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COALGEBRAS, COMODULES, PSEUDOCOMPACT ALGEBRAS AND TAME COMODULE TYPE

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Abstract. We develop a technique for the study of K-coalgebras and their representation types by applying a quiver technique and topologically pseudocompact modules over pseudocompact K-algebras in the sense of Gabriel [17], [19]. A definition of tame comodule type and wild comodule type for K-coalgebras over an algebraically closed field K is introduced. Tame and wild coalgebras are studied by means of their finite-dimensional subcoalgebras. A weak version of the tame-wild dichotomy theorem of Drozd [13] is proved for a class of K-coalgebras. By applying [17] and [19] it is shown that for any length K-category $\mathfrak{A}$ there exists a basic K-coalgebra C and an equivalence of categories $\mathfrak{A} \cong C$-comod. This allows us to define tame representation type and wild representation type for any abelian length K-category.

Hereditary coalgebras and path coalgebras KQ of quivers Q are investigated. Tame path coalgebras KQ are completely described in Theorem 9.4 and the following K-coalgebra analogue of Gabriel's theorem [18] is established in Theorem 9.3. An indecomposable basic hereditary K-coalgebra C is left pure semisimple (that is, every left C-comodule is a direct sum of finite-dimensional C-comodules) if and only if the quiver ${}_CQ^*$ opposite to the Gabriel quiver ${}_CQ$ of C is a pure semisimple locally Dynkin quiver (see Section 9) and C is isomorphic to the path K-coalgebra $K({}_CQ)$. Open questions are formulated in Section 10.

1. Introduction. Throughout this paper we fix field K. Given a K-coalgebra C we denote by C-Comod the category of left C-comodules, and by C-comod the full subcategory of C-Comod formed by comodules of finite K-dimension. The category of right C-comodules is denoted by Comod-C.

It is well known (see [28], [56]) that every C-comodule M is a directed union of finite-dimensional subcomodules and therefore the Grothendieck category C-Comod is locally finite and C-comod is the full subcategory of C-Comod consisting of objects of finite length (see [17], [34]).

One of the main aims of this paper is to introduce a concept of tame comodule type, polynomial growth and wild comodule type for K-coalgebras over an algebraically closed field K (see Definition 6.6), and to study tame and wild coalgebras by means of their finite-dimensional subcoalgebras. In particular, path coalgebras and hereditary left pure semisimple coalgebras C

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and their comodule categories C-comod are studied by means of the Gabriel quiver ${}_CQ$ of C and nilpotent K-linear representations of ${}_CQ$ of finite length (see Sections 8 and 9).

To make the exposition self-contained and elementary we collect in Sections 2, 3 and 4 the background facts on linear topological rings and modules, coalgebras, comodules and their functorial relations with pseudocompact algebras and pseudocompact modules in the sense of Gabriel [17]. Some of our main tools in the study of comodules are the category isomorphism $C\text{-Comod} \cong \mathrm{Dis}(C^*)$ and a pair of duality functors

$$C\text{-Comod} \underset{\widetilde{D}_2}{\overset{\widetilde{D}_1}{\rightleftarrows}} C^*\text{-PC}$$

described in Theorem 4.3, where $C^* = \mathrm{Hom}_K(C, K)$ is the K-algebra dual to C equipped with a profinite topology (3.4), $\mathrm{Dis}(C^*)$ is the category of discrete right C^*-modules and C^*-PC is the category of pseudocompact left C^*-modules. In particular, we get a pair of duality functors

$$C\text{-comod} \underset{D}{\overset{D}{\rightleftarrows}} \text{comod-}C,$$

where comod-C is the category of finite-dimensional right C-comodules. If C is finite-dimensional we get $C\text{-Comod} \cong \mathrm{Mod}(C^*)$.

In Section 5 we recall from [43] the definition of a basic K-coalgebra. By applying the duality functors $\widetilde{D}_1$ and $\widetilde{D}_2$ we associate a basic coalgebra C^b with any K-coalgebra C in such a way that $C\text{-comod} \cong C^b\text{-Comod}$. We prove in Proposition 5.6 that C^b is uniquely determined by C, up to K-coalgebra isomorphism. Moreover, we show in Corollary 5.10 that for any abelian length K-category $\mathfrak{A}$ there exist a directed family $\{\mathfrak{A}_\beta\}_\beta$ of full exact K-subcategories of $\mathfrak{A}$ and an inverse system $\{R_\beta, f_{\beta,\gamma}\}_{\beta\preceq\gamma}$ of finite-dimensional K-algebras R_β connected by K-algebra surjections $f_{\beta,\gamma} : R_\gamma \to R_\beta$ such that

$$\mathfrak{A} = \bigcup_\beta \mathfrak{A}_\beta,$$

$\mathfrak{A}_\beta \cong \mathrm{mod}(R_\beta)$, the embedding $\mathfrak{A}_\beta \subseteq \mathfrak{A}_\gamma$ is induced by $f_{\beta,\gamma}$ for all $\beta \preceq \gamma$, and $R = \varprojlim_\beta\{R_\beta, f_{\beta,\gamma}\}$ is a pseudocompact K-algebra.

This useful observation generalises a result in [43] and allows us to study tame representation type of $\mathfrak{A}$ by means of the representation type of the finite-dimensional K-algebras R_β. In general we would like to study global properties of $\mathfrak{A}$ by means of its local properties.

It is shown in Section 6 that any K-coalgebra C of tame comodule type is a directed union of finite-dimensional K-coalgebras C_β of tame comodule type, and the finite-dimensional K-algebras C_β^* are of tame representation type. If C is such that $\dim_K \operatorname{Ext}^1_C(S, S')$ is finite for every pair S, S' of simple left C-comodules, then C is of wild comodule type if and only if C is a directed union of finite-dimensional subcoalgebras C_β such that the finite-dimensional K-algebras C^*_β are of wild representation type. Hence we conclude that such K-coalgebras of tame comodule type are not of wild comodule type, and K-coalgebras of wild comodule type are not of tame comodule type. Unfortunately we are able to prove a tame-wild dichotomy type theorem of Drozd [13] only for a class of K-coalgebras.

A connection between finite representation type and pure semisimplicity for coalgebras is given in Section 7. In particular, it is shown that left pure semisimple coalgebras are of tame comodule type, and basic coalgebras C of finite comodule type are finite-dimensional and there is an equivalence of categories C-Comod $\cong \operatorname{Mod}(C^*)$.

Hereditary coalgebras and path coalgebras KQ of quivers Q are investigated in Section 9. It is shown in Proposition 8.13 that KQ is hereditary. We recall that a K-coalgebra C is called *hereditary* if the category C-Comod of left C-comodules is hereditary, that is, $\operatorname{Ext}^2_C(M, N) = 0$ for all M and N in C-Comod, or equivalently, epimorphic images of injective C-comodules are injective C-comodules. It was shown in [30] that the definition is left-right symmetric (see also [11]).

Path coalgebras KQ of tame comodule type are completely described in Theorem 9.4. Wild and fully wild path coalgebras are also investigated. A characterisation of left pure semisimple hereditary basic coalgebras is given in Theorem 9.3. This is a coalgebra analogue of Gabriel's well known characterisation [18] of representation-finite quivers Q asserting that the path K-algebra KQ of a finite connected quiver Q is representation-finite (or equivalently, left pure semisimple, see [1]) if and only if the underlying graph of Q is one of the Dynkin diagrams $\mathbb{A}_n$, $n \geq 1$, $\mathbb{D}_n$, $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.

A basic reduction tool for our study of comodules over path coalgebras KQ of arbitrary quivers Q is Proposition 8.1(d) asserting that the category KQ^*-comod is equivalent to the category of nilpotent K-linear representations of Q of finite length.

We recall from [32] that if Q is any of the pure semisimple infinite quivers $\mathbb{A}^{(s)}_\infty$, ${}_\infty\mathbb{A}^{(s)}_\infty$, $\mathbb{D}^{(s)}_\infty$ presented in Table 9.2 of Section 9 and $s \geq 1$, then there exists at least one indecomposable non-injective comodule X in KQ^*-comod such that there is no almost split sequence $0 \to X \to Y \to Z \to 0$ in the category KQ^*-comod. However, by Proposition 7.3, given a left pure semisimple coalgebra C, every indecomposable non-projective comodule Z in C-comod has an almost split sequence $0 \to X \to Y \to Z \to 0$ in C-comod.

The reader is referred to [11], [28] and [56] for the coalgebra and comodule terminology, to [6], [15], [17] and [34] for the category theory terminology, and to [2], [21] and [44] for the representation theory terminology.

Given a ring R with an identity element we denote by $J(R)$ the *Jacobson radical* of R. We recall that an artinian ring R is said to be *connected* if R is not decomposable in a product of rings; and R is said to be *basic* if $R/J(R) \cong F_1 \times \ldots \times F_m$, where $F_1, \ldots, F_m$ are division rings. We denote by $\mathrm{Mod}(R)$ the category of all right R-modules and by $\mathrm{mod}(R)$ the full subcategory of $\mathrm{Mod}(R)$ formed by finitely generated R-modules. If R is a K-algebra, we denote by $\mathrm{Mod}^{\mathrm{lf}}(R)$ the full subcategory of $\mathrm{Mod}(R)$ formed by the *locally finite-dimensional* R-modules, that is, the modules that are directed unions of finite-dimensional right R-submodules [23]. Given a right R-module M we denote by $\operatorname{soc} M$ the *socle* of M, that is, the sum of all simple R-submodules of M.

The concepts of tame comodule type, wild comodule type and the main results of this paper were presented during ICRA-IX at Beijing Normal University in August 2000 (see [48]).

2. Linear topological rings and modules. For convenience of the reader we collect from [3], [16], [17], [25] and [33] some facts on linear topological rings and modules, and on pseudocompact K-algebras and their pseudocompact modules (see also [59]).

By a *topological ring* we mean a ring R equipped with a topology such that addition and multiplication are continuous. A topological ring is said to be *right linear topological* if R has a basis (of neighborhoods of zero) consisting of right ideals. We state without proof the following useful results.

