COLLOQUIUM MATHEMATICUM

VOL. 90 2001 NO. 1

GENERALIZED CANONICAL ALGEBRAS AND STANDARD STABLE TUBES

BY

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Abstract. We introduce a new wide class of finite-dimensional algebras which admit families of standard stable tubes (in the sense of Ringel [17]). In particular, we prove that there are many algebras of arbitrary nonzero (finite or infinite) global dimension whose Auslander–Reiten quivers admit faithful standard stable tubes.

Introduction. Throughout the paper K will denote a fixed algebraically closed field. By an *algebra* we mean a finite-dimensional K-algebra (associative, with an identity), which we moreover assume to be basic. An algebra A can be written as a bound quiver algebra $A \cong KQ/I$, where $Q = Q_A$ is the Gabriel quiver of A and I is an admissible ideal in the path algebra KQ of Q. Equivalently, we will consider A as a K-category whose class of objects is the set of vertices of Q_A.

For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (over K) right A-modules, by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$ and by $\operatorname{rad}^\infty(\operatorname{mod} A)$ the infinite radical of $\operatorname{mod} A$. Recall that $\operatorname{rad}(\operatorname{mod} A)$ is generated by nonisomorphisms between indecomposable objects in $\operatorname{mod} A$, and $\operatorname{rad}^\infty(\operatorname{mod} A)$ is the intersection of all finite powers $\operatorname{rad}^i(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. By an *$A$-module* we mean an object of $\operatorname{mod} A$. For each vertex i of Q_A, we denote by $S_A(i)$ the simple A-module at i, and by $P_A(i)$ (respectively, $I_A(i)$) the projective cover (respectively, injective envelope) of $S_A(i)$ in $\operatorname{mod} A$. Moreover, we denote by D the standard duality $\operatorname{Hom}_K(-, K)$ on $\operatorname{mod} A$.

We shall denote by Γ_A the Auslander–Reiten quiver of A and by τ_A and τ_A^- the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$ in Γ_A, respectively. We do not distinguish between an indecomposable A-module and the vertex of Γ_A corresponding to it. By a *component* of Γ_A we mean a connected component of Γ_A. For a family $\mathcal{C}$ of components in Γ_A, we denote by $\operatorname{supp}_A \mathcal{C}$ the support of $\mathcal{C}$ and by $\operatorname{ann}_A \mathcal{C}$ the annihilator of $\mathcal{C}$. Recall that $\operatorname{supp}_A \mathcal{C}$ is the full subcategory of A given by all objects i such that $S_A(i)$ is a

2000 *Mathematics Subject Classification*: 16G10, 16G70, 18G05.

Supported by the Foundation for Polish Science.

composition factor of a module in $\mathcal{C}$, and $\operatorname{ann}_A \mathcal{C}$ is the intersection of the annihilators of all modules from $\mathcal{C}$. Then the family $\mathcal{C}$ is said to be *sincere* (respectively, *faithful*) if $\operatorname{supp}_A \mathcal{C} = A$ (respectively, $\operatorname{ann}_A \mathcal{C} = 0$). Clearly, if $\mathcal{C}$ is faithful then $\mathcal{C}$ is sincere. We also note that $\mathcal{C}$ is a family of components of $\Gamma_{A/\operatorname{ann}_A \mathcal{C}}$. Finally, $\mathcal{C}$ is said to be *regular* provided $\mathcal{C}$ contains neither a projective module nor an injective module. A component in Γ_A of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, is said to be a *stable tube* of rank r. Therefore, a stable tube of rank r in Γ_A is an infinite component consisting of τ_A-periodic indecomposable A-modules having period r. Moreover, a stable tube of rank 1 is said to be *homogeneous*. It has been proved in [29] (see also [11]) that a regular component $\mathcal{C}$ of Γ_A contains an oriented cycle if and only if $\mathcal{C}$ is a stable tube.

We are concerned with the problem of describing the structure of standard components in the Auslander–Reiten quiver Γ_A of an algebra A, raised more than 15 years ago by Ringel [18]. Recall that a component $\mathcal{C}$ of Γ_A is called *standard* if the full subcategory of $\operatorname{mod} A$ formed by modules from $\mathcal{C}$ is equivalent to the mesh category $K(\mathcal{C})$ of $\mathcal{C}$ (cf. [2]). Further, following [22], a component $\mathcal{C}$ of Γ_A is called *generalized standard* if $\operatorname{rad}^\infty(X, Y) = 0$ for all modules X and Y in $\mathcal{C}$. It is known [12] that every standard component of Γ_A is generalized standard. It has been proved in [22, Theorem 2.3] that every generalized standard component $\mathcal{C}$ in Γ_A is quasiperiodic, that is, all but finitely many τ_A-orbits in $\mathcal{C}$ are periodic. In particular, this implies that every regular (generalized) standard component of Γ_A is either a stable tube or is of the form $\mathbb{Z}\Delta$ for a finite connected quiver Δ without oriented cycles (solution of Problem 3 from [18]). It is also known that a component $\mathcal{C}$ of Γ_A is generalized standard if and only if $\mathcal{C}$ is a generalized standard component of $\Gamma_{A/\operatorname{ann}_A \mathcal{C}}$.

Hence, in order to describe the generalized standard components, it is enough to describe the faithful ones. We have proved in [22, Corollary 3.3] that the faithful generalized standard regular components without oriented cycles are exactly the connecting components of the Auslander–Reiten quivers of tilted algebras given by regular tilting modules over wild hereditary algebras. Moreover, such components are standard [16, Theorem 1.7]. We refer to [7] and [19] for properties of regular connecting components of tilted algebras, and to [21] for a complete description of faithful generalized standard components without oriented cycles.

It is expected that infinite faithful (generalized) standard components with oriented cycles can be obtained from faithful standard stable tubes by a sequence of admissible operations (see [13]). Therefore, description of algebras whose Auslander–Reiten quiver admits a faithful (generalized) standard stable tube is an important open problem (see [24, Problem 3]). Recently this problem has attracted much attention (see [7], [8], [9], [15],

[22]–[26]). The most general result in this direction is due to I. Reiten and the author [15] and says that an algebra A admits a sincere (equivalently, faithful) standard stable tube without external short cycles if and only if A is concealed canonical. We note that all concealed canonical algebras are quasitilted [4], that is, algebras of global dimension at most 2 and with every indecomposable module either of projective dimension at most one or of injective dimension at most one. This raised the following question: does existence of a faithful standard stable tube in the Auslander–Reiten quiver Γ_A of an algebra A force some strong homological properties of A?

The main aim of this paper is to exhibit a wide class of algebras, called generalized canonical algebras, whose Auslander–Reiten quivers admit infinite families of faithful standard stable tubes. We will also show that there are generalized canonical algebras of arbitrary nonzero (finite or infinite) global dimension, and that every basic algebra is a factor algebra of a generalized canonical algebra. This shows that the classification of all basic algebras with faithful standard tubes in the Auslander–Reiten quiver is a hopeless problem.

For basic background on the representation theory of algebras we refer to [1] and [17].

1. Standard stable tubes. The aim of this section is to present some characterizations and properties of standard stable tubes established so far.

