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THE MEAN VALUE OF $|L(k,\chi)|^2$ AT POSITIVE RATIONAL INTEGERS $k \geq 1$

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Abstract. Let $k \geq 1$ denote any positive rational integer. We give formulae for the sums

$$S_{\mathrm{odd}}(k,f) = \sum_{\chi(-1)=-1} |L(k,\chi)|^2$$

(where χ ranges over the $\phi(f)/2$ odd Dirichlet characters modulo $f > 2$) whenever $k \geq 1$ is odd, and for the sums

$$S_{\mathrm{even}}(k,f) = \sum_{\chi(-1)=+1} |L(k,\chi)|^2$$

(where χ ranges over the $\phi(f)/2$ even Dirichlet characters modulo $f > 2$) whenever $k \geq 1$ is even.

1. Introduction. The aim of this paper is to prove the following two results:

THEOREM 1. *Let $f > 2$, $k \geq 1$ and $l \geq 1$ denote rational integers. Set*

$$\phi_l(f) = \prod_{p \mid f} (1 - 1/p^l) \quad \textit{and} \quad \phi(f) = f\phi_1(f).$$

Then for any $k \geq 1$ there exists a polynomial $R_k(X) = \sum_{l=0}^{2k} r_{k,l} X^l$ of degree $2k$ with rational coefficients such that for all $f > 2$ we have

$$\frac{2}{\phi(f)} \sum_{\chi(-1)=(-1)^k} |L(k,\chi)|^2 = \frac{\pi^{2k}}{2((k-1)!)^2} \sum_{l=1}^{2k} r_{k,l}\phi_l(f) f^{l-2k}$$

where χ ranges over the $\phi(f)/2$ Dirichlet characters modulo f such that $\chi(-1) = (-1)^k$.

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THEOREM 2. *Assume $f > 2$ and $k \geq 1$. Let*

$$M_{\rm odd}(k,f) = \frac{2}{\phi(f)} \sum_{\chi(-1)=-1} |L(k,\chi)|^2 \quad (k \geq 1 \text{ odd}),$$

$$M_{\rm even}(k,f) = \frac{2}{\phi(f)} \sum_{\chi(-1)=+1} |L(k,\chi)|^2 \quad (k \geq 2 \text{ even})$$

denote the mean value of $|L(k,\chi)|^2$ where χ ranges over the $\phi(f)/2$ Dirichlet characters modulo f such that $\chi(-1) = (-1)^k$. Then

$$M_{\rm odd}(1,f) = \frac{\pi^2}{6}\phi_2(f) - \frac{\pi^2\phi_1(f)}{2f},$$

$$M_{\rm even}(2,f) = \frac{\pi^4}{90}\phi_4(f) + \frac{\pi^4}{9f^2}\phi_2(f),$$

$$M_{\rm odd}(3,f) = \frac{\pi^6}{945}\phi_6(f) - \frac{\pi^6}{45f^4}\phi_2(f),$$

$$M_{\rm even}(4,f) = \frac{\pi^8}{9450}\phi_8(f) + \frac{\pi^8}{2025f^4}\phi_4(f) + \frac{4\pi^8}{567f^6}\phi_2(f).$$

To prove these results, we follow the same line of reasoning as for proving [Lou1, Th. 2] (which is nothing else but our formula for $M_{\rm odd}(1,f)$ and generalizes [Wal] who only considered the case of prime modulus f). First, in (1) we generalize [Lou1, Th. 1] by giving a formula for the values $L(k,\chi)$ for the χ's that satisfy $\chi(-1) = (-1)^k$. Second, we generalize [Lou1, Lemma (a)] in Proposition 4. Third, we prove in Proposition 5 that Theorem 1 holds with the polynomials $R_k(X)$ defined in (5)–(7). Finally, Theorem 2 follows from Theorem 1 and the computation of the $R_i(X)$ for $1 \leq i \leq 4$.