Lemma 2.1. *Let R be a right linear topological ring. Then the open right ideals of R satisfy the following conditions.*

(a) *If I_1, I_2 are open right ideals, then $I_1 \cap I_2$ is open.*
(b) *If $I_1 \subseteq I_2$ are right ideals and I_1 is open then I_2 is open.*
(c) *If I_1 is an open right ideal, then so is $(I_1 : r) = \{s \in R;\ rs \in I_1\}$ for each $r \in R$.* ■

Lemma 2.2. *Let R be any ring and let $\mathcal{F}$ be a set of right ideals of R satisfying the following two conditions*:

(i) *For any I_1, I_2 in $\mathcal{F}$ there is $I_3 \in \mathcal{F}$ such that $I_3 \subseteq I_1 \cap I_2$.*
(ii) *If $I_1 \in \mathcal{F}$ and $r \in R$, then there is $I_2 \in \mathcal{F}$ such that $I_2 \subseteq (I_1 : r)$.*

Then there exists a unique right linear topology on R having $\mathcal{F}$ as a basis. ■

Let R be a right topological ring. By a *topological right R-module* we mean a right R-module M equipped with a topology such that addition and multiplication $M \times R \to M$ are continuous. If the topology on R is right linear, a topological right R-module M is said to be *right linear topological* if M has a basis (of neighborhoods of zero) consisting of right R-submodules. If R is a right linear topological ring, we denote by $\mathrm{Dis}(R)$ the full subcategory of $\mathrm{Mod}(R)$ formed by the *discrete R-modules*, that is, linear topological right R-modules with the discrete topology.

The following result is easily verified.

LEMMA 2.3. *Let R be a right linear topological ring with a basis $\mathcal{F}$.*

(a) *A right R-module M is discrete if and only if for each $m \in M$ the annihilator* $(0 : m) = \{s \in R;\ ms = 0\}$ *is open in R.*

(b) *The category* $\mathrm{Dis}(R)$ *is closed under subobjects*, *factor objects and arbitrary direct sums in* $\mathrm{Mod}(R)$.

(c) $\mathrm{Dis}(R)$ *is a Grothendieck category and the modules R/I, with $I \in \mathcal{F}$, form a set of generators of* $\mathrm{Dis}(R)$. *In particular*, $\mathrm{Dis}(R)$ *has enough injective objects.* ■

The final part of (c) follows from the well known fact that any Grothendieck category has enough injective objects (see [6], [34]).

DEFINITION 2.4 (Gabriel [17]). Let K be a field. A K-algebra R is said to be a *pseudocompact algebra* if R is a Hausdorff linear topological K-algebra, R admits a basis $\mathcal{F}$ consisting of two-sided ideals such that $\dim_K R/I$ is finite for all $I \in \mathcal{F}$ and the natural K-algebra homomorphism $R \to \varprojlim_{I\in\mathcal{F}} R/I$ is an isomorphism.

A pseudocompact K-algebra R is said to be *basic* if the factor algebra $R/J(R)$ of R modulo its Jacobson radical $J(R)$ is a product of division rings.

A right linear topological R-module M is called a *pseudocompact R-module* if M has a basis consisting of right R-submodules N such that $\dim_K M/N$ is finite and the natural R-homomorphism $M \to \varprojlim_N M/N$ is an isomorphism.

We denote by $\mathcal{PC}_K$ the category of all pseudocompact K-algebras. Given an algebra R in $\mathcal{PC}_K$, we denote by R-PC and PC-R the categories of pseudocompact left R-modules and right R-modules, respectively. Given pseudocompact R-modules M and N we denote by $\hom_R(M,N)$ the K-vector space of all continuous R-homomorphisms from M to N.

Following Gabriel [17] (see also Brumer [5, p. 448]) we define a pair of duality functors

$$\mathrm{Dis}(R) \underset{D_2}{\overset{D_1}{\rightleftarrows}} R\text{-PC} \tag{2.5}$$

as follows. Given a right R-module M in $\mathrm{Dis}(R)$ we set $D_1(M) = M^* = \mathrm{Hom}_K(M,K)$ and we view $D_1(M)$ as a left R-module in a natural way. We define a linear topology on $D_1(M)$ by taking as a basis the submodules

$$N_\gamma^\perp = \{\varphi \in \mathrm{Hom}_K(M,K);\ \varphi(N_\gamma) = 0\},$$

where N_γ runs through all finitely generated submodules of M. It follows from Lemma 2.3(a) that every such N is annihilated by an open ideal I of R, that is, N_γ is a finitely generated module over the finite-dimensional K-algebra R/I. Hence $\dim_K N_\gamma$ is finite and therefore $\dim_K M^*/N_\gamma^\perp = \dim_K N_\gamma^*$ is finite. Since obviously $M^* \cong \varprojlim_{N_\gamma} N_\gamma^* \cong \varprojlim_{N_\gamma} M^*/N_\gamma^\perp$, the left R-module $D_1(M) = M^*$ is pseudocompact.

Given a pseudocompact left R-module $L \cong \varprojlim_\beta L/L_\beta$ in R-PC, where L_β is an open submodule of L, we set $D_2(L) = \mathrm{hom}_K(L,K)$, that is, $D_2(L)$ consists of all $\varphi \in \mathrm{Hom}_K(L,K)$ such that $\mathrm{Ker}\,\varphi$ contains an open submodule L_β of L. It follows from Lemma 2.3(a) that the right R-module $D_2(L) \cong \varinjlim_\beta (L/L_\beta)^*$ is discrete. The functors D_1 and D_2 are defined for morphisms in a natural way.

For the proof of the following useful result we refer to [17] (see also [5, Proposition 2.3]).

PROPOSITION 2.6. *Let R be a pseudocompact K-algebra.*

(a) *The category R-*PC *of pseudocompact left R-modules is abelian with exact inverse limits and enough projective objects.*

(b) *A pseudocompact R-module P is projective in R-*PC *if and only if P is a direct summand of a direct product of copies of R with the product topology.*

(c) *Every finitely generated discrete R-module is of finite K-dimension.*

(d) *The contravariant functors D_1 and D_2 in* (2.5) *are dualities of categories such that $D_1 \circ D_2 \cong \mathrm{id}$ and $D_2 \circ D_1 \cong \mathrm{id}$.* ∎

An interesting application of pseudocompact algebras and their pseudocompact modules in the study of hereditary abelian categories with Serre duality can be found in a recent preprint [35].

3. Coalgebras and pseudocompact algebras. Let K be a field. We recall (see [28], [56]) that a (unitary) *K-coalgebra* C is a non-zero K-vector space C together with K-linear maps $\Delta : C \to C \otimes C$ (*comultiplication*) and $\varepsilon : C \to K$ (*counity*) satisfying the coassociativity condition $(\Delta \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \Delta)\Delta$ and the counity conditions $(\varepsilon \otimes \mathrm{id}_C)\Delta = \mathrm{id}_C$, $(\mathrm{id}_C \otimes \varepsilon)\Delta = \mathrm{id}_C$, under the identification $C \otimes K \cong C \cong K \otimes C$, where we set $\otimes = \otimes_K$. A *subcoalgebra* of a K-coalgebra C is a K-vector subspace D of C such that $\Delta(D) \subseteq D \otimes D \subseteq C \otimes C$. A coalgebra C is said to be *simple* if C has no non-zero subcoalgebras. The following result is often called the fundamental coalgebra structure theorem (see [28], [56]).

THEOREM 3.1. (a) *Every finite-dimensional subspace of a K-coalgebra C is contained in a finite-dimensional subcoalgebra of C.*

(b) *Any K-coalgebra is a directed union of finite-dimensional subcoalgebras.* ∎

A K-linear map $f : C \to C'$ between K-coalgebras C and C' is a *coalgebra homomorphism* if $\Delta' f = (f \otimes f)\Delta$ and $\varepsilon' f = \varepsilon$. We denote by $\mathcal{C}oalg_K$ the category of K-coalgebras together with coalgebra homomorphisms. It is well known that $\mathcal{C}oalg_K$ has arbitrary coproducts and direct limits. An important role in our study of coalgebras is played by the duality functors

$$\mathcal{C}oalg_K \underset{D_2}{\overset{D_1}{\rightleftarrows}} \mathcal{PC}_K \tag{3.2}$$

defined as follows. Given a K-coalgebra C we equip the K-vector space

$$C^* = \mathrm{Hom}_K(C, K)$$

with the K-algebra structure given by the induced maps

$$C^* \otimes C^* \to (C \otimes C)^* \xrightarrow{\Delta^*} C^*$$

(convolution product [28], [56]) and $\varepsilon^* : K \to C^*$. We also define a linear profinite topology on the algebra C^* taking for its basis the two-sided ideals

$$H^{\perp} = \{\varphi \in \mathrm{Hom}_K(C, K);\ \varphi(H) = 0\}, \tag{3.3}$$

where H runs through all finite-dimensional subcoalgebras of C. It follows from Lemma 2.2 that we have defined a linear topology such that C^* is a pseudocompact K-algebra. For this we note that according to Theorem 3.1 the coalgebra C is a directed union of its finite-dimensional subcoalgebras H_β and therefore

$$C^* \cong \varprojlim_{H_\beta} H_\beta^* \cong \varprojlim_{H_\beta} C^*/H_\beta^{\perp}. \tag{3.4}$$

Note also that $\dim_K C^*/H_\beta^{\perp} = \dim_K H_\beta^*$ is finite. We call C^* the *convolution pseudocompact K-algebra* associated with C. It is well known that all open one-sided ideals of C^* are closed (see [58] and [59, Section 1] for references).

We define the functor D_1 of (3.2) by assigning to any K-coalgebra C the K-algebra $D_1(C) = C^*$ equipped with the profinite topology defined above. The functor D_1 is defined on coalgebra homomorphisms in an obvious way.

