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THE PRINCIPAL SPECTRUM FOR LINEAR NONAUTONOMOUS PARABOLIC PDEs OF SECOND ORDER: SPACE-INDEPENDENT CASE

BY

JANUSZ MIERCZYŃSKI (Wrocław)

Abstract. We show that the study of the principal spectrum of a linear nonautonomous parabolic PDE of second order $u_t = \Delta u + a(t)u$ on a bounded domain, with the Dirichlet or Neumann boundary conditions, reduces to the investigation of the spectrum of the linear nonautonomous ODE $\dot{v} = a(t)v$.

In [3] the author presented a theory of principal spectrum for linear nonautonomous parabolic partial differential equations (PDEs) of second order

$$u_t = \Delta u + a(t,x)u, \quad x \in \Omega,\ t > 0,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. The equation is complemented with the boundary conditions either of the Dirichlet type or the Robin (regular oblique) type.

The keystone of the theory is the construction of a one-dimensional invariant subbundle $\mathcal{S}$ (the *Kreĭn–Rutman bundle*). The principal spectrum is defined now to be the dynamical (Sacker–Sell) spectrum of the linear skew-product flow generated on $\mathcal{S}$ by the equation. It follows from the general theory of linear skew-product flows that the principal spectrum is a nonempty compact interval.

Moreover, it was proved that both the supremum and infimum of that principal spectral interval are nondecreasing functions of the zero-order term a, which gives us a useful tool for estimates.

The purpose of the present paper is to prove that if a is an essentially bounded function depending on t only then the computation of the principal spectrum consists in finding the exponential growth rates of solutions of the scalar ordinary differential equation $\dot{v} = a(t)v$. The theory of growth rates for systems of ordinary differential equations (ODEs) was presented in the book [1] by B. F. Bylov *et al.*

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Consider a linear parabolic partial differential equation of second order

$$u_t = \Delta u + a(t)u, \qquad x \in \Omega,\ t > 0, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ of class C^2 and $a \in L^\infty(0,\infty)$. Equation (1) is complemented either with the Dirichlet boundary conditions

$$u(t,x) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{2a}$$

or with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}(t,x) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{2b}$$

where $\nu\colon \partial\Omega \to \mathbb{R}^N$ is the normal unit outward vector field.

For $t \ge 0$ put $(a\cdot t)(s) := a(t+s)$. Denote by $\mathbb{A}$ the closure in the weak-* topology of the set $\{a\cdot t : t \ge 0\} \subset L^\infty(0,\infty)$. It is well known that

(i) $\mathbb{A}$ is a compact metrizable space,
(ii) the mapping $\mathbb{A} \times [0,\infty) \ni (b,t) \mapsto b\cdot t \in \mathbb{A}$ is continuous.

For $b \in \mathbb{A}$ denote by B the solution of the initial value problem

$$\begin{cases} \dot v = b(t)v, \\ v(0) = 1. \end{cases} \tag{3}$$

It is well known that

$$B(t) = \exp\Big(\int_0^t b(s)\,ds\Big). \tag{4}$$

We denote by $C([0,\infty))$ the Fréchet space of continuous real functions on $[0,\infty)$ (endowed with the ordinary topology of uniform convergence on compact sets).

PROPOSITION 1. *The assignment* $\mathbb{A} \ni b \mapsto B \in C([0,\infty))$ *is continuous.*

Proof. Let $b_n \to b$ in $\mathbb{A}$. Fix $T > 0$. The set $\{B_n|_{[0,T]}\}$ is clearly bounded and equicontinuous in the Banach space $C([0,T])$ of continuous real functions with the supremum norm, so by the Ascoli–Arzelà theorem there is a subsequence converging to some $\widetilde{B} \in C([0,T])$. But for each $t \in [0,T]$ one has

$$\begin{aligned}\lim_{n\to\infty} B_n(t) &= \lim_{n\to\infty} \exp\Big(\int_0^t b_n(s)\,ds\Big) = \lim_{n\to\infty} \exp\Big(\int_0^\infty b_n(s)\mathbf{1}_{[0,t]}(s)\,ds\Big)\\ &= \exp\Big(\int_0^\infty b(s)\mathbf{1}_{[0,t]}(s)\,ds\Big) = \exp\Big(\int_0^t b(s)\,ds\Big) = B(t),\end{aligned}$$

from which it follows that $B_n|_{[0,T]}$ converges in $C([0,T])$ to $B|_{[0,T]}$. ∎