Let A be an algebra and n be the rank of the Grothendieck group $K_0(A)$ of A. Recall that an indecomposable A-module X is called a *brick* provided $\operatorname{End}_A(X) \cong K$. Further, by the *mouth* of a stable tube $\mathcal{T}$ we mean the unique τ_A-orbit in $\mathcal{T}$ formed by the modules having exactly one direct predecessor (and exactly one direct successor). The following fact proved by Ringel [17, (3.1)(2)] will be crucial in our considerations.

Proposition 1.1. *Let $E_1, \ldots, E_r$ be a family of pairwise orthogonal bricks in* $\operatorname{mod} A$ *satisfying the following conditions*:

(i) $\tau_A E_i \cong E_{i-1}$ *for all* $i \in \{1, \ldots, r\}$, *where* $E_0 = E_r$.
(ii) $\operatorname{Ext}^2_A(E_i, E_j) = 0$ *for all* $i, j \in \{1, \ldots, r\}$.

Then $E_1, \ldots, E_r$ form the mouth of a standard stable tube $\mathcal{T}$ of rank r in Γ_A.

For a module X in $\operatorname{mod} A$ we denote by $\operatorname{pd}_A X$ the projective dimension of X and by $\operatorname{id}_A X$ the injective dimension of X in $\operatorname{mod} A$. The following fact proved in [22, Lemma 5.9] describes the homological properties of indecomposable modules lying in faithful standard stable tubes.

Lemma 1.2. *Let $\mathcal{T}$ be a faithful standard stable tube in Γ_A and X be a module in $\mathcal{T}$. Then* $\operatorname{pd}_A X \le 1$ *and* $\operatorname{id}_A X \le 1$.

We may now establish the following characterization of standard stable tubes.

LEMMA 1.3. *Let $\mathcal{T}$ be a stable tube in Γ_A. The following conditions are equivalent*:

(i) *$\mathcal{T}$ is standard.*
(ii) *The mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks.*
(iii) $\operatorname{rad}^\infty(Z,Z)=0$ *for all modules Z in $\mathcal{T}$.*
(iv) *$\mathcal{T}$ is generalized standard.*

Proof. The implication (i)⇒(ii) is obvious. The equivalence of the conditions (ii), (iii) and (iv) is established in [22, Corollary 5.3] (see also [23, Lemma 3.1]). Assume (iv) holds and consider the algebra $B = A/\operatorname{ann}_A \mathcal{T}$. Then $\mathcal{T}$ is a faithful generalized standard stable tube in Γ_B. Moreover, it follows from Lemma 1.2 that $\operatorname{pd}_B X \le 1$ for any module X in $\mathcal{T}$. In particular, we conclude that $\operatorname{Ext}^2_B(E,E')=0$ for any two modules E and E' lying on the mouth of $\mathcal{T}$. Since (iv) is equivalent to (ii), we also know that the mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks. Finally, Proposition 1.1 yields that $\mathcal{T}$ is standard stable in Γ_B, and hence also in Γ_A. Thus (iv) implies (i).

We also have the following information on the ranks of generalized standard stable tubes.

PROPOSITION 1.4. *Let $\mathcal{T}_i$, $i\in I$, be a family of pairwise orthogonal generalized standard stable tubes of Γ_A. For each $i \in I$, denote by r_i the rank of the tube $\mathcal{T}_i$. Then*

$$\sum_{i\in I}(r_i-1)\le n-1.$$

Proof. [22, Lemma 5.10].

COROLLARY 1.5. *Let $\mathcal{T}$ be a standard stable tube of rank r in Γ_A. Then $r\le n$.*

Another relationship between a family of stable tubes and the rest of the module category is concerned with the existence or nonexistence of paths (or cycles). Recall that a sequence of nonzero nonisomorphisms $X = X_0 \to X_1 \to \ldots \to X_{n-1} \to X_n = Y$ between indecomposable A-modules is called a *path* of length n, and it is called a *cycle* if $X=Y$. If $n=2$ the path (cycle) is said to be *short* [1]. Moreover, if $\mathcal{C}$ is a family of indecomposable A-modules and X, Y are in $\mathcal{C}$, while $X_1,\ldots,X_{n-1}$ do not lie in $\mathcal{C}$, we say that the path or cycle is *external* (with respect to $\mathcal{C}$).

We now exhibit the class of canonical algebras introduced by Ringel [17, (3.7)], playing an important role in the representation theory of algebras. Let $m\ge 1$ be a positive integer, $\mathbf{p}=(p_0,\ldots,p_m)$ be an $(m+1)$-tuple of positive integers, and $\boldsymbol{\lambda}=(\lambda_0,\ldots,\lambda_m)$ be an $(m+1)$-tuple of pairwise different elements of $\mathbb{P}_1(K) = K \cup \{\infty\}$, normalized so that $\lambda_0 = \infty$, $\lambda_1 = 0$, $\lambda_2 = 1$. Consider the quiver $\Delta(\mathbf{p})$ of the form

$$\begin{tikzcd}
 & \bullet \arrow[dl, "\alpha_{01}"'] & \bullet \arrow[l, "\alpha_{02}"'] & \cdots \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l, "\alpha_{0p_0-1}"'] & \\
0 & \bullet \arrow[l, "\alpha_{11}"'] & \bullet \arrow[l, "\alpha_{12}"'] & \cdots \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l, "\alpha_{1p_1-1}"'] & \omega \arrow[l, "\alpha_{1p_1}"'] \arrow[ul, "\alpha_{0p_0}"'] \arrow[ddl, "\alpha_{mp_m}"] \\
 & \vdots & \vdots & & \vdots & \vdots & \\
 & \bullet \arrow[uul, "\alpha_{m1}"] & \bullet \arrow[l, "\alpha_{m2}"] & \cdots \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l, "\alpha_{mp_m-1}"] &
\end{tikzcd}$$

For $m = 1$, put $C(\mathbf{p}, \boldsymbol{\lambda}) = K\Delta(\mathbf{p})$. For $m \geq 2$, assume that $\mathbf{p}$ consists of integers $p_i \geq 2$, consider the ideal $I(\mathbf{p}, \boldsymbol{\lambda})$ in $K\Delta(\mathbf{p})$ generated by the elements

$$\alpha_{ip_i} \ldots \alpha_{i2}\alpha_{i1} + \alpha_{0p_0} \ldots \alpha_{02}\alpha_{01} + \lambda_i \alpha_{1p_1} \ldots \alpha_{12}\alpha_{11}, \quad i = 2, \ldots, m,$$

and the bound quiver algebra $C(\mathbf{p}, \boldsymbol{\lambda}) = K\Delta(\mathbf{p})/I(\mathbf{p}, \boldsymbol{\lambda})$. Then $C(\mathbf{p}, \boldsymbol{\lambda})$ is called the *canonical algebra* of type $(\mathbf{p}, \boldsymbol{\lambda})$, $\mathbf{p}$ the *weight sequence* of $C(\mathbf{p}, \boldsymbol{\lambda})$, and $\boldsymbol{\lambda}$ the *parameter sequence* of $C(\mathbf{p}, \boldsymbol{\lambda})$. It has been shown in [17, (3.7)] that the Auslander–Reiten quiver Γ_C of $C = C(\mathbf{p}, \boldsymbol{\lambda})$ has a decomposition