Now let $R \cong \varprojlim_{I \in \mathcal{F}} R/I$ be a pseudocompact K-algebra with a basis $\mathcal{F}$ of neighborhoods of zero consisting of two-sided ideals I such that $\dim_K R/I < \infty$. Consider the topologically K-dual space to R,

$$R^\circ = \mathrm{hom}_K(R, K),$$

where $\mathrm{hom}_K(R, K)$ consists of all K-linear functionals $\varphi : R \to K$ such that

$\operatorname{Ker}\varphi$ contains an open ideal from $\mathcal{F}$. It follows that

$$R^{\circ} = \hom_K(R,K) \cong \hom_K(\varprojlim_{I\in\mathcal{F}} R/I, K) \cong \varinjlim_{I\in\mathcal{F}} (R/I)^* \tag{3.5}$$

where $(R/I)^* = \operatorname{Hom}_K(R/I,K) = \hom_K(R/I,K)$, because $\dim_K R/I$ is finite. The K-algebra structure on the finite-dimensional K-algebra R/I induces a dual K-coalgebra structure on $(R/I)^*$ in such a way that the K-linear map $(R/I_2)^* \to (R/I_1)^*$ induced by the algebra surjection $R/I_1 \to R/I_2$ is a coalgebra embedding for all ideals $I_1 \subseteq I_2$ in $\mathcal{F}$. This defines a unique coalgebra structure on R° such that $(R/I)^*$ is a subcoalgebra of R° for all I in $\mathcal{F}$ (cf. [23, Section 3.3]). It follows from the definition of the profinite linear topology on C^* applied to $C = R^{\circ}$ that there are functorial isomorphisms

$$(R^{\circ})^* = (\varinjlim_{I\in\mathcal{F}} (R/I)^*)^* \cong \varprojlim_{I\in\mathcal{F}} (R/I)^{**} \cong \varprojlim_{I\in\mathcal{F}} R/I \cong R$$

and the composite isomorphism $(R^{\circ})^* \cong R$ is an isomorphism of pseudocompact K-algebras. Similarly we show that there is a functorial coalgebra isomorphism $C \cong (C^*)^{\circ}$ for any K-coalgebra C.

We define the functor D_2 of (3.2) by assigning to any pseudocompact K-algebra $R \cong \varprojlim_{I\in\mathcal{F}} R/I$ the K-vector space $D_2(R) = R^{\circ}$ equipped with the coalgebra structure defined above. The functor D_2 is defined on morphisms in $\mathcal{PC}_K$ in an obvious way.

The arguments given above yield the following important duality theorem (see [27] and [59, Section 2]).

THEOREM 3.6. (a) *For any K-coalgebra C the vector space $C^* = \operatorname{Hom}_K(C,K)$ is a pseudocompact K-algebra with respect to the induced linearly topological K-algebra structure defined above.*

(b) *The map $C \mapsto C^*$ defines a duality between the category of finite-dimensional K-coalgebras and the category of finite-dimensional K-algebras.*

(c) *For any pseudocompact K-algebra R the vector space $D_2(R) = R^{\circ} = \hom_K(R,K)$ is a K-coalgebra with respect to the induced coalgebra structure defined above. Moreover, there is a natural isomorphism $(R^{\circ})^* \cong R$ of pseudocompact K-algebras. If $R = C^* = D_1(C)$ is the pseudocompact K-algebra of a K-coalgebra C, then there is a natural K-coalgebra isomorphism $(C^*)^{\circ} \cong C$.*

(d) *The contravariant functors $\mathcal{C}oalg_K \underset{D_2}{\overset{D_1}{\rightleftarrows}} \mathcal{PC}_K$ of (3.2) are dualities of categories such that $D_1 \circ D_2 \cong \mathrm{id}$ and $D_2 \circ D_1 \cong \mathrm{id}$.* ∎

4. The category of comodules and its dual. Let C be a non-zero K-coalgebra, with comultiplication Δ and counity ε, where K is a field. We recall that a *left C-comodule* is a K-vector space M together with a

K-linear map $\delta_M : M \to C \otimes M$ such that $(\Delta \otimes \mathrm{id}_M)\delta_M = (\mathrm{id}_C \otimes \delta_M)\delta_M$ and $(\varepsilon \otimes \mathrm{id}_M)\delta_M$ is the canonical isomorphism $M \cong K \otimes M$. A K-linear map $f : M \to M'$ between left C-comodules is a *C-comodule homomorphism* if $\delta_{M'} f = (\mathrm{id}_C \otimes f)\delta_M$. Left C-subcomodules of the C-comodule C are called *left coideals* of C.

We denote by C-Comod the category of all left C-comodules, and by C-comod its full subcategory formed by the C-comodules of finite K-dimension. The corresponding categories of right C-comodules are denoted by Comod-C and comod-C, respectively.

If M is a left C-comodule and C^* is the pseudocompact K-algebra of C then the composite K-linear map

$$(4.1)\quad M \otimes C^* \xrightarrow{\delta_M \otimes \mathrm{id}} C \otimes M \otimes C^* \xrightarrow{\mathrm{id} \otimes \tau} C \otimes C^* \otimes M \xrightarrow{\mathrm{ev} \otimes \varepsilon} K \otimes M \cong M$$

defines a right C^*-module structure on M (called the *rational C^*-module structure* [56]), where $\tau : M \otimes C^* \to C^* \otimes M$ is the twist isomorphism and $\mathrm{ev} : C \otimes C^* \to K$ is the evaluation map $c \otimes \varphi \mapsto \varphi(c)$. It is shown in [56] that this correspondence defines a categorical isomorphism

$$(4.2)\qquad C\text{-Comod} \cong \mathrm{Rat}(C^*)$$

of the category of left C-comodules and the category $\mathrm{Rat}(C^*)$ of right C^*-modules in the sense of [56].

We shall need the following useful observation (see [27], [59, Section 2]).

THEOREM 4.3. *Let C be a K-coalgebra and $C^* = \mathrm{Hom}_K(C, K)$ the associated pseudocompact K-algebra* (3.4).

(a) *The map assigning to any left C-comodule M the underlying vector space M endowed with the rational right C^*-module structure defines category isomorphisms*

$$(4.4)\qquad C\text{-Comod} \cong \mathrm{Dis}(C^*) \quad \textit{and} \quad C\text{-comod} \cong \mathrm{dis}(C^*)$$

where $\mathrm{Dis}(C^*)$ *is the category of discrete right C^*-modules and* $\mathrm{dis}(C^*)$ *is its full subcategory formed by the finite-dimensional modules.*

(b) *Let H be a finite-dimensional subcoalgebra of C. Then there is a natural exact embedding H-*Comod $\subseteq C$-Comod, *and a comodule M in* C-Comod *lies in H-*Comod *if and only if M viewed as a discrete right C^*-module is annihilated by the ideal $H^\perp$ of C^*.*

(c) *For any finite-dimensional left C-comodule M there is a finite-dimensional subcoalgebra H of C such that M lies in H-*comod $\subseteq C$-comod.

(d) *There exist contravariant equivalences of categories*

$$(4.5)\qquad C\text{-Comod} \underset{\widetilde{D}_2}{\overset{\widetilde{D}_1}{\rightleftarrows}} C^*\text{-PC}$$

such that $\widetilde{D}_1 \circ \widetilde{D}_2 \cong \mathrm{id}$ *and* $\widetilde{D}_2 \circ \widetilde{D}_1 \cong \mathrm{id}$.

Proof. (a) Let M be a left C-comodule. Given any $m \in M$ we have $\delta_M(m) = c_1 \otimes m_1 + \ldots + c_r \otimes m_r$, where $c_1, \ldots, c_r \in C$ and $m_1, \ldots, m_r \in M$. By Theorem 3.1, there is a finite-dimensional subcoalgebra H of C containing $c_1, \ldots, c_r$. It follows from (4.2) that the two-sided ideal $H^\perp$ (see (3.3)) of C^* annihilates the element m of the right C^*-module M, that is, the annihilator $(0 : m)$ contains $H^\perp$. Hence, according to Lemma 2.1, the right ideal $(0 : m)$ is open. In view of Lemma 2.3, the right C^*-module M is discrete.

Conversely, assume M is a discrete right C^*-module and let N be a finitely generated submodule of M. It follows from Lemma 2.3 that N is annihilated by an open ideal of C^*, that is, there exists a finite-dimensional subcoalgebra H of C such that $N \cdot H^\perp = 0$. Hence N is a finite-dimensional right module over the finite-dimensional K-algebra $C^*/H^\perp$ isomorphic to the algebra H^*. Then N^* is a left H^*-module and the multiplication map $H^* \otimes N^* \to N^*$ induces $N \cong N^{**} \to (H^* \otimes N^*)^* \cong H^{**} \otimes N^{**} \cong H \otimes N$ defining an H-comodule structure on N such that the induced rational right module structure of N over the algebra $C^*/H^\perp \cong H^*$ coincides with the original one. Since N is arbitrary, we have defined a unique left C-comodule structure on M such that the induced rational right C^*-module structure on M is the original one. This finishes the proof of (a).

The statements (b) and (c) follow from the above considerations.

(d) Consider the duality functors $\mathrm{Dis}(C^*) \underset{D_2}{\overset{D_1}{\rightleftarrows}} C^*\text{-PC}$ of (2.5) and take for $\widetilde{D}_1$ and $\widetilde{D}_2$ in (4.5) the composition of the functors D_1 and D_2 with the category isomorphism $C\text{-Comod} \cong \mathrm{Dis}(C^*)$ defined above. Then (b) is an immediate consequence of Proposition 2.6 applied to the pseudocompact K-algebra $R = C^*$. ■

Corollary 4.6. *For any K-coalgebra C there exists a pair of K-linear duality functors*

$$C\text{-comod} \underset{D}{\overset{D}{\rightleftarrows}} \text{comod-}C. \tag{4.7}$$

Proof. Let $C^* = \mathrm{Hom}_K(C, K)$ be the pseudocompact K-algebra associated with C. Let $\mathrm{dis}(C^*)$ be the full subcategory of $\mathrm{Dis}(C^*)$ formed by finite-dimensional modules. It follows from Proposition 2.6 and Lemma 2.3(a) that the functors (2.5) with $R = C^*$ induce K-linear duality functors $\mathrm{dis}(C^*) \underset{D_2}{\overset{D_1}{\rightleftarrows}} \mathrm{dis}(C^{*\mathrm{op}})$. Then in view of the category isomorphisms $C\text{-comod} \cong \mathrm{dis}(C^*)$ and $\text{comod-}C \cong \mathrm{dis}(C^{*\mathrm{op}})$ (see (4.4)) there are K-linear equivalences of categories

$$C\text{-comod} \cong \mathrm{dis}(C^*) \underset{D_2}{\overset{D_1}{\rightleftarrows}} (\mathrm{dis}(C^{*\mathrm{op}}))^{\mathrm{op}} \cong (\text{comod-}C)^{\mathrm{op}}$$

and we are done. ■

Let us list the main properties of comodule categories we need throughout this paper (see [28] and [56]).

