text{odd}}(k, f) = \frac{2}{\phi(f)} \sum_{\chi(-1) = -1} |L(k, \chi)|^2 \quad (k \geq 1 \text{ odd}), \]

\[ M_{\text{even}}(k, f) = \frac{2}{\phi(f)} \sum_{\chi(-1) = +1} |L(k, \chi)|^2 \quad (k \geq 2 \text{ even}) \]

denote the mean value of $|L(k, \chi)|^2$ where $\chi$ ranges over the $\phi(f)/2$ Dirichlet characters modulo $f$ such that $\chi(-1) = (-1)^{k}$. Then

\[ M_{\text{odd}}(1, f) = \frac{\pi^2}{6} \phi_2(f) - \frac{\pi^2 \phi_1(f)}{2f}, \]

\[ M_{\text{even}}(2, f) = \frac{\pi^4}{90} \phi_4(f) + \frac{\pi^4}{9f^2} \phi_2(f), \]

\[ M_{\text{odd}}(3, f) = \frac{\pi^6}{945} \phi_6(f) - \frac{\pi^6}{45f^4} \phi_2(f), \]

\[ M_{\text{even}}(4, f) = \frac{\pi^8}{9450} \phi_8(f) + \frac{\pi^8}{2025f^4} \phi_4(f) + \frac{4\pi^8}{567f^6} \phi_2(f). \]

To prove these results, we follow the same line of reasoning as for proving [Lou1, Th. 2] (which is nothing else but our formula for $M_{\text{odd}}(1, f)$ and generalizes [Wal] who only considered the case of prime modulus $f$). First, in (1) we generalize [Lou1, Th. 1] by giving a formula for the values $L(k, \chi)$ for the $\chi$’s that satisfy $\chi(-1) = (-1)^{k}$. Second, we generalize [Lou1, Lemma (a)] in Proposition 4. Third, we prove in Proposition 5 that Theorem 1 holds with the polynomials $R_k(X)$ defined in (5)–(7). Finally, Theorem 2 follows from Theorem 1 and the computation of the $R_i(X)$ for $1 \leq i \leq 4$.

2. Proof of the results

2.1. Formulae for $M_{\text{odd}}(k, f)$ and $M_{\text{even}}(k, f)$

PROPOSITION 3. Let $k \geq 1$ and $f > 2$ denote positive rational integers. Let $\cot^{(k)}(x)$ denote the $k$th derivative of $x \mapsto \cot(x) = \cos(x)/\sin(x)$.

1. If $\chi$ is a Dirichlet character modulo $f > 2$ and if $\chi(-1) = (-1)^{k}$ then

\[ L(k, \chi) = \frac{(-1)^{k-1} \pi^k}{2f^k(k-1)!} \sum_{l=1}^{f-1} \chi(l) \cot^{(k-1)}(\pi l/f). \]

2. We have

\[ \sum_{\chi(-1) = (-1)^{k}} |L(k, \chi)|^2 = \frac{\pi^{2k} \phi(f)}{4((k-1)!)^2f^{2k}} \sum_{l=1}^{f-1} \left( \cot^{(k-1)}\left(\frac{\pi l}{f}\right) \right)^2 \]
where the first sum ranges over all the $\phi(f)/2$ Dirichlet characters modulo $f > 2$ which satisfy $\chi(-1) = (-1)^k$ and the second sum ranges over integers $l$ relatively prime to $f$.

Proof. Recall that for $0 < b < 1$ we have

$$\pi \cot(\pi b) = \sum_{n \geq 0} \left( \frac{1}{n + b} - \frac{1}{n + 1 - b} \right).$$

Therefore, for $k \geq 1$ we have

$$\frac{(-1)^{k-1} \pi^k}{(k-1)!} \cot^{(k-1)}(\pi b) = \sum_{n \geq 0} \left( \frac{1}{(n + b)^k} + \frac{(-1)^k}{(n + 1 - b)^k} \right).$$