$$\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{Q}^C$$

where $\mathcal{P}^C$ (respectively, $\mathcal{Q}^C$) is a family of components consisting of indecomposable C-modules of positive (respectively, negative) rank and $\mathcal{T}^C$ is a $\mathbb{P}_1(K)$-family $\mathcal{T}^C_\lambda$, $\lambda \in \mathbb{P}_1(K)$, of faithful, standard, pairwise orthogonal, stable tubes separating $\mathcal{P}^C$ from $\mathcal{Q}^C$ (in the sense of [17, (3.1)]). Following [8], by a *concealed canonical algebra* (of type $(\mathbf{p}, \boldsymbol{\lambda})$) we mean an algebra of the form $A = \mathrm{End}_C(T)$, where T is a tilting module from the additive category $\mathrm{add}\, \mathcal{P}^C$ of $\mathcal{P}^C$, that is, a direct sum T of n (= rank of $K_0(C)$) pairwise nonisomorphic indecomposable modules from $\mathcal{P}^C$ such that $\mathrm{pd}_C\, T \leq 1$ and $\mathrm{Ext}^1_C(T, T) = 0$. Then again Γ_A has a decomposition

$$\Gamma_A = \mathcal{P}^A \vee \mathcal{T}^A \vee \mathcal{Q}^A$$

where $\mathcal{T}^A = \mathrm{Hom}_C(T, \mathcal{T}^C)$ is a $\mathbb{P}_1(K)$-family $\mathcal{T}^A_\lambda$, $\lambda \in \mathbb{P}_1(K)$, of faithful, standard, pairwise orthogonal, stable tubes separating $\mathcal{P}^A$ from $\mathcal{Q}^A$. In particular, $\mathcal{T}^A$ is a family of sincere stable tubes without external paths. We also note that $\mathrm{gl.dim}\, A \leq 2$, $\mathrm{pd}_A\, X \leq 1$ for any indecomposable module X from $\mathcal{P}^A \vee \mathcal{T}^A$, and $\mathrm{id}_A\, X \leq 1$ for any indecomposable module Y from $\mathcal{T}^A \vee \mathcal{Q}^A$, and so A is quasitilted in the sense of [4].

It is shown in [15, Theorem 3.1] that a connected algebra A is concealed canonical if and only if Γ_A has a sincere family of pairwise orthogonal standard stable tubes without external short paths. In fact, this can be deepened (slightly) as follows:

THEOREM 1.6. *Let A be a connected algebra. Then A is concealed canonical if and only if Γ_A has a sincere family of pairwise orthogonal stable tubes without external short paths.*

Proof. The necessity part follows from the properties of the Auslander–Reiten quivers of concealed canonical algebras described above. Assume Γ_A admits a sincere family $\mathcal{T}$ of pairwise orthogonal stable tubes without external short paths. Then it follows from [15, Corollary 1.6] that the family $\mathcal{T}$ is faithful. Applying now arguments from the proof of [26, Proposition 1.1] we deduce that the algebra A is quasitilted. Invoking now the recent result of Happel [3] we conclude that A is a tilted algebra or a quasitilted algebra of canonical type. Finally, using the known structure of Auslander–Reiten components of tilted algebras [5, 6, 17, 18] and quasitilted algebras of canonical type [10, 14, 17] we deduce that A is a concealed canonical algebra.

As a direct consequence (see also [15, Corollaries 3.2 and 3.3]) we obtain the following fact.

COROLLARY 1.7. *Let A be a connected algebra and $\mathcal{C}$ be a sincere stable tube in Γ_A without external short cycles. Then $\mathcal{C}$ is a faithful standard stable tube and A is a concealed canonical algebra.*

2. The main result. The aim of this section is to define a new wide class of algebras, containing the class of all canonical algebras [17], and prove that their Auslander–Reiten quivers admit an infinite family of standard stable tubes.

Let $m \geq 1$ be a fixed positive integer, $\mathcal{B} = \{B_0, \ldots, B_m\}$ a family of basic connected nonsimple algebras, and $\mathcal{P} = \{P_0, \ldots, P_m\}$ a family of modules such that, for each $i \in \{0, \ldots, m\}$, P_i is an indecomposable projective-injective B_i-module with injective top and projective socle. Moreover, let $\Lambda = \{\lambda_0, \ldots, \lambda_m\}$ be a set of $m+1$ pairwise different elements of $\mathbb{P}_1(K) = K \cup \{\infty\}$, normalized so that $\lambda_0 = \infty$, $\lambda_1 = 0$, $\lambda_2 = 1$. Write each B_i as a bound quiver algebra
$$B_i = K\Delta^{(i)}/I^{(i)},$$
where $K\Delta^{(i)}$ is the path algebra of a connected quiver $\Delta^{(i)}$ and $I^{(i)}$ is an admissible ideal in $K\Delta^{(i)}$. Then
$$P_i = P_{B_i}((\omega, i)) = I_{B_i}((0, i))$$
for some vertices (ω, i) and $(0, i)$ of $\Delta^{(i)}$. Further, denote by u_i a fixed path in $\Delta^{(i)}$ with source (ω, i) and target $(0, i)$. Finally, denote by $\Delta = \Delta(\mathcal{B}, \mathcal{P})$ the quiver obtained from the disjoint union of the quivers $\Delta^{(0)}, \ldots, \Delta^{(m)}$ by identifying the vertices $(\omega, 0), \ldots, (\omega, m)$ with a vertex ω, and the vertices $(0, 0), \ldots, (0, m)$ with a vertex 0. Then in Δ we have paths $u_0, \ldots, u_m$ with source ω and target 0.

For $m=1$, let $C(\mathcal{B},\mathcal{P},\Lambda)$ be the bound quiver algebra $K\Delta(\mathcal{B},\mathcal{P})/I(\mathcal{B},\mathcal{P})$, where $I(\mathcal{B},\mathcal{P})$ is the ideal in $K\Delta(\mathcal{B},\mathcal{P})$ generated by $I^{(0)}$ and $I^{(1)}$.

For $m \geq 2$, we assume additionally that each $\Delta^{(i)}$ is different from the quiver $(\omega, i) \to (0, i)$. Consider the ideal $I(\mathcal{B},\mathcal{P},\Lambda)$ in $K\Delta(\mathcal{B},\mathcal{P})$ generated by $I^{(0)}, \dots, I^{(m)}$, and the elements

$$u_i + u_0 + \lambda_i u_1, \quad i = 2, \dots, m,$$

and put $C(\mathcal{B},\mathcal{P},\Lambda) = K\Delta(\mathcal{B},\mathcal{P})/I(\mathcal{B},\mathcal{P},\Lambda)$.

If, additionally, each projective-injective B_i-module P_i is faithful, the algebra $C(\mathcal{B},\mathcal{P},\Lambda)$ is said to be a *generalized canonical algebra* of type $(\mathcal{B},\mathcal{P},\Lambda)$. Observe that, if each B_i is the path algebra of an equioriented linear quiver

$$(\omega, i) \to (p_i - 1, i) \to \dots \to (1, i) \to (0, i)$$

and P_i is the unique indecomposable (faithful) projective-injective B_i-module, then $C(\mathcal{B},\mathcal{P},\Lambda)$ is just the canonical algebra $C(\mathbf{p},\boldsymbol{\lambda})$ of type $(\mathbf{p},\boldsymbol{\lambda})$, where $\mathbf{p} = (p_0, \dots, p_m)$ and $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m)$.