THEOREM 4.8. *Let C be a K-coalgebra, where K is a field.*

(a) *Every left C-comodule M is a directed union of finite-dimensional subcomodules.*

(b) *The Grothendieck category C*-Comod *is locally finite and C*-comod *is the full subcategory of C*-Comod *consisting of objects of finite length* (see [17], [34]). *The category C*-comod *is a skeletally small abelian Krull–Schmidt K-category.*

(c) *The coalgebra C, viewed as a left C-comodule, is an injective cogenerator in the category C*-Comod. *If M is a left C-comodule, then the left comultiplication map $\delta_M : M \to C \otimes M$ is a C-comodule embedding and the left C-comodule $C \otimes M$ is injective.*

(d) *The category C*-Comod *has enough injective objects and every injective object in C*-Comod *is a direct sum of injective envelopes $E(S)$ of simple C-comodules S. Every simple left C-comodule is isomorphic to a subcomodule of the left comodule ${}_C C$.*

(e) *The direct sum $\bigoplus_\beta E_\beta$ of any family of comodules E_β is injective in* C-Comod *if and only if each summand E_β is injective.* ∎

In general the category C-Comod does not have enough projectives, and sometimes it has no non-zero projective object (see [32]).

An important role in the study of comodule categories is played by the following version of the Yoneda lemma (see [56]).

LEMMA 4.9. *Let C be a K-coalgebra and M a left C-comodule. The Yoneda K-linear map*

$$\widetilde{\varepsilon}_M : \mathrm{Hom}_C(M, C) \xrightarrow{\sim} \mathrm{Hom}_K(M, K) = M^*,$$

$\varphi \mapsto \varepsilon\varphi$, is an isomorphism. $\widetilde{\varepsilon}_M$ is functorial with respect to C-comodule homomorphisms $f : M' \to M$, and its inverse is given by $\psi \mapsto (\mathrm{id}_C \otimes \psi)\delta_M$. ∎

5. Basic coalgebras. Given a left C-comodule M we denote by $\mathrm{soc}_C M$ the *socle* of M, that is, the sum of all simple C-subcomodules of M. We recall that the *coradical* C_0 of a coalgebra C is the sum of all simple subcoalgebras of C (see [28], [56]).

We start with the following useful observation.

LEMMA 5.1. *Let C be a K-coalgebra and C^* the associated pseudocompact K-algebra* (3.4).

(a) $\mathrm{soc}_C C = \mathrm{soc}\, C_C = C_0$, *the sum of all simple subcoalgebras of C.*

(b) *The Jacobson radical $J(C^*)$ of the K-algebra C^* is the intersection of all open two-sided ideals of C^* and $J(C^*) = (\mathrm{soc}_C C)^\perp$.*

(c) *The ideal $J(C^*)$ is closed in C^* and $\bigcap_{m=0}^{\infty} J(C^*)^m = 0$.*

Proof. It is clear that under the identification C-Comod $= \mathrm{Dis}(C^*)$ of Theorem 4.3 the socle $\mathrm{soc}_C C$ coincides with the socle $\mathrm{soc}\, C_{C^*}$ of the right C^*-module C. Then (a) is a consequence of [28, Corollary 5.1.8].

Let $C_0 = \sum_\lambda H_\lambda$, where H_λ is a simple subcoalgebra of C. Then $J(C^*) = C^\perp = \bigcap_\lambda H_\lambda^\perp$ and the open two-sided ideals $H_\lambda^\perp$ are closed (see [58] and [59, Section 1] for references). Then the statements (b) and (c) are easily deduced from [19, Section 7], [24] and [28, Proposition 5.2.9], because the algebra C^* is pseudocompact with respect to the profinite topology defined by the ideals $H^\perp$. ■

DEFINITION 5.2 [43, p. 404]. A K-coalgebra C is called *basic* if the left C-subcomodule $\mathrm{soc}_C C$ of C has a direct sum decomposition $\mathrm{soc}_C C = \bigoplus_{j\in I_C} S(j)$, where I_C is a set, $S(j)$ are simple comodules and $S(i) \not\cong S(j)$ for all $i \neq j$. ■

The following lemma shows that the definition is left-right symmetric and the notion of basic coalgebra introduced in [9] is equivalent to the above one.

LEMMA 5.3. *Let C be a K-coalgebra, where K is a field. The following conditions are equivalent.*

(a) *The coalgebra C is basic.*

(b) *The left C-comodule C has a direct sum decomposition $C = \bigoplus_{i\in I} E_i$, where each left C-comodule E_i is indecomposable*, $\mathrm{soc}\, E_i$ *is simple and $E_i \not\cong E_j$ for $i \neq j$.*

(c) *The left C-comodule C is a minimal injective cogenerator in the category C-*Comod.

(d) *If D is a simple subcoalgebra of C then D^* is a division K-algebra.*

Proof. The equivalence of (a), (b) and (c) is immediate, by Theorem 4.8(d) and the obvious observation that simple comodules S_1 and S_2 are isomorphic if and only if their injective envelopes $E(S_1)$ and $E(S_2)$ are.

(a)⇒(d). Let $H \subseteq C$ be a simple subcoalgebra of C. Then $\dim_K H$ is finite and according to [28, Lemma 5.1.4(3)], the K-algebra H^* is simple and therefore of the full matrix algebra form $H^* \cong \mathbb{M}_r(D)$, where $r \geq 1$ and D is a division K-algebra. Moreover, by [28, Lemma 5.1.4(1)], the map $N \mapsto N^\perp$ defines a bijection between the left subcomodules of H and the left ideals of the algebra H^*. It follows that H^* is a division K-algebra, because otherwise $r \geq 2$ and the matrix algebra $\mathbb{M}_r(D)$ contains two different isomorphic simple left ideals and therefore the coalgebra H contains two different isomorphic simple left subcomodules. This contradicts the assumption that C is basic.

(d)⇒(a). Assume to the contrary that S_1 and S_2 are different non-isomorphic simple left C-comodules. It follows that S_1 and S_2 are of finite dimension and according to Theorem 3.2, the K-vector space $S_1 + S_2$ is contained in a finite-dimensional subcoalgebra C_1 of C. Since, according to

[28, Lemma 5.1.4(3)], the map $N \mapsto N^{\perp}$ defines a bijection between the left subcomodules of C_1 and the left ideals of the algebra C_1^*, the semisimple K-algebra $C_1^*/J(C_1^*)$ contains two different non-isomorphic simple left ideals and therefore has an epimorphic image of the form $R \cong \mathbb{M}_r(D)$, where $r \geq 2$ and D is a division K-algebra. By Theorem 3.6, the K-algebra surjection $C_1^* \to R$ induces a coalgebra injection $R^* \subseteq C_1$. This is a contradiction, because the coalgebra R^* is simple [28, Lemma 5.1.4(3)]. ■

Since simple coalgebras are finite-dimensional, Lemma 5.3 yields

COROLLARY 5.4. *If the field K is algebraically closed, then a K-coalgebra C is basic if and only if C is pointed, that is, every simple subcoalgebra of C is one-dimensional.* ■

In view of the well known results of Gabriel [17] and [19, 7.2], Lemma 5.3 yields

COROLLARY 5.5. *Let K be a field, let C be a K-coalgebra and C^* the pseudocompact K-algebra* (3.4) *associated with C. The following conditions are equivalent.*

(a) *C is basic.*

(b) *C^* is basic.*

(c) *C^* has product decompositions $C^* \cong \prod_{i \in I_C} e_i C^* \cong \prod_{i \in I_C} C^* e_i$ in the category* PC-C^* *and* C^*-PC, *respectively, where e_i is a primitive idempotent of C^* and $e_i C^*$ (resp. $C^* e_i$) is an indecomposable pseudocompact right (resp. left) ideal of C^* for $i \in I_C$.* ■

The following useful fact was proved in [43, p. 404] by applying an idea of Gabriel [17] and [19, 7.2].

PROPOSITION 5.6. *For every K-coalgebra C there exists a basic K-coalgebra C^b of the form* (5.8) *such that C*-Comod $\cong C^b$-Comod, *and the conditions determine C^b uniquely, up to K-coalgebra isomorphism.* ■

Proof. Let C^* be the pseudocompact K-algebra (3.4) associated with C. By Gabriel [17] and [19, 7.2], there is a basic pseudocompact K-algebra $\Lambda_C = (C^*)_b$ associated with C^*, and it can be constructed as follows. Let $E = \bigoplus_{i \in I_C} E(S_i)$ be a minimal injective cogenerator in C-Comod, where I_C is a set and $E(S_i)$ is an injective envelope of a simple left C-comodule S_i. Let $\{E_\beta\}$ be a directed family of finite-dimensional subcomodules of E such that $E = \bigcup_\beta E_\beta$ (see Theorem 3.1). By Gabriel [17], the induced directed family of two-sided ideals $\mathrm{Hom}_C(E/E_\beta, E)$ defines a profinite linear topology on the K-algebra

(5.7) $$\Lambda_C = \mathrm{End}_C(E) \cong \varprojlim_{\beta} \Lambda_C/\mathrm{Hom}_C(E/E_\beta, E)$$

such that the pseudocompact K-algebra $\varLambda_C$ is basic and the contravariant functor

$$h_E = \mathrm{Hom}_C(-, E) : C\text{-Comod} \to \varLambda_C\text{-PC}$$

is a duality. Let

$$C^b = \varLambda_C^{\circ} = \mathrm{hom}_K(\varLambda_C, K) \tag{5.8}$$

be the topological dual K-coalgebra of $\varLambda_C$ (see (3.5)). By Theorem 3.6, $C^b = \varLambda_C^{\circ}$ is a K-coalgebra such that the pseudocompact K-algebra (3.4) associated with C^b is isomorphic to $\varLambda_C$. By applying the duality (3.5) and the composite contravariant equivalence $C^b\text{-Comod} \cong \mathrm{Dis}(C^{b*}) \cong \mathrm{Dis}(\varLambda_C) \cong (\varLambda_C\text{-PC})^{\mathrm{op}}$ we get an equivalence of categories $\varPhi : C\text{-Comod} \cong C^b\text{-Comod}$. We note that $\varPhi(E) \cong \bigoplus_{i\in I_C} E'_i$, where $E'_i = \varPhi(E(S_i))$, $i \in I_C$, is a family of indecomposable left C^b-subcomodules of C^b such that $E'_i \not\cong E'_j$ for $i \neq j$. It follows from Lemma 5.3 that C^b is a basic coalgebra.