For $1 < p < \infty$ denote by $\{e^{\Delta_p t}\}_{t\ge 0}$ the analytic semigroup of bounded linear operators generated on $L^p(\Omega)$ by the closure of the Laplace operator Δ with the corresponding boundary conditions (2). Let $\mathbf{e}$ stand for the principal eigenfunction (we choose $\mathbf{e}$ so that $\mathbf{e}(x) > 0$ for $x \in \Omega$ and $\|\mathbf{e}\|_{L^2(\Omega)} = 1$), and let Λ stand for the principal eigenvalue of the Laplacian with the boundary conditions (2).

Consider the equation

$$u_t = \Delta u + b(t)u, \qquad x \in \Omega,\ t > 0, \tag{5}$$

where $b \in \mathbb{A}$, endowed with the corresponding boundary conditions (2) and the initial condition

$$u(0,x) = u_0(x), \qquad x \in \Omega. \tag{6}$$

For $b \in \mathbb{A}$, $1 < p < \infty$, $u_0 \in L^p(\Omega)$ and $t \ge 0$ put

$$u_p(t; b, u_0) := B(t) e^{\Delta_p t} u_0. \tag{7}$$

The theory of principal spectrum was presented in [3] in an axiomatic way. We now briefly analyze the fulfillment of those axioms in the present setting.

Axiom (A1) states that the parameter space $\mathbb{A}$ is a convex compact metrizable subset of a topological vector space consisting of (equivalence classes of) Lebesgue measurable functions from $(0,\infty) \times \Omega$ into $\mathbb{R}^{N^2+N+1}$. As regards the domain and the dimensionality of the target space, now only the zero-order term depends on t (and is independent of x), so the choice of functions from $(0,\infty)$ to $\mathbb{R}$ is natural here. In the present setting $\mathbb{A}$ need not be convex, since in [3] the convexity of $\mathbb{A}$ was used only in Part 3, whereas in the main part of the theory (used in the present paper) it was not necessary.

Axiom (A2) states the translation invariance of $\mathbb{A}$ and (A3) asserts that the translation is continuous. Both are clearly satisfied (see (i) and (ii)).

The next six axioms (A4)–(A9) concern the properties of the solution. Axiom (A4) establishes the existence of a solution considered a function from $[0,\infty)$ to $L^p(\Omega)$.

Axiom (A5) states that the solution depends continuously (as an element of the Banach space $C([0,T], L^p(\Omega))$) on initial conditions and parameters. Its fulfillment follows from the following

LEMMA 2. *Let $T > 0$. Assume that b_n converges in $\mathbb{A}$ to b and u_n converges in $L^p(\Omega)$ to u_0. Then $u_p(\cdot; b_n, u_n)$ converges in $C([0,T], L^p(\Omega))$ to $u_p(\cdot; b_n, u_n)$.*

Proof. We have

$$\begin{aligned}\sup_{t\in[0,T]} \|B_n(t)e^{\Delta_p t}u_n - B(t)e^{\Delta_p t}u_0\|_{L^p(\Omega)} \\ \le \|B_n - B\|_{C([0,T])} \cdot \sup_{t\in[0,T]} \|e^{\Delta_p t}u_0\|_{L^p(\Omega)} \\ + \|B_n\|_{C([0,T])} \cdot \sup_{t\in[0,T]} \|e^{\Delta_p t}(u_n - u_0)\|_{L^p(\Omega)}.\end{aligned}$$

The convergence to zero follows by Proposition 1 and the fact that $\{e^{\Delta_p t}\}_{t\ge 0}$ is a C_0-semigroup of bounded linear operators on $L^p(\Omega)$. ■

Axiom (A6) states that $u_p(t;u_0,b)$ is in fact an element of $C^1(\overline{\Omega})$ for $t>0$ (this allows us to write simply $u(t;u_0,b)$), and (A7) states that for $t>0$ fixed the mapping $L^p(\Omega) \ni u_0 \mapsto u(t;u_0,b) \in C^1(\overline{\Omega})$ is completely continuous. The fulfillment of both these axioms follows from the fact that they are satisfied for the Laplacian.