To formulate our main result we distinguish a family of indecomposable $C(\mathcal{B},\mathcal{P},\Lambda)$-modules and a family of components in $\Gamma_{\operatorname{mod} C(\mathcal{B},\mathcal{P},\Lambda)}$. For each vertex i of $\Delta(\mathcal{B},\mathcal{P})$ we denote by e_i the corresponding primitive indempotent of $C(\mathcal{B},\mathcal{P},\Lambda)$. Next, for each path v in $\Delta(\mathcal{B},\mathcal{P})$, we denote by $\overline{v}$ its image $v + I(\mathcal{B},\mathcal{P},\Lambda)$ in $C(\mathcal{B},\mathcal{P},\Lambda) = K\Delta(\mathcal{B},\mathcal{P})/I(\mathcal{B},\mathcal{P},\Lambda)$. Moreover, we put $C = C(\mathcal{B},\mathcal{P},\Lambda)$.

Let ξ be an element of $\mathbb{P}_1(K) \setminus \Lambda$. Then $u_\xi = \overline{u}_0 + \xi \overline{u}_1$ is an element of $e_\omega C e_0$, and we define

$$E^\xi := e_\omega C / u_\xi C.$$

Observe that E^ξ is an indecomposable C-module with simple top isomorphic to $S_C(\omega)$ and simple socle isomorphic to $S_C(0)$. Moreover, the heart $\operatorname{rad} E^\xi / \operatorname{soc} E^\xi$ of E^ξ is the direct sum $\operatorname{rad} P_0/\operatorname{soc} P_0 \oplus \dots \oplus \operatorname{rad} P_m / \operatorname{soc} P_m$ of the hearts of the projective-injective modules $P_0, \dots, P_m$. Denote by $\mathcal{T}_\xi$ the component of $\Gamma_{\operatorname{mod} C}$ containing the module E^ξ.

Denote by $\Phi = \Phi(\mathcal{B})$ the set of all $i \in \{0, \dots, m\}$ for which the algebra B_i is the path algebra $K\Delta^{(i)}$ of an equioriented linear quiver

$$\Delta^{(i)} : (\omega, i) \xrightarrow{\alpha_{ip_i}} (p_{i-1}, i) \to \dots \to (1, i) \xrightarrow{\alpha_{1i}} (0, i),$$

and by Ω the set of all $\lambda_i \in \Lambda$ with $i \in \Phi$. Moreover, put $\Sigma = \Lambda \setminus \Omega$. For each $\lambda_i \in \Omega$, define the C-module

$$E^{\lambda_i} := e_\omega C / \overline{\alpha}_{i,p_i} C.$$

Observe that E^{λ_i} is an indecomposable C-module with simple top isomorphic to $S_C(\omega)$ and simple socle isomorphic to $S_C(0)$. Moreover, the heart $\operatorname{rad} E^{\lambda_i} / \operatorname{soc} E^{\lambda_i}$ of E^{λ_i} is the direct sum $\bigoplus_{j \neq i} \operatorname{rad} P_j / \operatorname{soc} P_j$ of the hearts of all projective-injective modules from $\mathcal{P}$ different from P_i. Denote by $\mathcal{T}_{\lambda_i}$ the component of $\Gamma_{\operatorname{mod} C}$ containing the module E^{λ_i}, $i \in \Phi$.

We may now formulate the main result of the paper.

THEOREM 2.1. *Let $\mathcal{T} = (\mathcal{T}_\mu)_{\mu\in\mathbb{P}_1(K)\setminus\Sigma}$ be the family of components of $\Gamma_{\operatorname{mod} C(\mathcal{B},\mathcal{P},\Lambda)}$ defined above. Then*:

(i) *$\mathcal{T}$ is a family of pairwise orthogonal standard stable tubes of $\Gamma_{\operatorname{mod} C(\mathcal{B},\mathcal{P},\Lambda)}$.*

(ii) *The tubes $\mathcal{T}_\xi$, $\xi \in \mathbb{P}_1(K) \setminus \Lambda$, are homogeneous.*

(iii) *For each $\lambda_i \in \Omega$, $\mathcal{T}_{\lambda_i}$ is a stable tube of rank p_i.*

(iv) *The tubes of $\mathcal{T}$ are all faithful* (*respectively*, *sincere*) *if and only if the B_i-modules P_i, $0 \le i \le m$, are all faithful* (*respectively*, *sincere*).

Proof. Let $C = C(\mathcal{B},\mathcal{P},\Lambda)$, $e = e_0 + e_\omega$ and $A = eCe$. Then A is the path algebra of the Kronecker quiver $0 \underset{u_1}{\overset{u_0}{\leftleftarrows}} \omega$, and we have two natural K-linear covariant functors

$$\operatorname{mod} C \underset{T}{\overset{R}{\rightleftarrows}} \operatorname{mod} A,$$

where R is the restriction functor which assigns to each C-module X the A-module Xe, and T is the extension functor which assigns to each A-module Y the C-module $Y \otimes_A eC$. It is known that:

(a) R is exact, right adjoint to T, and $RT \cong \mathbf{1}_{\operatorname{mod} A}$.

(b) T is full, faithful, right exact, preserves indecomposability and carries projectives to projectives.

(c) A C-module X belongs to the image of T if and only if there exists an exact sequence $P'' \to P' \to X \to 0$ in $\operatorname{mod} C$, where P' and P'' are direct sums of modules $e_\omega C$ and $e_0 C$.

We first prove that $\mathcal{T}_\xi$, $\xi \in \mathbb{P}_1(K) \setminus \Lambda$, are standard homogeneous tubes. Observe that, for each $\xi \in \mathbb{P}_1(K) \setminus \Lambda$, the module $F^\xi = R(E^\xi)$ is an indecomposable regular A-module of the form $e_\omega A/v_\xi A$, where $v_\xi = u_0 + \xi u_1 \in e_\omega A e_0$. Hence F^ξ is an indecomposable module lying on the mouth of a homogeneous tube of $\Gamma_{\operatorname{mod} A}$. In particular, we have $F^\xi \cong \tau_A F^\xi$.

We now claim that $E^\xi \cong \tau_C E^\xi$. Observe that E^ξ admits a minimal projective resolution of the form

$$0 \to e_0 C \xrightarrow{f_\xi} e_\omega C \to E^\xi \to 0, \tag{1}$$

where f_ξ is given by left multiplication by $u_\xi \in e_\omega C e_0$. Applying the Nakayama functor $\nu_C = D\operatorname{Hom}_C(-, C) : \operatorname{mod} C \to \operatorname{mod} C$ we obtain an exact sequence (see [17, (2.4)] or [1, Chapter V])

$$0 \to \tau_C E^\xi \to \nu_C(e_0 C) \xrightarrow{\nu_C(f_\xi)} \nu_C(e_\omega C) \to \nu_C(E^\xi) \to 0, \tag{2}$$

where $\nu_C(e_0C) = D\operatorname{Hom}_C(e_0C, C) = D(Ce_0) = I_C(0)$, $\nu_C(e_\omega C) = D\operatorname{Hom}_C(e_\omega C, C) = D(Ce_\omega) = I_C(\omega)$, and $\nu_C(f_\xi) \in \operatorname{Hom}_C(D(Ce_0), D(Ce_\omega)) = \operatorname{Hom}_C(Ce_\omega, Ce_0) \cong e_\omega Ce_0$ assigns to each $\varphi \in \operatorname{Hom}_K(Ce_0, K) = D(Ce_0)$ the composition $\varphi f_\xi \in \operatorname{Hom}_K(Ce_\omega, K) = D(Ce_\omega)$. Applying the restriction functor R to the exact sequence (1), we obtain a minimal projective resolution