Now, for $b > 0$ we set $\zeta(s, b) = \sum_{n \geq 0} (n + b)^{-s}$ for $\Re(s) > 1$ (Hurwitz’s zeta function). For $\Re(s) > 1$ we have

$$L(s, \chi) = f^{-s} \sum_{l=1}^{f-1} \chi(l) \zeta(s, l/f)$$

$$= f^{-s} \sum_{l=1}^{f-1} \chi(f - l) \zeta(s, 1 - (l/f))$$

$$= f^{-s} \chi(-1) \sum_{l=1}^{f-1} \chi(l) \zeta(s, 1 - (l/f))$$

$$= \frac{f^{-s}}{2} \sum_{l=1}^{f-1} \chi(l) (\zeta(s, l/f) + \chi(-1) \zeta(s, 1 - (l/f)))$$

$$= \frac{f^{-s}}{2} \sum_{l=1}^{f-1} \chi(l) \sum_{n \geq 0} \left( \frac{1}{(n + (l/f))^s} + \frac{\chi(-1)}{(n + 1 - (l/f))^s} \right).$$

Moreover, if $\chi(-1) = -1$ then it is easily seen that this last equality is valid for $\Re(s) > 0$. Therefore, if $k \geq 1$ and $\chi(-1) = (-1)^k$ then, using (3) and (4), we do obtain (1). Let us recall that for $f > 2$ and $\varepsilon = \pm 1$ we have

$$\sum_{\chi(-1) = \varepsilon} \chi(l) \chi(l') = \frac{\phi(f)}{2} \langle l, l' \rangle_{\varepsilon}$$

where

$$\langle l, l' \rangle_{\varepsilon} := \begin{cases} 1 & \text{if } l' \equiv l \pmod{f} \text{ and } \gcd(l, f) = 1, \\
\varepsilon & \text{if } l' \equiv -l \pmod{f} \text{ and } \gcd(l, f) = 1, \\
0 & \text{otherwise.} \end{cases}$$

We deduce the second point from these relations, (1) and $\cot^{(k)}(-x) = (-1)^{k-1} \cot^{(k)}(x)$.  ■
2.2. Evaluation of the sums \( \sum_{l=1}^{d-1} (\cot^{(k-1)}(\pi l/d))^2 \). The derivative of \( \cot \) is \(-1 - \cot^2\). Therefore, if we define inductively polynomials \( P_k(X) \in \mathbb{Z}[X] \) by means of \( P_1(X) = X \) and \( P_{k+1}(X) = (X^2 + 1)P'_k(X) \), then \( \cot^{(k-1)}(X) = (-1)^{k-1}P_k(\cot) \) and \( (\cot^{(k-1)})^2 = Q_k(\cot^2) \) where \( Q_k(X) = \sum_{l=0}^{k} q_{k,l}X^l \in \mathbb{Z}[X] \) is defined by \( Q_k(X^2) = (P_k(X))^2 \).

For \( j \geq 1 \) we define polynomials \( s_j(X) \in \mathbb{Q}[Y] \) of degree \( 2j \) by

\[
s_j(X) = \frac{(X-1)(X-2)(X-3)\ldots(X-2j)}{(2j+1)!} \quad (j \geq 1),
\]

and use them to define inductively on \( j \geq 1 \) polynomials \( F_j(X) \in \mathbb{Q}[X] \) of degree \( \leq 2j \) by means of

\[
F_j(X) = s_1(X)F_{j-1}(X) + \ldots + (-1)^{j-1}s_{j-1}(X)F_1(X) + (-1)^j js_j(X) = 0.
\]

We finally set

\[
R_k(X) = q_{k,0}(X-1) + 2 \sum_{j=1}^{k} q_{k,j}F_j(X) \in \mathbb{Q}[X].
\]

Notice that for any \( k \geq 1 \) the degree of \( R_k(X) \) is \( \leq 2k \) and that \( R_k(1) = 0 \) (this is because (5) yields \( s_j(1) = 0 \) for all \( j \geq 1 \) and (6) then yields \( F_j(1) = 0 \) for all \( j \geq 1 \)). We will write

\[
R_k(X) = \sum_{l=0}^{2k} r_{k,l}X^l
\]

and we will prove that these \( R_k(X) \) are the polynomials which appear in the statement of Theorem 1. Using (7), (6) and (5) we computed Table 1 opposite, according to which we deduce Theorem 2 from Theorem 1.