2. Proof of the results

2.1. *Formulae for $M_{\rm odd}(k,f)$ and $M_{\rm even}(k,f)$*

PROPOSITION 3. *Let $k \geq 1$ and $f > 2$ denote positive rational integers. Let $\cot^{(k)}$ denote the kth derivative of $x \mapsto \cot(x) = \cos(x)/\sin(x)$.*

1. *If χ is a Dirichlet character modulo $f > 2$ and if $\chi(-1) = (-1)^k$ then*

$$L(k,\chi) = \frac{(-1)^{k-1}\pi^k}{2f^k(k-1)!} \sum_{l=1}^{f-1} \chi(l)\cot^{(k-1)}(\pi l/f). \tag{1}$$

2. *We have*

$$\sum_{\chi(-1)=(-1)^k} |L(k,\chi)|^2 = \frac{\pi^{2k}\phi(f)}{4((k-1)!)^2 f^{2k}} \sum_{\substack{l=1\\(l,f)=1}}^{f-1} \left(\cot^{(k-1)}\left(\frac{\pi l}{f}\right)\right)^2 \tag{2}$$

where the first sum ranges over all the $\phi(f)/2$ *Dirichlet characters modulo* $f > 2$ *which satisfy* $\chi(-1) = (-1)^k$ *and the second sum ranges over integers* l *relatively prime to* f.

Proof. Recall that for $0 < b < 1$ we have

$$\pi\cot(\pi b) = \sum_{n\geq 0}\left(\frac{1}{n+b} - \frac{1}{n+1-b}\right).$$

Therefore, for $k \geq 1$ we have

$$\frac{(-1)^{k-1}\pi^k}{(k-1)!}\cot^{(k-1)}(\pi b) = \sum_{n\geq 0}\left(\frac{1}{(n+b)^k} + \frac{(-1)^k}{(n+1-b)^k}\right). \tag{3}$$

Now, for $b > 0$ we set $\zeta(s,b) = \sum_{n\geq 0}(n+b)^{-s}$ for $\Re(s) > 1$ (Hurwitz's zeta function). For $\Re(s) > 1$ we have

$$\begin{aligned} L(s,\chi) &= f^{-s}\sum_{l=1}^{f-1}\chi(l)\zeta(s,l/f) \\ &= f^{-s}\sum_{l=1}^{f-1}\chi(f-l)\zeta(s,1-(l/f)) \\ &= f^{-s}\chi(-1)\sum_{l=1}^{f-1}\chi(l)\zeta(s,1-(l/f)) \\ &= \frac{f^{-s}}{2}\sum_{l=1}^{f-1}\chi(l)(\zeta(s,l/f) + \chi(-1)\zeta(s,1-(l/f))) \\ &= \frac{f^{-s}}{2}\sum_{l=1}^{f-1}\chi(l)\sum_{n\geq 0}\left(\frac{1}{(n+(l/f))^s} + \frac{\chi(-1)}{(n+1-(l/f))^s}\right). \end{aligned} \tag{4}$$

Moreover, if $\chi(-1) = -1$ then it is easily seen that this last equality is valid for $\Re(s) > 0$. Therefore, if $k \geq 1$ and $\chi(-1) = (-1)^k$ then, using (3) and (4), we do obtain (1). Let us recall that for $f > 2$ and $\varepsilon = \pm 1$ we have

$$\sum_{\chi(-1)=\varepsilon}\chi(l)\overline{\chi(l')} = \frac{\phi(f)}{2}\langle l,l'\rangle_\varepsilon$$

where

$$\langle l,l'\rangle_\varepsilon := \begin{cases} 1 & \text{if } l' \equiv l \pmod f \text{ and } \gcd(l,f) = 1, \\ \varepsilon & \text{if } l' \equiv -l \pmod f \text{ and } \gcd(l,f) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We deduce the second point from these relations, (1) and $\cot^{(k)}(-x) = (-1)^{k-1}\cot^{(k)}(x)$. ∎

2.2. *Evaluation of the sums* $\sum_{l=1}^{d-1}(\cot^{(k-1)}(\pi l/d))^2$. The derivative of cot is $-1-\cot^2$. Therefore, if we define inductively polynomials $P_k(X) \in \mathbb{Z}[X]$ by means of $P_1(X) = X$ and $P_{k+1}(X) = (X^2+1)P_k'(X)$, then $\cot^{(k-1)} = (-1)^{k-1}P_k(\cot)$ and $(\cot^{(k-1)})^2 = Q_k(\cot^2)$ where $Q_k(X) = \sum_{l=0}^{k} q_{k,l}X^l \in \mathbb{Z}[X]$ is defined by $Q_k(X^2) = (P_k(X))^2$.