$$(1') \qquad 0 \to e_0A \xrightarrow{g_\xi} e_\omega A \to F^\xi \to 0$$

of F^ξ in $\operatorname{mod} A$, where g_ξ is given by left multiplication by the element $v_\xi = u_\xi \in e_\omega Ae_0 = e_\omega Ce_0$. Application of R to the exact sequence (2) yields an exact sequence

$$(2') \qquad 0 \to R(\tau_C E^\xi) \to \nu_A(e_0A) \xrightarrow{\nu_A(g_\xi)} \nu_A(e_\omega A) \to R\nu_C(E_\xi) \to 0,$$

where $\nu_A(g_\xi)$ assigns to each $\psi \in \operatorname{Hom}_K(Ae_0, K) = D(Ae_0)$ the composition $\psi g_\xi \in \operatorname{Hom}_K(Ae_\omega, K) = D(Ae_\omega)$. In particular, we conclude that $R(\tau_C E^\xi) \cong \tau_A F^\xi \cong F^\xi$. Further, it follows from (1) and (c) that E^ξ belongs to the image of T, and hence $E^\xi \cong T(F^\xi)$, because $R(E^\xi) = F^\xi$. On the other hand, it follows from (2) that $\tau_C E^\xi$ is an indecomposable C-module with simple socle isomorphic to $S_C(0)$ and simple top isomorphic to $S_C(\omega)$, and its heart $\operatorname{rad}\tau_C E^\xi/\operatorname{soc}\tau_C E^\xi$ is isomorphic to the direct sum $\operatorname{rad} P_0/\operatorname{soc} P_0 \oplus \ldots \oplus \operatorname{rad} P_n/\operatorname{soc} P_m$ of the hearts of the modules $P_0, \ldots, P_m$. This implies that $\tau_C E^\xi$ also belongs to the image of the functor T, and consequently $\tau_C E^\xi \cong T(F^\xi)$, because $R(\tau_C E^\xi) \cong F^\xi$. Therefore, we conclude that $E^\xi \cong \tau_C E^\xi$. Moreover, every nonzero endomorphism $h \in \operatorname{End}_C(E^\xi)$ induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_C(0) & \xrightarrow{f_\xi} & P_C(\omega) & \longrightarrow & E^\xi & \longrightarrow & 0 \\
 & & {\scriptstyle h''}\downarrow & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & P_C(0) & \xrightarrow{f_\xi} & P_C(\omega) & \longrightarrow & E^\xi & \longrightarrow & 0
\end{array}$$

Since $\operatorname{End}_C(P_C(\omega)) \cong K$, we obtain $\operatorname{End}_C(E^\xi) \cong K$, and hence E^ξ is a brick. Finally, the exact sequence (1) implies $\operatorname{pd}_C E^\xi = 1$, and consequently $\operatorname{Ext}^2_C(E^\xi, E^\xi) = 0$. Applying now Proposition 1.1 we conclude that E^ξ lies on the mouth of a standard stable tube of rank 1, and so $\mathcal{T}_\xi$ is a standard homogeneous tube.

Assume now the set Φ is nonempty, and let λ_i be an element of Ω. Then B_i is the path algebra of the quiver

$$(\omega, i) \xrightarrow{\alpha_{ip_i}} (p_{i-1}, i) \to \ldots \to (1, i) \to (0, i).$$

We shall prove that $\mathcal{T}_{\lambda_i}$ is a standard stable tube whose mouth consists of the modules $S_C((1,i)), \ldots, S_C((p_i - 1, i)), E^{\lambda_i}$. We have two cases to consider.

Assume first that $p_i \geq 2$. It follows from the definition of E^{λ_i} that its minimal projective resolution in $\operatorname{mod} C$ is of the form

$$0 \to e_{p_i-1,i}C \xrightarrow{f_i} e_\omega C \to E^{\lambda_i} \to 0$$

where the monomorphism f_i is given by left multiplication by the element $\overline{\alpha}_{i,p_i} \in e_\omega C e_{(p_{i-1},i)}$. Then we get an exact sequence of the form

$$0 \to \tau_C E^{\lambda_i} \to \nu_C(e_{p_i-1,i}C) \xrightarrow{\nu_C(f_i)} \nu_C(e_\omega C) \to \nu_C(E^{\lambda_i}) \to 0.$$

Since $\nu_C(e_{p_i-1,i}C) \cong I_C((p_i-1,i))$ and $\nu_C(e_\omega C) = I_C(\omega)$, we infer that $\tau_C E^{\lambda_i} \cong \operatorname{soc} I_C((p_i-1)) = S_C((p_i-1,i))$. Further, for $r \in \{2,\dots,p_i-1\}$, the simple module $S_C((r,i))$ has a minimal projective resolution of the form

$$0 \to e_{r-1,i}C \to e_{r,i}C \to S_C((r,i)) \to 0.$$

Then $\tau_C S_C((r,i))$ is given by an exact sequence

$$0 \to \tau_C S_C((r,i)) \to \nu_C(e_{r-1,i}C) \to \nu_C(e_{r,i}C) \to \nu_C S_C((r,i)) \to 0.$$

Since $\nu_C(e_{r-1,i}C) = I_C((r-1,i))$ and $\nu_C(e_{r,i}C) = I_C((r,i))$, we deduce that $\tau_C S_C((r,i)) \cong \operatorname{soc} I_C((r-1,i)) = S_C((r-1,i))$. Finally, we claim that $\tau_C S_C((1,i)) \cong E^{\lambda_i}$. It is enough to show that $\tau_C^- E^{\lambda_i} \cong S_C((1,i))$. Observe that E^{λ_i} admits a minimal injective resolution of the form

$$0 \to E^{\lambda_i} \to I_C(0) \to I_C((1,i)) \to 0.$$

Applying the Nakayama functor $\nu_C^- = \operatorname{Hom}_C(DC,-)$ we then obtain an exact sequence of the form (see [17, (2.4)])

$$0 \to \nu_C^- E^{\lambda_i} \to \nu_C^- I_C(0) \to \nu_C^- I_C((1,i)) \to \tau_C^- E^{\lambda_i} \to 0.$$

Since $\nu_C^- I_C(0) \cong P_C(0)$ and $\nu_C^- I_C((1,i)) \cong P_C((1,i))$ we infer that $\tau_C^- E^{\lambda_i} \cong \operatorname{top} P_C((1,i)) = S_C((1,i))$. Further, observe that E^{λ_i} is a brick. Indeed, again any nonzero endomorphism $h \in \operatorname{End}_C(E^{\lambda_i})$ induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & e_{p_{i-1},i}C & \xrightarrow{f_i} & e_\omega C & \longrightarrow & E^{\lambda_i} & \longrightarrow & 0 \\
 & & {\scriptstyle h''}\downarrow & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & e_{p_{i-1},i}C & \xrightarrow{f_i} & e_\omega C & \longrightarrow & E^{\lambda_i} & \longrightarrow & 0
\end{array}$$

Then $\operatorname{End}_C(P_C(\omega)) \cong K$ implies $\operatorname{End}_C(E^{\lambda_i}) \cong K$. Therefore, $S_C((1,i)),\dots$ $\dots, S_C((p_{i-1},i)), E^{\lambda_i}$ is a family of pairwise orthogonal bricks forming a periodic τ_C-orbit. It also follows from our considerations that all these modules have projective dimension one, and consequently $\operatorname{Ext}_C^2(S_C((r,i)), S_C((t,i))$ $= 0$, $\operatorname{Ext}_C^2(S_C((r,i)), E^{\lambda_i}) = 0$, $\operatorname{Ext}_C^2(E^{\lambda_i}, S_C((r,i)) = 0$ and $\operatorname{Ext}_C^2(E^{\lambda_i}, E^{\lambda_i})$ $= 0$ for all $r, s \in \{1,\dots,p_i-1\}$. Applying Proposition 1.1 we conclude that $\mathcal{T}_{\lambda_i}$ is a standard stable tube of rank p_i whose mouth consists of the modules $S_C((1,i)),\dots,S_C((p_1-1,i)), E^{\lambda_i}$.