**Proposition 4.** Let \( k \geq 1 \) be a given rational integer and let \( R_k(X) \) be as in (7). Then for any rational integer \( d > 1 \) we have

\[
R(k,d) := \sum_{l=1}^{d-1} (\cot^{(k-1)}(\pi l/d))^2 = R_k(d).
\]

**Proof.** Let \( k \geq 1 \) be a given integer. Let \( j \) range from 1 to \( k \). Let \( D \) range over the integers \( D \geq k \). We set

\[
S_j(X_1, \ldots, X_D) = \sum_{a=1}^{D} X_a^j
\]

and

\[
\sigma_j(X_1, \ldots, X_D) = \sum_{1 \leq a_1 < \ldots < a_j \leq D} X_{a_1} \ldots X_{a_j}.
\]
For each \( j \in \{1, \ldots, k\} \) there exists \( f_j = f_j(X_1, \ldots, X_j) \in \mathbb{Z}[X_1, \ldots, X_j] \) such that for all \( D \geq k \) we have

\[
S_j(X_1, \ldots, X_D) = f_j(\sigma_1(X_1, \ldots, X_D), \ldots, \sigma_i(X_1, \ldots, X_D)).
\]

Newton’s formulae

\[
f_j - X_1 f_{j-1} + X_2 f_{j-2} + \ldots + (-1)^{j-1} X_{j-1} f_1 + (-1)^j j X_j = 0 \quad \text{(for } j \leq D \text{)}
\]

\( (\text{and } f_1(X_1) = X_1) \) allow us to compute inductively these polynomials \( f_j = f_j(X_1, \ldots, X_j) \) for \( 1 \leq j \leq k \). In particular, the polynomials defined in (6) are given by

\[
F_j(X) = f_j(s_1(X), \ldots, s_j(X)) \quad (j \geq 1).
\]

According to (7), (10) and (8), to complete the proof, we only have to show that if \( k \geq 1 \) is given, then for any \( d > 1 \) we have
\( R(k, d) = q_{k,0}(d - 1) + 2 \sum_{j=1}^{k} q_{k,j}f_j(s_1(d), \ldots, s_j(d)). \) (11)

Set \( d' = (d - 1)/2 \) if \( d \geq 3 \) is odd, \( d/2 \) if \( d \geq 2 \) is even. Choose \( D \) such that \( D \geq d' \) and \( D \geq k \), set

\[
\alpha_l(d) = \begin{cases} 
\cot^2(\pi l/d) & \text{for } 1 \leq l \leq d', \\
0 & \text{for } d' < l \leq D,
\end{cases}
\]

and for \( 1 \leq j \leq k \) set

\[
\sigma_j(d) = \sigma_j(\alpha_1(d), \ldots, \alpha_D(d))
\]

and

\[
S_j(d) := S_j(\alpha_1(d), \ldots, \alpha_D(d)) = f_j(\sigma_1(d), \ldots, \sigma_j(d)) \quad \text{(by (9)).}
\]

Since for \( d > 1 \) and \( j \geq 1 \) we have

\[
\sum_{l=1}^{d-1} \cot^2(\pi l/d) = 2 \sum_{l=1}^{D} (\alpha_l(d))^j = 2S_j(d) = 2f_j(\sigma_1(d), \ldots, \sigma_j(d)),
\]

we obtain

\( R(k, d) = q_{k,0}(d - 1) + 2 \sum_{j=1}^{k} q_{k,j}f_j(\sigma_1(d), \ldots, \sigma_j(d)). \) (12)

Therefore, according to (11) and (12), it only remains to show that for any \( d > 1 \) we have \( \sigma_j(d) = s_j(d) \) for \( 1 \leq j \leq k \). Since the \( \cot(\pi l/d) \) for \( 1 \leq l \leq d - 1 \) are the roots of the polynomial \((X + i)^d - (X - i)^d)/(2id) = X^{d-1} - s_1(d)X^{d-3} + s_2(d)X^{d-5} - \ldots \) (where \( i^2 = -1 \)), we see that the \( \alpha_l(d) \) for \( 1 \leq l \leq D \) are the roots of the polynomial \( X^D - s_1(d)X^{D-1} + s_2(d)X^{D-2} - \ldots \) (for \( s_j(d) = 0 \) for \( 2j \geq d \)), and we do obtain \( \sigma_j(d) = s_j(d) \).

2.3. Proof of the main theorem

Proposition 5 (proves Theorem 1). Let \( \mu \) denote Möbius’ function.