To finish the proof, we assume that H is a basic K-coalgebra such that there exists an equivalence of categories $\varPsi : C^b\text{-Comod} \to H\text{-Comod}$. By Lemma 5.3, C^b and H are minimal injective cogenerators in C^b-Comod and H-Comod, respectively. Then the equivalence $\varPsi$ induces a H-comodule isomorphism $\varPsi(C^b) \cong H$ and an isomorphism

$$\varLambda_{C^b} = \mathrm{Hom}_{C^b}(C^b, C^b) \cong \mathrm{Hom}_H(\varPsi(C^b), \varPsi(C^b)) \cong \mathrm{Hom}_H(H, H) = \varLambda_H$$

of pseudocompact K-algebras with respect to the Gabriel topology defined above (see (5.7)), because $\varPsi$ carries finite-dimensional comodules to finite-dimensional ones. Since obviously the map $r \mapsto (x \mapsto xr)$ defines an isomorphism $R \cong \mathrm{Hom}_R(R, R) = \mathrm{hom}_R(R, R)$ of pseudocompact K-algebras for any pseudocompact K-algebra R, the above composite K-algebra isomorphism, together with the algebra isomorphisms

$$\mathrm{Hom}_H(H, H) \cong \mathrm{hom}_{H^*}(H^*, H^*) \cong H^*,$$

$$\mathrm{Hom}_{C^b}(C^b, C^b) \cong \mathrm{hom}_{C^{b*}}(C^{b^*}, C^{b^*}) = \mathrm{hom}_{\varLambda_C}(\varLambda_C, \varLambda_C) \cong \varLambda_C$$

defined by the duality (4.5), applied to $C = C^b$ and $C = H$, gives an isomorphism $H^* \cong \varLambda_C$ of pseudocompact K-algebras. In view of Theorem 3.6 we get a coalgebra isomorphism $H \cong (H^*)^{\circ} \cong (\varLambda_C)^{\circ} = C^b$. This finishes the proof. ∎

A slightly different construction of a basic K-coalgebra associated with a given one is presented in [9].

Following Gabriel [19], we call an abelian K-category $\mathfrak{A}$ a *length K-category* if every object of $\mathfrak{A}$ has a finite composition series and $\mathrm{End}(X)$ is a finite-dimensional K-algebra for any object X of $\mathfrak{A}$. Now we are able to prove the following realisation result (see also [57] and [60]).

PROPOSITION 5.9. *For every abelian length K-category $\mathfrak{A}$ there exists a basic K-coalgebra C and an equivalence of K-categories $\mathfrak{A} \cong C$-comod.*

Proof. Let $\mathfrak{A}$ be an abelian length K-category. It is well known (see [17], [34]) that there exists a fully faithful exact embedding of $\mathfrak{A}$ in the locally finite Grothendieck category $\mathcal{A} = \operatorname{Lex} \mathfrak{A}$ of all left exact contravariant additive functors from $\mathfrak{A}$ into the category of K-vector spaces such that $\mathfrak{A}$ coincides with the full subcategory of $\mathcal{A}$ formed by all objects of finite length. It was proved by Gabriel in [17] that there is a duality functor $\mathcal{A} \cong (R\text{-PC})^{\mathrm{op}}$, where R is a basic pseudocompact K-algebra. Let $C = R^{\circ}$ be the basic K-coalgebra (3.5) associated with R. It follows from Theorem 3.6 that there is an isomorphism $C^* \cong R$ of pseudocompact K-algebras and according to Theorem 4.3 there is a duality $C\text{-Comod} \cong (R\text{-PC})^{\mathrm{op}}$. Hence we get an equivalence $\mathcal{A} \cong C\text{-Comod}$, and the required equivalence of categories $\mathfrak{A} \cong C\text{-comod}$. ■

The following useful observation generalises a result in [43].

Corollary 5.10. *Let $\mathfrak{A}$ be an abelian length K-category. Then there exist a directed family $\{\mathfrak{A}_\beta\}_\beta$ of full exact K-subcategories of $\mathfrak{A}$ and an inverse system $\{R_\beta, f_{\beta,\gamma}\}_{\beta\preceq\gamma}$ of finite-dimensional K-algebras R_β connected by K-algebra surjections $f_{\beta,\gamma} : R_\gamma \to R_\beta$ such that $R = \varprojlim_\beta\{R_\beta, f_{\beta,\gamma}\}$ is a pseudocompact K-algebra,*

$$\mathfrak{A} = \bigcup_\beta \mathfrak{A}_\beta$$

is a directed union, $\mathfrak{A}_\beta \cong \operatorname{mod}(R_\beta)$ and the embedding $\mathfrak{A}_\beta \subseteq \mathfrak{A}_\gamma$ is induced by the K-algebra surjection $f_{\beta,\gamma} : R_\gamma \to R_\beta$ for all $\beta \preceq \gamma$, that is, the diagram

$$\begin{tikzcd}
\mathfrak{A}_\beta \arrow[r, hook] \arrow[d, "\cong"] & \mathfrak{A}_\gamma \arrow[d, "\cong"] \\
\operatorname{mod}(R_\beta) \arrow[r, hook] & \operatorname{mod}(R_\gamma)
\end{tikzcd}$$

is commutative, where the lower inclusion functor is induced by the K-algebra surjection $f_{\beta,\gamma} : R_\gamma \to R_\beta$.

Proof. By Proposition 5.9, there exist a basic K-coalgebra C and an equivalence of K-categories $\mathbb{F} : \mathfrak{A} \xrightarrow{\simeq} C\text{-comod}$. We know from Theorem 3.1 that C is a directed union of finite-dimensional subcoalgebras H_β and therefore $C\text{-comod} = \bigcup_\beta H_\beta\text{-comod}$ is a directed union. Let $\mathfrak{A}_\beta$ be the preimage of $H_\beta\text{-comod} \subseteq C\text{-comod}$ under the equivalence $\mathbb{F}$, let $R_\beta = H_\beta^*$ and take for $f_{\beta,\gamma} : R_\gamma \to R_\beta$ the K-algebra surjection induced by the coalgebra embedding $H_\beta \subseteq H_\gamma$ for all $\beta \preceq \gamma$. It follows from Theorem 4.3 that there is a category isomorphism $H_\beta\text{-comod} \cong \operatorname{mod}(R_\beta)$ and consequently an equivalence $\mathfrak{A}_\beta \cong \operatorname{mod}(R_\beta)$. The remaining part of the corollary follows easily by applying Theorems 3.6 and 4.3. ■

The following two "localisation" corollaries have important applications.

Corollary 5.11. *Let $\mathcal{A}$ be an abelian K-category and let $\mathcal{S} = \{S_i\}_{i\in I}$ be a set of pairwise non-isomorphic objects of $\mathcal{A}$ such that* $\mathrm{End}(S_j)$ *is a division ring for all $i \in I$ and* $\mathrm{Hom}_{\mathcal{A}}(S_i, S_j) = 0$ *for all $i \neq j$. Assume that* $\mathrm{End}(X)$ *is a finite-dimensional K-algebra for any object X of $\mathcal{A}$ with $\mathcal{S}$-filtration, that is, with a finite filtration* $0 = X_0 \subset X_1 \subset \ldots \subset X_s = X$ *such that $X_t/X_{t-1} \in \mathcal{S}$ for $1 \le t \le s$. Denote by $\mathcal{A}\|_{\mathcal{S}}$ the full subcategory of $\mathcal{A}$ formed by all objects with $\mathcal{S}$-filtration.*

(a) *$\mathcal{A}\|_{\mathcal{S}}$ is an exact abelian length K-subcategory of $\mathcal{A}$ closed under extensions and $\mathcal{S}$ is a complete set of simple objects of $\mathcal{A}\|_{\mathcal{S}}$.*

(b) *There exists a basic K-coalgebra $C_{\mathcal{S}}$ and a K-linear equivalence of categories* $\mathcal{A}\|_{\mathcal{S}} \cong C_{\mathcal{S}}$-comod. *The number of pairwise non-isomorphic simple subcomodules of $C_{\mathcal{S}}$ is $|\mathcal{S}| = |I|$.*

Proof. The statement (a) is a consequence of [36, Theorem 1.2]. To get (b) apply (a) and Proposition 5.9 to the length category $\mathfrak{A} = \mathcal{A}\|_{\mathcal{S}}$. ∎

Corollary 5.12. *Let C be a basic K-coalgebra with a decomposition* $\operatorname{soc}_C C = \bigoplus_{j\in I_C} S(j)$, *and let $\mathcal{S} \subseteq \{S(i)\}_{i\in I_C}$ be a set of pairwise non-isomorphic left simple C-comodules. Denote by* C-comod$\|_{\mathcal{S}}$ *the full subcategory of* C-comod *formed by all comodules whose composition factors are in $\mathcal{S}$. Then there exists a basic K-coalgebra $C_{\mathcal{S}}$ and a K-linear equivalence of categories* C-comod$\|_{\mathcal{S}} \cong C_{\mathcal{S}}$-comod. *The socle of C is a direct sum of $|\mathcal{S}|$ pairwise non-isomorphic simple comodules.* ∎

We recall that for any K-algebra Λ (in general, infinite-dimensional) the category $\mathrm{Mod}^{\mathrm{lf}}(\Lambda)$ of *locally finite-dimensional* right R-modules is defined to be the full subcategory of $\mathrm{Mod}(\Lambda)$ formed by modules that are directed unions of finite-dimensional right Λ-submodules [23].

Proposition 5.9 and its proof yield the following result of J. A. Green [23].