Assume now that $p_i = 1$. Then B_i is the path algebra of the quiver $\Delta^{(i)}$: $(\omega, i) \xrightarrow{\alpha_{i,1}} (0, i)$, $m = 1$, and C is the bound quiver algebra $K\Delta(\mathcal{B},\mathcal{P})/I(\mathcal{B},\mathcal{P})$, where $I(\mathcal{B},\mathcal{P})$ is the ideal in $K\Delta(\mathcal{B},\mathcal{P})$ generated by $I^{(j)}$ for $j \in \{0,1\} \setminus \{i\}$. The module E^{λ_i} has a minimal projective resolution of the form

$$0 \to e_0C \xrightarrow{f_i} e_\omega C \to E^{\lambda_i} \to 0$$

where f_i is given by left multiplication by $\overline{\alpha}_{i,1} \in e_\omega Ce_0$. Then $\tau_C E^{\lambda_i}$ is determined by the exact sequence

$$0 \to \tau_C E^{\lambda_i} \to \nu_C(e_0C) \xrightarrow{\nu_C(f_i)} \nu_C(e_\omega C) \to \nu_C E^{\lambda_i} \to 0.$$

Invoking again the functors $R : \operatorname{mod} C \to \operatorname{mod} A$ and $T : \operatorname{mod} A \to \operatorname{mod} C$ we conclude as in the first part of our proof that $R(E^{\lambda_i}) \cong F^{\lambda_i}$, $R(\tau_C E^{\lambda_i}) \cong \tau_A F^{\lambda_i} \cong F^{\lambda_i}$ for some indecomposable module F^{λ_i} lying on the mouth of a homogeneous tube of $\Gamma_{\operatorname{mod} A}$, and finally that $E^{\lambda_i} \cong T(F^{\lambda_i}) \cong \tau_C E^{\lambda_i}$. Moreover, $\operatorname{End}_C(E^{\lambda_i}) \cong \operatorname{End}_{B_j}(P_{B_j}(\omega, j))$, hence E^{λ_i} is a brick. Finally, $\operatorname{Ext}^2_C(E^{\lambda_i}, E^{\lambda_i}) = 0$, since $\operatorname{pd}_C E^{\lambda_i} = 1$. Therefore, applying Proposition 1.1, we conclude that $\mathcal{T}_{\lambda_i}$ is a standard homogeneous tube having E^{λ_i} on the mouth.

We now prove that the stable tubes $\mathcal{T}_\mu$, $\mu \in \mathbb{P}_1(K) \setminus \Sigma$, are pairwise orthogonal. It is known (see [23, Lemma 3.9]) that $\operatorname{Hom}_C(\mathcal{T}_\mu, \mathcal{T}_\eta) \neq 0$ for $\mu, \eta \in \mathbb{P}_1(K) \setminus \Sigma$, $\mu \neq \eta$, if and only if $\operatorname{Hom}_C(M, N) \neq 0$ for M lying on the mouth of $\mathcal{T}_\mu$ and N lying on the mouth of $\mathcal{T}_\eta$. Observe that the modules E^ξ, $\xi \in \mathbb{P}_1(K) \setminus \Sigma$, have simple socles isomorphic to $S_C(0)$ and simple tops isomorphic to $S_C(\omega)$. Hence, these modules are orthogonal to the simple modules $S_C((r, i))$ given by the middle vertices (r, i) of the linear quivers $\Delta^{(i)}$, $i \in \Phi$. Thus it is enough to show that $\operatorname{Hom}_C(E^\xi, E^\varrho) = 0$ for all $\xi \neq \varrho$ from $\mathbb{P}_1(K) \setminus \Sigma$. Suppose there is a nonzero homomorphism $g : E^\xi \to E^\varrho$ for some $\xi \neq \varrho$ in $\mathbb{P}_1(K) \setminus \Sigma$. Then the induced homomorphism $\operatorname{top}(g) : \operatorname{top}(E^\xi) \to \operatorname{top}(E^\eta)$ on the tops in an isomorphism, and hence the restriction $R(g) : R(E^\xi) \to R(E^\varrho)$ is a nonzero homomorphism of A-modules. This leads to a contradiction since $R(E^\xi) \cong F^\xi$ and $R(E^\varrho) \cong F^\varrho$ are pairwise orthogonal indecomposable regular A-modules due to the fact that the elements $u_0 + \xi u_1$ and $u_0 + \varrho u_1$ of $e_\omega Ae_0$ are linearly independent for $\xi \neq \varrho$. This finishes the proof of (i), (ii) and (iii).

For (iv), we first show that the tubes of $\mathcal{T}$ are sincere if and only if the B_i-modules P_i, $1 \leq i \leq m$, are sincere. Observe that a stable tube $\mathcal{T}_\mu$, $\mu \in \mathbb{P}_1(K) \setminus \Sigma$, is sincere if and only if the direct sum of all modules lying on the mouth of $\mathcal{T}_\mu$ is a sincere C-module. Recall also that, for $\xi \in \mathbb{P}_1(K) \setminus \Lambda$, we have $\operatorname{rad} E^\xi / \operatorname{soc} E^\xi \cong (\operatorname{rad} P_0 / \operatorname{soc} P_0) \oplus \ldots \oplus (\operatorname{rad} P_m / \operatorname{soc} P_m)$, while $\operatorname{rad} E^{\lambda_i} / \operatorname{soc} E^{\lambda_i} \cong \bigoplus_{j \neq i} (\operatorname{rad} P_j / \operatorname{soc} P_j)$ for each element $\lambda_i \in \Omega$. Moreover, for $\lambda_i \in \Omega$, $\operatorname{rad} P_i/\operatorname{soc} P_i$ is an indecomposable module lying in the tube $\mathcal{T}_{\lambda_i}$. Therefore, all tubes $\mathcal{T}_\mu$, $\mu \in \mathbb{P}_1(K) \setminus \Sigma$, are sincere in $\operatorname{mod} C$ if and only if the B_i-modules P_i, $0 \le i \le m$, are sincere.