For $j \geq 1$ we define polynomials $s_j(X) \in \mathbb{Q}[Y]$ of degree $2j$ by

$$s_j(X) = \frac{(X-1)(X-2)(X-3)\ldots(X-2j)}{(2j+1)!} \quad (j \geq 1), \tag{5}$$

and use them to define inductively on $j \geq 1$ polynomials $F_j(X) \in \mathbb{Q}[X]$ of degree $\leq 2j$ by means of

$$F_j(X) - s_1(X)F_{j-1}(X) + \ldots + (-1)^{j-1}s_{j-1}(X)F_1(X) + (-1)^j j s_j(X) = 0. \tag{6}$$

We finally set

$$R_k(X) = q_{k,0}(X-1) + 2\sum_{j=1}^{k} q_{k,j}F_j(X) \in \mathbb{Q}[X]. \tag{7}$$

Notice that for any $k \geq 1$ the degree of $R_k(X)$ is $\leq 2k$ and that $R_k(1) = 0$ (this is because (5) yields $s_j(1) = 0$ for all $j \geq 1$ and (6) then yields $F_j(1) = 0$ for all $j \geq 1$). We will write

$$R_k(X) = \sum_{l=0}^{2k} r_{k,l}X^l$$

and we will prove that these $R_k(X)$ are the polynomials which appear in the statement of Theorem 1. Using (7), (6) and (5) we computed Table 1 opposite, according to which we deduce Theorem 2 from Theorem 1.

PROPOSITION 4. *Let $k \geq 1$ be a given rational integer and let $R_k(X)$ be as in* (7). *Then for any rational integer $d > 1$ we have*

$$R(k,d) := \sum_{l=1}^{d-1}(\cot^{(k-1)}(\pi l/d))^2 = R_k(d). \tag{8}$$

Proof. Let $k \geq 1$ be a given integer. Let j range from 1 to k. Let D range over the integers $D \geq k$. We set

$$S_j(X_1,\ldots,X_D) = \sum_{a=1}^{D} X_a^j$$

and

$$\sigma_j(X_1,\ldots,X_D) = \sum_{1\leq a_1<\ldots<a_j\leq D} X_{a_1}\ldots X_{a_j}.$$

Table 1

$P_1(X) = X$ $Q_1(X) = X$ $F_1(X) = s_1(X) = (X^2 - 3X + 2)/6$ $R_1(X) = 2F_1(X) = (X^2 - 3X + 2)/3$
$P_2(X) = 1 + X^2$ $Q_2(X) = 1 + 2X + X^2$ $F_2(X) = s_1(X)F_1(X) - 2s_2(X) = (X^4 - 20X^2 + 45X - 26)/90$ $R_2(X) = (X-1) + 4F_1(X) + 2F_2(X) = (X^4 + 10X^2 - 11)/45$
$P_3(X) = 2X + 2X^3$ $Q_3(X) = 4X + 8X^2 + 4X^3$ $F_3(X) = s_1(X)F_2(X) - s_2(X)F_1(X) + 3s_3(X)$ $= (2X^6 - 42X^4 + 483X^2 - 945X + 502)/1890$ $R_3(X) = 8F_1(X) + 16F_2(X) + 8F_3(X) = 8(X^6 - 21X^2 + 20)/945$
$P_4(X) = 2 + 8X^2 + 6X^4$ $Q_4(X) = 4 + 32X + 88X^2 + 96X^3 + 36X^4$ $F_4(X) = s_1(X)F_3(X) - s_2(X)F_2(X) + s_3(X)F_1(X) - 4s_4(X)$ $= (3X^8 - 80X^6 + 924X^4 - 7920X^2 + 14175X - 7102)/28350$ $R_4(X) = 4(X-1) + 64F_1(X) + 176F_2(X) + 192f_3(X) + 72F_4(X)$ $= 4(3X^8 + 14X^4 + 200X^2 - 217)/1575$