Corollary 5.13. *Let K be a field and let Λ be an arbitrary K-algebra. There exists a basic K-coalgebra C_Λ and an equivalence of K-categories* $\mathrm{Mod}^{\mathrm{lf}}(\Lambda) \cong C_\Lambda$-Comod. ∎

6. Comodule types of coalgebras. Let K be any field and C a basic K-coalgebra. Throughout this section we fix a left comodule decomposition

$$\operatorname{soc}_C C = \bigoplus_{j\in I_C} S(j)^{t_j} \tag{6.1}$$

where I_C is a set, $t_j \ge 1$ and $S(j)$ are pairwise non-isomorphic simple comodules for $j \in I_C$. For every comodule M in C-comod we define the *composition length vector*

$$\mathbf{length}\, M = (m(j))_{j\in I_C} \in \mathbb{Z}^{(I_C)} \tag{6.2}$$

where $m(j) \in \mathbb{N}$ is defined to be the number of simple composition factors of M isomorphic to $S(j)$. Here $\mathbb{Z}^{(I_C)}$ is the direct sum of I_C copies of the free abelian group $\mathbb{Z}$.

We also associate with M its *dimension vector*

$$\mathbf{dim}\, M = (\widehat{m}(j))_{j \in I_C} \in \mathbb{Z}^{(I_C)} \tag{6.3}$$

where $\widehat{m}(j) = m(j) \cdot \dim_K \operatorname{End} S(j)$. Note that if the field K is algebraically closed then $\dim_K \operatorname{End} S(j) = 1$ for all $j \in I_C$ and therefore $\mathbf{dim}\, M = \mathbf{length}\, M$.

The *Grothendieck group* $K_0(C) = K_0(C\text{-comod})$ of the coalgebra C (or of the category C-comod) is defined to be the group $K_0(C) = \mathcal{F}/\mathcal{F}_0$, where $\mathcal{F}$ is the free abelian group having as a basis the set of the isomorphism classes $\widetilde{M}$ of modules M in C-comod and $\mathcal{F}_0$ is the subgroup of $\mathcal{F}$ generated by the elements $\widetilde{M} - \widetilde{L} - \widetilde{N}$ corresponding to all exact sequences $0 \to L \to M \to N \to 0$ in C-comod. We denote by $[M]$ the image of the isomorphism class $\widetilde{M}$ of the module M under the canonical group epimorphism $\mathcal{F} \to \mathcal{F}/\mathcal{F}_0$.

Since, by Corollary 4.6, there exists a K-linear duality $(C\text{-comod})^{\mathrm{op}} \cong \text{comod-}C$, there is an induced group isomorphism $K_0(C\text{-comod}) \cong K_0(\text{comod-}C)$.

Note that $M \mapsto \mathbf{length}\, M$ is an additive function on C-comod, that is, $\mathbf{length}\, M = \mathbf{length}\, L + \mathbf{length}\, N$ for any exact sequence $0 \to L \to M \to N \to 0$ in C-comod. It follows that $[M] \mapsto \mathbf{length}\, M$ extends to a group homomorphism

$$\mathbf{length} : K_0(C) \xrightarrow{\simeq} \mathbb{Z}^{(I_C)}. \tag{6.4}$$

This is an isomorphism, because obviously $K_0(C)$ is generated by the elements $[S(j)]$ with $j \in I_C$, and $\mathbf{length}\, S(j)$ is the jth standard $\mathbb{Z}$-basis vector of the free group $\mathbb{Z}^{(I_C)}$ for all $j \in I_C$. This together with Lemmas 4.9 and 5.3 proves the following lemma.

LEMMA 6.5. *For any K-coalgebra C with a decomposition $\operatorname{soc}_C C = \bigoplus_{j \in I_C} S(j)^{t_j}$, where $t_j \geq 1$ and $S(j)$ are pairwise non-isomorphic simple comodules for $j \in I_C$, the left C-comodule C has a decomposition*

$$C = \bigoplus_{j \in I_C} E(S(j))^{t_j}, \qquad t_j = \frac{\dim_K S(j)}{\dim_K \operatorname{End}_C S(j)},$$

the Grothendieck group $K_0(C)$ of C is freely generated by the classes $[S(j)]$ of the simple subcomodules $S(j)$, $j \in I_C$, of C, and the homomorphism (6.4) *defines a group isomorphism $K_0(C) \cong \mathbb{Z}^{(I_C)}$. Here $E(S(j))$ is the injective envelope of the simple C-comodule $S(j)$.* ∎

Given $v \in K_0(C)$ we denote by $\operatorname{ind}_v(C\text{-comod})$ the full subcategory of C-comod formed by the indecomposable comodules X with $\mathbf{length}\, X = v$.

Let R be a K-algebra. By a *C-R-bimodule* ${}_CL_R$ we mean a K-vector space L equipped with a left C-comodule structure and a right R-module structure satisfying the condition $\delta_L(xr) = \delta_L(x)r$ for all $x \in L$ and $r \in R$.

Following Drozd [13] we introduce tameness and wildness for coalgebras as follows (see also [44, p. 368] and [45]).

DEFINITION 6.6. Assume that K is an algebraically closed field.

(a) A K-coalgebra C is of *wild comodule type* (or briefly *wild*) if the category C-comod of finite-dimensional left C-comodules is of wild representation type, that is, there exists an exact representation embedding K-linear functor (see [45]) $T : \operatorname{mod} \Gamma_3(K) \to C\text{-comod}$, where

$$\Gamma_3(K) = \begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix}.$$

If, in addition, the functor T is fully faithful, we call C of *fully wild comodule type*, or strictly wild comodule type (see [13], [45]).

(b) A K-coalgebra C is of *tame comodule type* (or briefly *tame*) if the category C-comod is of tame representation type, that is, for every $v \in K_0(C)$ there exist C-$K[t]$-bimodules $L^{(1)}, \dots, L^{(r_v)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable left C-comodules M with $\mathbf{length}\, M = v$ are of the form $M \cong L^{(s)} \otimes K^1_\lambda$, where $s \le r_v$, $K^1_\lambda = K[t]/(t-\lambda)$ and $\lambda \in K$.

If there is a common bound for the number r_v for all vectors v, the tame coalgebra C is called *domestic* (see [54, (2.1)], [44, Section 14.4]).

In other words, C is of tame comodule type if the indecomposable left C-comodules of any fixed length vector v form a finite set of at most one-parameter families (see [13], [44, Section 14.4], [45]).

Following [51]–[53] (see also [44, p. 368]) we introduce the polynomial growth for tame coalgebras as follows.

DEFINITION 6.7. Assume that K is an algebraically closed field and let C be a K-coalgebra of tame comodule type.

(a) We define two *growth functions*

$$\boldsymbol{\mu}^1_C, \boldsymbol{\mu}^0_C : K_0(C) \to \mathbb{N}$$

as follows. Given a vector $v \in K_0(C)$ we define $\boldsymbol{\mu}^1_C(v)$ to be the minimal number r_v of C-$K[t]$-bimodules $L^{(1)}, \dots, L^{(r_v)}$ satisfying the conditions in 6.6(b). We let $\boldsymbol{\mu}^0_C(v)$ be the minimal number of isoclasses of indecomposable *discrete C-comodules* M with $\mathbf{length}\, M = v$, that is, C-comodules M which are not of the form $M \cong L^{(s)} \otimes K^1_\lambda$, where $s \le \boldsymbol{\mu}^1_C(v)$, $K^1_\lambda = K[t]/(t-\lambda)$, $\lambda \in K$ and $L^{(1)}, \dots, L^{(\boldsymbol{\mu}^1_C(v))}$ is a minimal family of C-$K[t]$-bimodules satisfying the conditions in 6.6(b).

(b) The tame K-coalgebra C is said to be of *polynomial growth* if there exists a formal power series

$$G(t)=\sum_{m=1}^{\infty}\sum_{j_1,\dots,j_m\in I_C} g_{j_1,\dots,j_m}t_{j_1}\dots t_{j_m}$$

with $t=(t_j)_{j\in I_C}$ and non-negative coefficients $g_{j_1,\dots,j_m}\in\mathbb{Z}$ such that $\boldsymbol{\mu}_C^1(v)\le G(v)$ for all $v=(v(j))_{j\in I_C}\in K_0(C)\cong\mathbb{Z}^{(I_C)}$ such that $\|v\|:=\sum_{j\in I_C}v(j)\ge 2$.

If $G(t)=\sum_{j\in I_C}g_jt_j$, where $g_j\in\mathbb{N}$, then C is said to be of *linear growth.*

We note that $G(v)$ is a polynomial of the coordinates $v(j)$ of the vector $v\in K_0(C)\cong\mathbb{Z}^{(I_C)}$, because $v(j)=0$ for all but a finite number of $j\in I_C$. Note also that tame domestic coalgebras are of linear growth.

The following lemma shows that tame comodule type, polynomial growth and linear growth are invariant under Morita equivalence of coalgebras.

LEMMA 6.8. *Assume that K is an algebraically closed field, C, H are K-coalgebras and there is a K-linear equivalence of categories* C-comod $\cong$ H-comod. *If C is of tame comodule type, then so is H and there is an isomorphism $\Phi: K_0(C)\xrightarrow{\simeq}K_0(H)$ of Grothendieck groups such that $\boldsymbol{\mu}_C^1=\boldsymbol{\mu}_H^1\circ\Phi$ and $\boldsymbol{\mu}_C^0=\boldsymbol{\mu}_H^0\circ\Phi$.*

Proof. Assume that $F: C$-comod $\xrightarrow{\simeq}$ H-comod is a K-linear equivalence of categories. It extends in the standard way to a K-linear equivalence $F: C$-Comod $\xrightarrow{\simeq}$ H-Comod, and according to [57] the functor F is of the cotensor product form $F(-)={}_HU_C\square_C(-)$, where ${}_HU_C$ is an H-C-bicomodule. It is clear that the map $[M]\mapsto[F(M)]$ defines an isomorphism $\Phi: K_0(C)\xrightarrow{\simeq}K_0(H)$ of Grothendieck groups.

Assume that C is of tame comodule type. Let $v\in K_0(C)$ and let $L^{(1)},\dots,L^{(r_v)}$ be as in 6.6(b). It is easy to show that the C-$K[t]$-bimodules ${}_HU_C\square_CL^{(1)},\dots,{}_HU_C\square_CL^{(r_v)}$ define a family of H-$K[t]$-bimodules satisfying the conditions in 6.6(b) for the vector $\Phi(v)\in K_0(H)$. It follows that H is of tame comodule type. Hence we also get the equalities $\boldsymbol{\mu}_C^1=\boldsymbol{\mu}_H^1\circ\Phi$ and $\boldsymbol{\mu}_C^0=\boldsymbol{\mu}_H^0\circ\Phi$. ■

Let us show that the definition of wildness for coalgebras is left-right symmetric.