Suppose now that there is $i \in \{0, \dots, m\}$ such that P_i is not a faithful B_i-module. Since P_i is an indecomposable projective-injective B_i-module, there exist two vertices x and y in $\Delta^{(i)}$, different from (ω, i) and $(0, i)$, and a nonzero element $e_x b e_y \in e_x B_i e_y$ such that $P_i e_x b e_y = 0$. Then, for any $\xi \in \mathbb{P}_1(K) \setminus \Lambda$, we have $E^\xi e_x b e_y = 0$ and $0 \ne e_x b e_y \in e_x B_i e_y = e_x C e_y$. Since E^ξ lies on the mouth of the homogeneous tube $\mathcal{T}_\xi$, this implies that $M e_x b e_y = 0$ for any module M from $\mathcal{T}_\xi$. Therefore, the tubes $\mathcal{T}_\xi$, $\xi \in \mathbb{P}_1(K) \setminus \Lambda$, are not faithful.

Finally, assume that each P_i is a faithful B_i-module. We now prove that all stable tubes $\mathcal{T}_\mu$, $\mu \in \mathbb{P}_1(K) \setminus \Sigma$, are faithful. In fact, we now show that each tube $\mathcal{T}_\mu$, $\mu \in \mathbb{P}_1(K) \setminus \Sigma$, contains an indecomposable faithful C-module. Take $\xi \in \mathbb{P}_1(K) \setminus \Lambda$. Then there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_C(0) & \xrightarrow{f_\xi} & P_C(\omega) & \xrightarrow{\pi_\xi} & E^\xi & \longrightarrow & 0 \\
 & & {\scriptstyle h'}\big\downarrow & & \big\downarrow{\scriptstyle h} & & \big\| & & \\
0 & \longrightarrow & E^\xi & \xrightarrow{w} & E^\xi[2] & \xrightarrow{v} & E^\xi & \longrightarrow & 0
\end{array}$$

where the upper sequence is the canonical minimal projective resolution of E^ξ and the lower sequence is an almost split sequence with the right (equivalently, left) term E^ξ. We claim that $h' \ne 0$. Suppose this is not the case. Then there exists a homomorphism $g : E^\xi \to E^\xi[2]$ such that $h = g\pi_\xi$. But then $\pi_\xi = vh = vg\pi_\xi$, and hence $vg = 1_{E^\xi}$, a contradiction because v is not a split epimorphism. Therefore, $h' \ne 0$ and consequently h' is a monomorphism, since $S_C(0)$ is a simple module. This implies that h is a monomorphism. Since P_i is a faithful B_i-module, there is a monomorphism $B_i \to P_i^{n_i}$ for some $n_i \ge 1$. Then, if $B_i = P_i \oplus P_i'$, there is a monomorphism $P_i' \to (\operatorname{rad} P_i)^{n_i}$. Further, observe that, for each $i \in \{0, \dots, m\}$, $\operatorname{rad} P_i$ is a submodule of $\operatorname{rad} E^\xi$. Since there are monomorphisms $w : E^\xi \to E^\xi[2]$ and $h : P_C(\omega) \to E^\xi[2]$, we conclude that there is a monomorphism $C \to (E^\xi[2])^n$ of C-modules for some $n \ge 1$. This shows that $E^\xi[2]$ is a faithful C-module in $\mathcal{T}_\xi$, and hence $\mathcal{T}_\xi$ is a faithful stable tube.

Let Ω be nonempty and λ_i an element of Ω. If $\Delta^{(i)}$ is the quiver $(\omega, i) \to (0, i)$, then we show as above that the indecomposable C-module $E^\xi[2]$ in $\mathcal{T}_{\lambda_i}$, being the middle term of an almost split sequence with right term E^ξ, is a faithful C-module. Therefore, we may assume that the equioriented linear quiver $\Delta^{(i)}$ consists of at least 2 arrows, or equivalently, $\mathcal{T}_{\lambda_i}$ is of rank $p_i \ge 2$. Denote by M^{λ_i} the indecomposable module in $\mathcal{T}_{\lambda_i}$ of quasilength p_i lying on the intersection of the infinite sectional path starting at E^{λ_i} and the infinite sectional path ending in $S_C((p_i - 1, i))$. Further, denote by N^{λ_i} the indecomposable module in $\mathcal{T}_{\lambda_i}$ of quasilength p_i+1 lying on the intersection of the infinite sectional path starting at E^{λ_i} and the infinite sectional path ending in E^{λ_i}. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_C((p_i-1,i)) & \xrightarrow{f_i} & P_C(\omega) & \xrightarrow{\pi_i} & E^{\lambda_i} & \longrightarrow & 0 \\ & & {\scriptstyle h'}\big\downarrow & & \big\downarrow{\scriptstyle h} & & \big\| & & \\ 0 & \longrightarrow & M^{\lambda_i} & \xrightarrow{w} & N^{\lambda_i} & \xrightarrow{v} & E^{\lambda_i} & \longrightarrow & 0 \end{array}$$

where the upper sequence is the canonical minimal projective resolution of E^{λ_i} and the lower one is the canonical nonsplittable short exact sequence with w being an irreducible monomorphism. Since v is not a split epimorphism, we conclude as above that $h' \neq 0$.

We claim that h' is a monomorphism. Indeed, if h' is not a monomorphism then its image is a uniserial C-module whose socle is not isomorphic to $S_C(0)$. On the other hand, it follows from our choice of M^{λ_i} that this module has a simple socle isomorphic to $S_C(0)$. Therefore, h' is a monomorphism, and consequently h is also a monomorphism. Moreover, it follows from the choice of M^{λ_i} that there are monomorphisms $P_C((r,i)) \to M^{\lambda_i}$ for all $r \in \{1,\dots,p_{i-1}\}$. We also note that there is a monomorphism $E^{\lambda_i} \to M^{\lambda_i}$, $\operatorname{rad} P_j$ is a submodule of $\operatorname{rad} E^{\lambda_i}$, and there is a monomorphism $P'_j \to (\operatorname{rad} P_j)^{n_j}$ for any $j \in \{0,\dots,m\} \setminus \{i\}$. Since we also have monomorphisms $w : M^{\lambda_i} \to N^{\lambda_i}$ and $h : P_C(\omega) \to N^{\lambda_i}$, we conclude that there is a monomorphism $C \to (N^{\lambda_i})^n$ of C-modules for some $n \geq 1$. This shows that N^{λ_i} is a faithful C-module lying in $\mathcal{T}_{\lambda_i}$, and consequently, $\mathcal{T}_{\lambda_i}$ is a faithful stable tube. Therefore, the family $\mathcal{T} = (\mathcal{T}_\mu)_{\mu \in \mathbb{P}_1(K)\setminus\Sigma}$ consists of faithful stable tubes. This finishes the proof of (iv), and hence of Theorem 2.1.

We obtain the following immediate consequences of the theorem.

Corollary 2.2. *Let $C(\mathcal{B},\mathcal{P},\Lambda)$ be a generalized canonical algebra. Then the Auslander–Reiten quiver $\Gamma_{\operatorname{mod} C(\mathcal{B},\mathcal{P},\Lambda)}$ admits an infinite family of pairwise orthogonal faithful standard stable tubes.*

Corollary 2.3. *Let $C = C(\mathcal{B},\mathcal{P},\Lambda)$ be a generalized canonical algebra. Then $\operatorname{soc}(C_C)$ and $\operatorname{soc}({}_CC)$ are projective modules.*

Proof. It follows from the definition of C that there are a monomorphism $C_C \to P_C(\omega)^r$ and an epimorphism $I_C(0)^s \to D({}_CC)$ for some $r,s \geq 1$. Since $\operatorname{soc} P_C(\omega)$ is projective and $\operatorname{top} I_C(0)$ is injective, we conclude that $\operatorname{soc}(C_C)$ and $\operatorname{soc}({}_CC)$ are projective.