For each $j \in \{1, \dots, k\}$ there exists $f_j = f_j(X_1, \dots, X_j) \in \mathbb{Z}[X_1, \dots, X_j]$ such that for all $D \geq k$ we have

$$S_j(X_1, \dots, X_D) = f_j(\sigma_1(X_1, \dots, X_D), \dots, \sigma_i(X_1, \dots, X_D)). \tag{9}$$

Newton's formulae

$$f_j - X_1 f_{j-1} + X_2 f_{j-2} + \dots (-1)^{j-1} X_{j-1} f_1 + (-1)^j j X_j = 0 \quad (\text{for } j \leq D)$$

(and $f_1(X_1) = X_1$) allow us to compute inductively these polynomials $f_j = f_j(X_1, \dots, X_j)$ for $1 \leq j \leq k$. In particular, the polynomials defined in (6) are given by

$$F_j(X) = f_j(s_1(X), \dots, s_j(X)) \quad (j \geq 1). \tag{10}$$

According to (7), (10) and (8), to complete the proof, we only have to show that if $k \geq 1$ is given, then for any $d > 1$ we have

$$R(k,d) = q_{k,0}(d-1) + 2\sum_{j=1}^{k} q_{k,j} f_j(s_1(d),\dots,s_j(d)). \tag{11}$$

Set $d' = (d-1)/2$ if $d \geq 3$ is odd, $d/2$ if $d \geq 2$ is even. Choose D such that $D \geq d'$ and $D \geq k$, set

$$\alpha_l(d) = \begin{cases} \cot^2(\pi l/d) & \text{for } 1 \leq l \leq d', \\ 0 & \text{for } d' < l \leq D, \end{cases}$$

and for $1 \leq j \leq k$ set

$$\sigma_j(d) = \sigma_j(\alpha_1(d),\dots,\alpha_D(d))$$

and

$$S_j(d) := S_j(\alpha_1(d),\dots,\alpha_D(d)) = f_j(\sigma_1(d),\dots,\sigma_j(d)) \quad \text{(by (9))}.$$

Since for $d > 1$ and $j \geq 1$ we have

$$\sum_{l=1}^{d-1} \cot^{2j}(\pi l/d) = 2\sum_{l=1}^{D} (\alpha_l(d))^j = 2S_j(d) = 2f_j(\sigma_1(d),\dots,\sigma_j(d)),$$

we obtain

$$R(k,d) = q_{k,0}(d-1) + 2\sum_{j=1}^{k} q_{k,j} f_j(\sigma_1(d),\dots,\sigma_j(d)). \tag{12}$$

Therefore, according to (11) and (12), it only remains to show that for any $d > 1$ we have $\sigma_j(d) = s_j(d)$ for $1 \leq j \leq k$. Since the $\cot(\pi l/d)$ for $1 \leq l \leq d-1$ are the roots of the polynomial $((X+i)^d - (X-i)^d)/(2id) = X^{d-1} - s_1(d)X^{d-3} + s_2(d)X^{d-5} - \dots$ (where $i^2 = -1$), we see that the $\alpha_l(d)$ for $1 \leq l \leq D$ are the roots of the polynomial $X^D - s_1(d)X^{D-1} + s_2(d)X^{D-2} - \dots$ (for $s_j(d) = 0$ for $2j \geq d$), and we do obtain $\sigma_j(d) = s_j(d)$. ■

2.3. *Proof of the main theorem*

PROPOSITION 5 (proves Theorem 1). *Let μ denote Möbius' function.*

1. *For $f > 2$ and $k \geq 1$ we have*

$$\sum_{\chi(-1)=(-1)^k} |L(k,\chi)|^2 = \frac{\phi(f)}{4f^{2k}}\left(\frac{\pi^k}{(k-1)!}\right)^2 \sum_{\substack{d|f \\ d>1}} \mu(f/d) R_k(d). \tag{13}$$