LEMMA 6.9. *Let K be an algebraically closed field and let C be a K-coalgebra. Then C is of wild (resp. fully wild) comodule type if and only if the category* comod-C *of finite-dimensional right C-comodules is of wild (resp. fully wild) representation type, that is, there exists an exact (resp. fully faithful) representation embedding K-linear functor* $\operatorname{mod}\Gamma_3(K)\to$ comod-C.

Proof. Assume that C is wild, that is, there exists an exact representation embedding K-linear functor $F : \operatorname{mod} \Gamma_3(K) \to C\text{-comod}$. It follows from Corollary 4.6 that there exists a K-linear duality $(C\text{-comod})^{\mathrm{op}} \cong \text{comod-}C$. Since the algebra $\Gamma_3(K)$ is self-dual, there is a K-linear duality $\operatorname{mod} \Gamma_3(K) \cong (\operatorname{mod} \Gamma_3(K))^{\mathrm{op}}$, which together with the representation embedding F and the duality $(C\text{-comod})^{\mathrm{op}} \cong \text{comod-}C$ induces a representation embedding K-linear functor $F' : \operatorname{mod} \Gamma_3(K) \to \text{comod-}C$. ∎

THEOREM 6.10. *Let K be an algebraically closed field and let C be a K-coalgebra such that* $\dim_K \operatorname{Ext}^1_C(S, S')$ *is finite for every pair S, S' of simple left C-comodules. The following conditions are equivalent.*

(a) *C is of wild comodule type* (*resp. fully wild comodule type*).

(b) *There exists a finite-dimensional subcoalgebra H of C of wild comodule type* (*resp. fully wild comodule type*).

(c) *C is a directed union of finite-dimensional subcoalgebras of wild comodule type* (*resp. fully wild comodule type*).

Proof. (a)⇒(b). Assume (a) holds. Let $F : \operatorname{mod} \Gamma_3(K) \to C\text{-comod}$ be an exact representation embedding K-linear functor. Let S_1 (resp. S_2) be the unique simple injective (resp. projective) right $\Gamma_3(K)$-module, up to isomorphism. Since $\dim_K \operatorname{Ext}^1_C(S, S')$ is finite for every pair S, S' of simple left C-comodules, $\dim_K \operatorname{Ext}^1_C(U_1, U_2)$ is finite for every pair U_1, U_2 of finite-dimensional left C-comodules. This follows by an easy induction on the length of U_1 and U_2.

By applying this to the finite-dimensional left C-comodules $U_1 = F(S_1)$ and $U_2 = F(S_2)$ we conclude that $\dim_K \operatorname{Ext}^1_C(F(S_1), F(S_2))$ is finite.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_r$ be a K-basis of the K-vector space $\operatorname{Ext}^1_C(F(S_1), F(S_2))$ and assume that the exact sequence

$$\overline{\mathbf{e}}_j : \quad 0 \to F(S_2) \xrightarrow{u_j} E_j \xrightarrow{\pi_j} F(S_1) \to 0$$

in C-comod represents $\mathbf{e}_j$ for $j = 1, \ldots, r$. It is easy to see that there exists a finite-dimensional subcoalgebra H of C such that $\delta_{E_j}(E_j) \subseteq H \otimes E_j \subseteq C \otimes E_j$ for $j = 1, \ldots, r$, and $\delta_{U_i}(U_i) \subseteq H \otimes U_i \subseteq C \otimes U_i$ for $i = 1, 2$ (see [56]). It follows that the left C-comodules $E_1, \ldots, E_r$, $F(S_1)$ and $F(S_2)$ belong to the full exact subcategory H-comod of C-comod. We claim that F carries $\operatorname{mod} \Gamma_3(K)$ to H-comod.

Let $C^* = \operatorname{Hom}_K(C, K)$ be the pseudocompact K-algebra (3.4) associated with C. In view of Theorem 4.3, the category isomorphism (4.4) allows us to identify the comodules in $H\text{-comod} \subseteq C\text{-comod}$ with the right C^*-modules annihilated by the ideal $H^\perp$ of (3.3). Therefore it remains to show that for every right $\Gamma_3(K)$-module X in $\operatorname{mod} \Gamma_3(K)$ the right C^*-module $F(X)$ is annihilated by $H^\perp$.

For this we note that for every such X there exist integers $n_1, n_2 \geq 0$ and an exact sequence

$$0 \to S_2^{n_2} \to X \to S_1^{n_1} \to 0$$

in $\mathrm{mod}\, \Gamma_3(K)$. Since the functor F is exact, we get the induced exact sequence

$$\overline{\mathbf{e}}: \quad 0 \to F(S_2)^{n_2} \to F(X) \to F(S_1)^{n_1} \to 0$$

in C-comod, and $\overline{\mathbf{e}}$ represents an element $\mathbf{e} \in \mathrm{Ext}^1_C(F(S_1)^{n_1}, U_2^{n_2})$.

Let $p_j : F(S_1)^{n_1} \to F(S_1)$ be the jth coordinate projection and let $v_i : F(S_2) \to F(S_2)^{n_2}$ be the ith coordinate injection for $j = 1, \ldots, n_1$ and $i = 1, \ldots, n_2$. Let $\varphi_{ij} : \mathrm{Ext}^1_C(F(S_1), F(S_2)) \to \mathrm{Ext}^1_C(F(S_1)^{n_1}, F(S_2)^{n_2})$ be the induced K-linear embedding $\mathrm{Ext}^1_C(p_j, v_i)$.

It follows that the elements $\varphi_{ij}(\mathbf{e}_1), \ldots, \varphi_{ij}(\mathbf{e}_r)$ with $j = 1, \ldots, n_1$ and $i = 1, \ldots, n_2$ form a K-basis of the K-vector space $\mathrm{Ext}^1_C(F(S_1)^{n_1}, F(S_2)^{n_2})$. We shall show that the exact sequence

$$\overline{\varphi_{ij}(\mathbf{e}_t)}: \quad 0 \to F(S_2)^{n_2} \to E_{ijt} \to F(S_1)^{n_1} \to 0$$

in C-comod representing $\varphi_{ij}(\mathbf{e}_t)$ is annihilated by $H^\perp$ for any i, j, t.

To see this we note that the element

$$\mathrm{Ext}^1_C(p_j, \mathrm{id})(\mathbf{e}_t) \in \mathrm{Ext}^1_C(F(S_1)^{n_1}, F(S_2)^{n_2})$$

is represented by an exact sequence

$$\overline{\varphi_j(\mathbf{e}_t)}: \quad 0 \to F(S_2)^{n_2} \to E_{jt} \to F(S_1)^{n_1} \to 0,$$

where E_{jt} is the pull-back of $\pi_t : E_t \to F(S_1)$ and $p_j : F(S_1)^{n_1} \to F(S_1)$. Since the right C^*-modules E_t and $F(S_1)$ are annihilated by $H^\perp$, so is their pull-back E_{jt}. Since $\varphi_{ij}(\mathbf{e}_t) = \mathrm{Ext}^1_C(\mathrm{id}, v_i)(\mathrm{Ext}^1_C(p_j, \mathrm{id})(\mathbf{e}_t))$, the C^*-module E_{ijt} is the push-out of $v_i : F(S_2) \to F(S_2)^{n_2}$ and $F(S_2)^{n_2} \to E_{jt}$. It follows that E_{ijt} is annihilated by $H^\perp$, because E_{jt} and $F(S_2)$ are. This proves our claim.

Since $\overline{\mathbf{e}}$ is a K-linear combination of the elements $\overline{\varphi_{ij}(\mathbf{e}_t)}$ and obviously for any $\lambda \in K$ the exact sequence $\overline{\lambda\varphi_{ij}(\mathbf{e}_t)}$ representing $\lambda\varphi_{ij}(\mathbf{e}_t)$ is annihilated by $H^\perp$, it remains to show that given $\mathbf{e}', \mathbf{e}'' \in \mathrm{Ext}^1_C(F(S_1)^{n_1}, F(S_2)^{n_2})$ such that the corresponding exact sequences $\overline{\mathbf{e}'} : 0 \to F(S_2)^{n_2} \xrightarrow{u'} E' \xrightarrow{\pi'} F(S_1)^{n_1} \to 0$ and $\overline{\mathbf{e}''} : 0 \to F(S_2)^{n_2} \xrightarrow{u''} E'' \xrightarrow{\pi''} F(S_1)^{n_1} \to 0$ are annihilated by $H^\perp$, the exact sequence $\overline{\mathbf{e}^+} : \ 0 \to F(S_2)^{n_2} \to E^+ \to F(S_1)^{n_1} \to 0$ representing $\mathbf{e}' + \mathbf{e}''$ in $\mathrm{Ext}^1_C(F(S_1)^{n_1}, F(S_2)^{n_2})$ is also annihilated by $H^\perp$. This follows from the well known fact that the right C^*-module E^+ is isomorphic to W/W_0, where $W = \{(x', x'') \in E' \oplus E'';\ \pi'(x') = \pi''(x'')\}$ and $W_0 = \{(u'(x), -u''(x)) \in E' \oplus E'';\ x \in F(S_2)^{n_2}\}$.

Hence we conclude that for every right $\Gamma_3(K)$-module X in $\mathrm{mod}\, \Gamma_3(K)$ the right C^*-module $F(X)$ is annihilated by $H^\perp$, that is, the functor F

carries $\operatorname{mod} \Gamma_3(K)$ to H-comod. It follows that the full exact subcategory H-comod of C-comod is of wild representation type and (b) follows.

Since the converse implication (b)$\Rightarrow$(a) is obvious, we easily conclude that (a) is equivalent to (c), because C is a directed union of finite-dimensional subcoalgebras (Theorem 3.1). This completes the proof. ∎

Now we are able to prove the following weak analogue of the tame-wild dichotomy theorem of Drozd [13] for a class of K-coalgebras (see *Note added in proof*).