The following corollary shows that there are generalized canonical algebras of arbitrary global dimension.

Corollary 2.4. *Let $p_1, \ldots, p_m$, $m \geq 1$, be a sequence of integers from $\mathbb{N}_2 = \mathbb{N} \setminus \{0,1\}$ and $n \in \mathbb{N}_2 \cup \{\infty\}$. Then there is a generalized canonical algebra $C(\mathcal{B}, \mathcal{P}, \Lambda)$ such that* $\operatorname{gl.dim} C(\mathcal{B}, \mathcal{P}, \Lambda) = n$ *and $\Gamma_{\operatorname{mod} C(\mathcal{B},\mathcal{P},\Lambda)}$ admits a family $\mathcal{T} = (\mathcal{T}_\mu)_{\mu \in K}$ of pairwise orthogonal faithful standard stable tubes having m tubes of ranks $p_1, \ldots, p_m$, respectively, and the remaining tubes homogeneous.*

Proof. For each $i \in \{1, \ldots, m\}$, let B_i be the path algebra of the quiver $(\omega, i) \to (p_i - 1, i) \to \cdots \to (1, i) \to (0, i)$ and P_i the unique indecomposable projective-injective (faithful) module. We also define an algebra B_0 and a faithful indecomposable projective-injective B_0-module P_0, depending on $n \in \mathbb{N}_2 \cup \{\infty\}$. For $n \in \mathbb{N}_2$, we define B_0 to be the incidence algebra of the partially ordered set

$$\begin{tikzcd} & 1 \arrow[dl] & \\ 0 & & \omega \arrow[ul] \arrow[dl] \\ & 1' \arrow[ul] & \end{tikzcd}$$

if $n = 2$, and the incidence algebra of the partially ordered set

$$\begin{tikzcd} & 1 \arrow[dl] & 2 \arrow[l] \arrow[ddl] & \cdots \arrow[l] \arrow[ddl] & n-2 \arrow[l] \arrow[ddl] & n-1 \arrow[l] \arrow[ddl] & \\ 0 & & & & & & \omega \arrow[ul] \arrow[dl] \\ & 1' \arrow[ul] & 2' \arrow[l] \arrow[uul] & \cdots \arrow[l] \arrow[uul] & (n-2)' \arrow[l] \arrow[uul] & (n-1)' \arrow[l] \arrow[uul] & \end{tikzcd}$$

for $3 \leq n < \infty$, and denote by P_0 the unique (faithful) indecomposable projective-injective B_0-module. Further, for $n = \infty$, let B_0 be the bound quiver algebra $K\Delta^{(0)}/I^{(0)}$ where $\Delta^{(0)}$ is the quiver

$$\begin{tikzcd} 0 & 1 \arrow[l, "\alpha"'] \arrow[loop below, "\beta"] & \omega \arrow[l, "\gamma"'] \end{tikzcd}$$

and $I^{(0)}$ is generated by β^2 and $\gamma\alpha - \gamma\beta\alpha$, and P_0 be the unique (faithful) indecomposable projective-injective B_0-module. Observe that $\operatorname{gl.dim} B_0 = n$. Put $\mathcal{B} = \{B_0, \ldots, B_m\}$, $\mathcal{P} = \{P_0, \ldots, P_m\}$ and take an arbitrary normalized set $\Lambda = \{\lambda_0, \ldots, \lambda_m\}$ of pairwise different elements of $\mathbb{P}_1(K)$. Then it follows from Theorem 2.1 that $C(\mathcal{B}, \mathcal{P}, \Lambda)$ is a generalized canonical algebra with the required properties.

Our final corollary shows that an arbitrary basic algebra can be a factor algebra of a generalized canonical algebra.

Corollary 2.5. *Let B be a basic algebra and M be an arbitrary faithful B-module. Then there is a generalized canonical algebra $C = C(\mathcal{B}, \mathcal{P}, \Lambda)$*

such that B is a factor algebra of C and M is a factor module of the quotient $\operatorname{rad} P_C(\omega)/\operatorname{soc} P_C(\omega)$.

Proof. Consider the one-point extension

$$B' = B[M] = \begin{bmatrix} K & M \\ 0 & B \end{bmatrix}$$

of B by M. Since M is a faithful B-module, B' is a basic connected algebra having an indecomposable projective faithful module Q with $\operatorname{rad} Q \cong M$. Next consider the one-point coextension

$$B'' = [Q]B' = \begin{bmatrix} B' & D(Q) \\ 0 & K \end{bmatrix}$$

of B' by Q. Then B'' is a basic connected algebra having a faithful indecomposable projective-injective module P such that $P/\operatorname{soc} P \cong Q$. Observe that B is a factor algebra of B'' and $M \cong \operatorname{rad} P/\operatorname{soc} P$. Take now an arbitrary generalized canonical algebra $C = C(\mathcal{B}, \mathcal{P}, \Lambda)$ with $\mathcal{B}$ containing B'' and $\mathcal{P}$ containing P. Clearly then B is a factor algebra of C and M is a factor module of $\operatorname{rad} P_C(\omega)/\operatorname{soc} P_C(\omega)$.

We end the paper with several remarks.

REMARKS 2.6. (1) Following Thrall [27], an algebra A is called a (right) QF-3 *algebra* if there is a unique (up to isomorphism) minimal faithful (right) A-module. It has been shown in [27] (see also [28, Proposition 3.1.1]) that A is a QF-3 algebra if and only if there is a projective-injective faithful right A-module, and if and only if the injective hull of A_A is projective. Therefore, the algebras B_i with faithful indecomposable projective-injective modules P_i used in our construction of the algebras $C(\mathcal{B}, \mathcal{P}, \Lambda)$ are special types of QF-3 algebras.

(2) It is known that a stable tube $\mathcal{C}$ is faithful if and only if all but finitely many indecomposable modules in $\mathcal{C}$ are faithful. Hence all but finitely many indecomposable modules of any faithful standard stable tube $\mathcal{T}_\mu$, $\mu \in \mathbb{P}_1(K) \setminus \Sigma$, of the Auslander–Reiten quiver $\Gamma_{\operatorname{mod} C(\mathcal{B}, \mathcal{P}, \Lambda)}$ of a generalized canonical algebra $C(\mathcal{B}, \mathcal{P}, \Lambda)$ are faithful.

(3) Our assumption on the field K to be algebraically closed is only for simplicity of construction of the algebras $C(\mathcal{B}, \mathcal{P}, \Lambda)$ and the proof of Theorem 2.1. In [20] C. M. Ringel introduced the class of canonical algebras over an arbitrary (commutative) field K, invoking an arbitrary tame bimodule instead of the Kronecker quiver, and proved that their Auslander–Reiten quivers admit an infinite family of faithful generalized standard stable tubes. Invoking [20] one can extend our construction of the algebras $C(\mathcal{B}, \mathcal{P}, \Lambda)$ in an obvious way and prove the corresponding version of Theorem 2.1 for algebras over an arbitrary field. In particular, we obtain generalized canonical algebras over arbitrary fields.

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Received 12 April 2001 (4049)