2. *If $R_k(X) = \sum_{l=0}^{2k} r_{k,l}X^l$ is a polynomial of degree $\leq 2k$ such that $R_k(1) = 0$ then*

$$\sum_{\substack{d|f \\ d>1}} \mu(f/d) R_k(d) = \sum_{d|f} \mu(f/d) R_k(d) = \sum_{l=1}^{2k} r_{k,l}\phi_l(f) f^l. \tag{14}$$

Proof. Only the first point needs a proof. Since $\sum_{d|n}\mu(d)=1$ if $n=1$ and 0 if $n>1$, we deduce (13) from (2) and the following computation:

$$\sum_{\substack{a=1\\(a,f)=1}}^{f-1}\Big(\cot^{(k-1)}\Big(\frac{\pi a}{f}\Big)\Big)^2=\sum_{a=1}^{f-1}\Big(\cot^{(k-1)}\Big(\frac{\pi a}{f}\Big)\Big)^2\Big(\sum_{\substack{d|a\\d|f}}\mu(d)\Big)$$
$$=\sum_{\substack{d|f\\d<f}}\mu(d)\sum_{b=1}^{f/d-1}\Big(\cot^{(k-1)}\Big(\frac{\pi db}{f}\Big)\Big)^2$$
$$=\sum_{\substack{d|f\\d<f}}\mu(d)R_k\Big(\frac{f}{d}\Big)=\sum_{\substack{d|f\\d>1}}\mu\Big(\frac{f}{d}\Big)R_k(d). \ ■$$

3. Remarks. 1. According to our proof, the polynomial

$$((2k+1)!)^{2k}R_k(X)\in\mathbb{Z}[X]$$

has integral coefficients. Therefore, $((2k+1)!)^{2k}R(k,d)=((2k+1)!)^{2k}R_k(d)$ is a rational integer (see (8)), and any entry $R_k(X)$ of Table 1 can be easily checked: verify that the polynomial $((2k+1)!)^{2k}R_k(X)$ of degree $2k$ has integral coefficients and that the $2k+1$ rational integers $((2k+1)!)^{2k}R(k,d)-((2k+1)!)^{2k}R_k(d)$ are equal to zero for $1\le d\le 2k+1$.

2. After the publication of [Lou1], Qi Minggao sent us another proof of [Lou1, Th. 2] (see [QiM]). However, his proof was much more complicated than ours and cannot be generalized for computing the mean value of $|L(k,\chi)|^2$ where χ ranges over the Dirichlet characters modulo f such that $\chi(-1)=(-1)^k$.

3. Since the values at non-positive integers of Dirichlet L-functions are generalized Bernoulli numbers (see [Was, Th. 4.2]), and since according to their functional equations these values at non-positive integers are related to their values at positive integers, one might think it would be easier to prove Theorem 1 by dealing with these values at non-positive integers. However, this approach is doomed to failure because functional equations are valid only for primitive characters, and according to [Lou3], there is no hope for ever finding similar simple formulae for the mean value of $|L(k,\chi)|^2$ where χ ranges over the primitive Dirichlet characters modulo f such that $\chi(-1)=(-1)^k$.

4. Whereas for any positive rational integer $n\ge 1$ asymptotic expansions exist of the type

$$\sum_{\chi\neq 1}|L(1,\chi)|^2=\frac{\pi^2}{6}p-\log^2 p+\sum_{k=0}^{n-1}a_kp^{-k}+O(p^{-n}) \tag{15}$$

for mean values of primitive L-functions modulo primes $p \geq 3$ (see [KM]), there is no known formula for such mean values. Hence, there is no hope of finding formulae for the mean values

$$M(k,f) := \frac{1}{\phi(f)} \sum_{\chi} |L(k,\chi)|^2 = \frac{1}{2} M_{\mathrm{odd}}(k,f) + \frac{1}{2} M_{\mathrm{even}}(k,f)$$

where χ ranges over the $\phi(f)$ Dirichlet characters modulo $f > 2$ (and where $k \geq 1$ is a positive rational integer). However, asymptotic formulae similar to (15) for these $M(k,\chi)$ are given in [KM].

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