The same argument holds in the case of Axiom (A8), which says that for $t>0$ and $b\in\mathbb{A}$ the linear mapping $u_0\mapsto u(t;u_0,b)$ extends to an operator in $\mathcal{L}(C^1(\overline{\Omega})^*, C^1(\overline{\Omega}))$, where * stands for the dual.

Axiom (A9) stipulates that for $0<T_1\le T_2$ the mapping assigning to a parameter $b\in\mathbb{A}$ the function $u_0\mapsto u(\cdot;u_0,b)$ considered an element of $C([T_1,T_2],\mathcal{L}(C^1(\overline{\Omega})^*, C^1(\overline{\Omega})))$, $0\le T_1<T_2$, is continuous. Its fulfillment again follows easily by the construction of the solution.

Axiom (A10), stating that $u(t;u_0,b)$ belongs to the interior of the corresponding nonnegative cone provided $u_0(x)>0$ for a.e. $x\in\Omega$ and $u_0\ne 0$, is satisfied as it holds for the Laplacian.

Axiom (A11), regarding the monotone dependence of the solution with respect to initial values and zero-order terms, is satisfied by the form (7) of the solution and the maximum principles. (In fact, (A11) is not needed in the definition of principal spectrum.)

Denote by $v\colon[0,\infty)\to L^2(\Omega)$ the solution $v(t):=u_2(t;a,\mathbf{e})$. From now on, let $\|\cdot\|$ stand for the $L^2(\Omega)$-norm.

By Thm. 2.12 in Mierczyński [3], $\lambda\in\mathbb{R}$ belongs to the principal spectrum of (1) if and only if there are sequences $0\le s_n<t_n$, $s_n\to\infty$, $t_n-s_n\to\infty$, such that

$$\lim_{n\to\infty}\frac{\log\|v(t_n)\|-\log\|v(s_n)\|}{t_n-s_n}=\lambda.$$

As $v(t)=A(t)e^{\Delta_2 t}\mathbf{e}=A(t)e^{-\Lambda t}\mathbf{e}$, where $A(t)=\exp(\int_0^t a(\theta)\,d\theta)$, we obtain our main result.

THEOREM 3. *$\lambda\in\mathbb{R}$ belongs to the principal spectrum of* (1) *if and only if there are sequences $s_n<t_n$, $s_n\to\infty$, $t_n-s_n\to\infty$, such that*

$$\lim_{n\to\infty} \frac{1}{t_n - s_n} \int\limits_{s_n}^{t_n} a(\theta)\, d\theta = \lambda + \Lambda.$$

In particular, under the Neumann boundary conditions the principal spectrum equals the set

$$\Big\{ \lim_{n\to\infty} \frac{1}{t_n - s_n} \int\limits_{s_n}^{t_n} a(\theta)\, d\theta : s_n < t_n,\ \lim_{n\to\infty} s_n = \infty,\ \lim_{n\to\infty} (t_n - s_n) = \infty \Big\}.$$

An important special case arises when a is (the restriction to $(0,\infty)$ of) a Bohr almost periodic function on $(-\infty,\infty)$. Then the set of all possible limits

$$\lim \frac{1}{t_n - s_n} \int\limits_{s_n}^{t_n} a(\theta)\, d\theta$$

reduces to $\{\overline{a}\}$, where $\overline{a}$ is the average of a,

$$\overline{a} := \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T a(\theta)\, d\theta.$$

According to Theorem 3 the principal spectrum is the singleton $\{\overline{a} - \Lambda\}$. (For more on the theory of principal spectrum in the almost periodic case see Hutson, Shen and Vickers [2].)

Our last remark concerns the situation when the equation (1) is asymptotically autonomous in the sense that there is $a^* \in \mathbb{R}$ such that

$$\frac{1}{t-s} \int\limits_s^t |a(\theta) - a^*|\, d\theta \to 0 \quad \text{as } s \to \infty,\ t - s \to \infty. \tag{8}$$

Then it follows in a straightforward way from Theorem 3 that the principal spectrum equals $\{a^* - \Lambda\}$.

Condition (8) is satisfied for example if

$$\lim_{t\to\infty} a(t) = a^*$$

or if

$$\int\limits_t^\infty |a(\theta) - a^*|\, d\theta \to 0 \quad \text{as } t \to \infty.$$

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Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: mierczyn@ulam.im.pwr.wroc.pl
Web: http://www.im.pwr.wroc.pl/~mierczyn

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