1. For \( f > 2 \) and \( k \geq 1 \) we have

\( \sum_{\chi(-1) = (-1)^k} |L(k, \chi)|^2 = \frac{\phi(f)}{4f^{2k}} \left( \frac{\pi^k}{(k-1)!} \right)^2 \sum_{d|f \atop d > 1} \mu(f/d)R_k(d). \) (13)

2. If \( R_k(X) = \sum_{l=0}^{2k} r_{k,l}X^l \) is a polynomial of degree \( \leq 2k \) such that \( R_k(1) = 0 \) then

\( \sum_{\substack{d|f \atop d > 1}} \mu(f/d)R_k(d) = \sum_{d|f} \mu(f/d)R_k(d) = \sum_{l=1}^{2k} r_{k,l}\varphi_l(f)f^l. \) (14)
Proof. Only the first point needs a proof. Since \( \sum_{d|n} \mu(d) = 1 \) if \( n = 1 \) and 0 if \( n > 1 \), we deduce (13) from (2) and the following computation:

\[
\sum_{a=1}^{f-1} \left( \cot^{(k-1)} \left( \frac{\pi a}{f} \right) \right)^2 = \sum_{a=1}^{f-1} \left( \cot^{(k-1)} \left( \frac{\pi a}{f} \right) \right)^2 \left( \sum_{d|a} \mu(d) \right).
\]

\[
= \sum_{d|f, d<f} \mu(d) \sum_{b=1}^{f/d-1} \left( \cot^{(k-1)} \left( \frac{\pi db}{f} \right) \right)^2
\]

\[
= \sum_{d|f, d<f} \mu(d) R_k \left( \frac{f}{d} \right) = \sum_{d|f, d>f} \mu \left( \frac{f}{d} \right) R_k(d).
\]

3. Remarks. 1. According to our proof, the polynomial

\[
((2k+1)!)^{2k} R_k(X) \in \mathbb{Z}[X]
\]

has integral coefficients. Therefore, \( ((2k+1)!)^{2k} R(k, d) = ((2k+1)!)^{2k} R_k(d) \) is a rational integer (see (8)), and any entry \( R_k(X) \) of Table 1 can be easily checked: verify that the polynomial \( ((2k+1)!)^{2k} R_k(X) \) of degree \( 2k \) has integral coefficients and that the \( 2k+1 \) rational integers \( ((2k+1)!)^{2k} R(k, d) - ((2k+1)!)^{2k} R_k(d) \) are equal to zero for \( 1 \leq d \leq 2k + 1 \).

2. After the publication of [Lou1], Qi Minggao sent us another proof of [Lou1, Th. 2] (see [QiM]). However, his proof was much more complicated than ours and cannot be generalized for computing the mean value of \( |L(k, \chi)|^2 \) where \( \chi \) ranges over the Dirichlet characters modulo \( f \) such that \( \chi(-1) = (-1)^k \).

3. Since the values at non-positive integers of Dirichlet \( L \)-functions are generalized Bernoulli numbers (see [Was, Th. 4.2]), and since according to their functional equations these values at non-positive integers are related to their values at positive integers, one might think it would be easier to prove Theorem 1 by dealing with these values at non-positive integers. However, this approach is doomed to failure because functional equations are valid only for primitive characters, and according to [Lou3], there is no hope for ever finding similar simple formulae for the mean value of \( |L(k, \chi)|^2 \) where \( \chi \) ranges over the primitive Dirichlet characters modulo \( f \) such that \( \chi(-1) = (-1)^k \).

4. Whereas for any positive rational integer \( n \geq 1 \) asymptotic expansions exist of the type

\[
\sum_{\chi \neq 1} |L(1, \chi)|^2 = \frac{\pi^2}{6} p - \log^2 p + \sum_{k=0}^{n-1} a_k p^{-k} + O(p^{-n})
\]
for mean values of primitive \(L\)-functions modulo primes \(p \geq 3\) (see [KM]), there is no known formula for such mean values. Hence, there is no hope of finding formulae for the mean values

\[
M(k, f) := \frac{1}{\phi(f)} \sum_{\chi} |L(k, \chi)|^2 = \frac{1}{2} M_{\text{odd}}(k, f) + \frac{1}{2} M_{\text{even}}(k, f)
\]

where \(\chi\) ranges over the \(\phi(f)\) Dirichlet characters modulo \(f > 2\) (and where \(k \geq 1\) is a positive rational integer). However, asymptotic formulae similar to (15) for these \(M(k, \chi)\) are given in [KM].

**REFERENCES**


Institut de Mathématiques de Luminy, UPR 9016
163, avenue de Luminy
Case 907
13288 Marseille Cedex 9, France
E-mail: loubouti@iml.univ-mrs.fr

Received 25 October 2000;
revised 20 March 2001