THEOREM 6.11. *Let K be an algebraically closed field and assume that C is a K-coalgebra of tame comodule type.*

(a) *Every finite-dimensional subcoalgebra of C is of tame comodule type.*

(b) *C is a directed union of finite-dimensional subcoalgebras of tame comodule type.*

(c) *If $\dim_K \operatorname{Ext}^1_C(S, S')$ is finite for every pair S, S' of simple left C-comodules, then C is not of wild comodule type.*

Proof. (a) Fix a decomposition (6.1). Let H be a finite-dimensional subcoalgebra of C. Then there exists a finite subset J of I_C such that $\operatorname{soc}_H H = \bigoplus_{j\in J} S(j)$ and $K_0(H) \cong \mathbb{Z}^{(J)}$.

It follows from Theorem 4.3(b) that H-Comod is a full exact subcategory of C-Comod, and for every vector $v \in K_0(H) \cong \mathbb{Z}^{(J)} \subseteq \mathbb{Z}^{(I_C)}$ all left H-comodules M such that $v = \mathbf{length}\, M$ are the C-comodules M such that $v = \mathbf{length}\, M$.

Let $C^* = \operatorname{Hom}_K(C, K)$ be the pseudocompact K-algebra (3.4) associated with C. By Theorem 4.3 the category C-Comod may be identified with the category $\operatorname{Dis}(C^*)$ of discrete right C^*-modules, and a comodule M in C-Comod lies in H-Comod if and only if M viewed as a discrete right C^*-module is annihilated by the ideal $H^\perp$ of C^*.

Fix a vector $v \in K_0(H) \cong \mathbb{Z}^{(J)} \subseteq \mathbb{Z}^{(I_C)}$. Since C is of tame comodule type, there exist C-$K[t]$-bimodules $L^{(1)}, \ldots, L^{(r_v)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable left H-comodules M with $\mathbf{length}\, M = v$ are of the form $M \cong L^{(s)} \otimes K^1_\lambda$, where $s \le r_v$, $K^1_\lambda = K[t]/(t-\lambda)$ and $\lambda \in K$. Hence it remains to find H-$K[t]$-bimodules $L^{(1)}, \ldots, L^{(r_v)}$ with the above properties.

View $L^{(1)}, \ldots, L^{(r_v)}$ as $K[t]$-C^*-bimodules and view the factor $K[t]$-H^*-bimodules $\overline{L}^{(1)}, \ldots, \overline{L}^{(r_v)}$ as H-$K[t]$-bimodules, where $\overline{L}^{(s)} = L^{(j)}/L^{(j)}H^\perp$ for $s = 1, \ldots, r_v$. Hence $\overline{L}^{(j)}$ is a finitely generated $K[t]$-module, and if $M \cong L^{(s)} \otimes K^1_\lambda$ is in H-comod then $L^{(s)} \otimes K^1_\lambda \cong \overline{L}^{(s)} \otimes K^1_\lambda$. It follows that H is of tame comodule type, because in view of [44, Theorem 14.18] applied to the finite-dimensional K-algebra $H^* \cong C^*/H^\perp$, the $K[t]$-H^*-bimodules

$\overline{L}^{(1)}, \ldots, \overline{L}^{(r_v)}$ can be replaced by bimodules which are finitely generated free over $K[t]$.

The statement (b) is a consequence of (a) and Theorem 3.1.

(c) Assume to the contrary that C is of wild comodule type. By Theorem 6.10, there exists a finite-dimensional subcoalgebra H of C of wild comodule type. By (a), H is of tame comodule type. Since the category isomorphism H-comod $\cong \operatorname{mod}(H^*)$ preserves tame representation type and wild representation type, we get a contradiction with the tame-wild dichotomy theorem of Drozd [13] applied to the algebra H^*. This finishes the proof. ∎

The proof of Theorem 6.11(a) yields

COROLLARY 6.12. *If C is a tame coalgebra of polynomial growth then every finite-dimensional subcoalgebra D of C is tame of polynomial growth.* ∎

The following proposition shows that a finite-dimensional K-coalgebra C is of tame comodule type (resp. tame of polynomial growth) if and only if the associated finite-dimensional K-algebra C^* is tame (resp. tame of polynomial growth).

PROPOSITION 6.13. *Let K be an algebraically closed field and let C be a K-coalgebra with a decomposition* $\operatorname{soc}_C C = \bigoplus_{j \in I_C} S(j)^{t_j}$ (see (6.1)) *such that the set I_C is finite.*

(a) *C is of tame comodule type if and only if for every $d \in \mathbb{N}$ there exist C-$K[t]$-bimodules $N^{(1)}, \ldots, N^{(r_d)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable left C-comodules M with* $\dim_K M = d$ *are of the form $M \cong N^{(s)} \otimes K^1_\lambda$, where $s \leq r_d$, $K^1_\lambda = K[t]/(t-\lambda)$ and $\lambda \in K$.*

(b) *C is tame of polynomial growth if and only if there exists an integer $g \geq 1$ such that $\boldsymbol{\mu}^1_C(v) \leq \|v\|^g$ for all $v = (v(j))_{j \in I_C} \in K_0(C) \cong \mathbb{Z}^{(I_C)}$ with $\|v\| = \sum_{j \in I_C} v(j) \geq 2$.*

Proof. (a) Apply the definition of tameness and the fact that given $d \in \mathbb{N}$ the number of vectors $v = (v(j))_{j \in I_C} \in K_0(C) \cong \mathbb{Z}^{(I_C)}$ such that $\|v\| = d$ is finite.

(b) Let C be tame of polynomial growth and let $G(t)$ be a power series as in Definition 6.7 such that $\boldsymbol{\mu}^1_C(v) \leq G(v)$ for all $v \in K_0(C) \cong \mathbb{Z}^{(I_C)}$ with $\|v\| \geq 2$. We have $I_C = \{j_1, \ldots, j_n\}$ for some $n \geq 1$ and $G(z)$ is a polynomial of the form

$$G(z) = \sum_{m \geq 0} \sum_{s_1 + \ldots + s_n = m} g_{s_1, \ldots, s_n} z_{j_1}^{s_1} z_{j_2}^{s_2} \ldots z_{j_n}^{s_n}$$

with non-negative coefficients $g_{s_1, \ldots, s_m} \in \mathbb{Z}$. It is easy to see that there exists $g_0 \geq 1$ such that $g_{s_1, \ldots, s_n} v(j_1)^{s_1} \ldots v(j_n)^{s_n} \leq [v(j_1) + \ldots + v(j_n)]^{g_0}$ for all $v =$

$[v(j_1), \ldots, v(j_n)] \in \mathbb{Z}^{(I_C)}$ such that $\|v\| \geq 2$. It follows that $G(v) \leq n_0 \|v\|^{g_0}$ for some $n_0 \geq 2$, and therefore $G(v) \leq \|v\|^g$ for some $g \geq g_0$. ∎

We say that a vector $v \in K_0(C) \cong \mathbb{Z}^{(I_C)}$ has a *finite-dimensional support subcoalgebra* H_v of C if every indecomposable C-comodule M in C-comod with **length** $M = v$ lies in H_v-comod $\subseteq C$-comod, that is, $\mathrm{ind}_v(H_v\text{-comod}) = \mathrm{ind}_v(C\text{-comod})$.

Now we are able to prove a weak version of tame-wild dichotomy.

PROPOSITION 6.14. *Let C be a basic K-coalgebra with a decomposition* (6.1). *Assume that K is algebraically closed and every $v \in K_0(C) \cong \mathbb{Z}^{(I_C)}$ has a finite-dimensional support subcoalgebra H_v.*

(a) *The following three conditions are equivalent.*

(a1) *C is of tame comodule type.*

(a2) *Every finite-dimensional subcoalgebra of C is of tame comodule type.*

(a3) *C is a directed union of finite-dimensional subcoalgebras of tame comodule type.*

(b) *If C is not of tame comodule type then it is of wild comodule type.*

(c) *If* $\dim_K \mathrm{Ext}^1_C(S, S')$ *is finite for every pair S, S' of simple left C-comodules*, *then C is of tame comodule type if and only if it is not of wild comodule type.*

Proof. The implication (a1)⇒(a2) is a consequence of Theorem 6.11(a). In view of Theorem 3.1, the implication (a2)⇒(a3) is obvious.

(a3)⇒(a1). Assume that C is a directed union of finite-dimensional subcoalgebras of tame comodule type. Then, by assumption, for every $v \in K_0(C) \cong \mathbb{Z}^{(I_C)}$ there exists a finite-dimensional subcoalgebra H_v of C such that every indecomposable C-comodule M with **length** $M = v$ lies in H_v-comod $\subseteq$ C-comod. Then H_v is of tame comodule type and therefore there exist H_v-$K[t]$-bimodules $L^{(1)}, \ldots, L^{(r_v)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable left H_v-comodules M with **length** $M = v$ are of the form $M \cong L^{(s)} \otimes K^1_\lambda$, where $s \leq r_v$, $K^1_\lambda = K[t]/(t - \lambda)$ and $\lambda \in K$. This proves that C is of tame comodule type, because the H_v-$K[t]$-bimodules $L^{(1)}, \ldots, L^{(r_v)}$ are C-$K[t]$-bimodules in a natural way. Since the converse implication is a consequence of Theorem 6.11(b), the proof of (a) is complete.

(b) If $\dim_K C$ is finite then, in view of Proposition 6.13, the tame-wild dichotomy holds for C, because C-comod $\cong \mathrm{mod}(C^*)$ and the tame-wild dichotomy of Drozd [13] applies to the finite-dimensional K-algebra C^*.

Assume now that C is arbitrary, and fix a directed family C_β of finite-dimensional K-subcoalgebras of C such that $C = \bigcup_\beta C_\beta$. Moreover, assume that C is not of tame comodule type. By the equivalence of (a1) and (a2), there exists β such that C_β is not of tame comodule type. By the observation above, C_β is of wild comodule type, and therefore C is of wild comodule type.

The statement (c) is a consequence of (b) and Theorem 6.11(c). This finishes the proof. ∎

REMARK 6.15. Definitions 6.6 and 6.7 define tame representation type, polynomial growth and wild representation type for any abelian length K-category [19], because by Proposition 5.9, for any length K-category $\mathfrak{A}$ there exists a K-coalgebra C and an equivalence of K-categories $\mathfrak{A} \cong C$-comod. In view of Lemma 6.8, the definitions do not depend on the choice of the coalgebra C.

REMARK 6.16. By applying the arguments used in the proof of Lemmata 1.2 and 1.3 of [10] one can show that the hypothesis of Proposition 6.14 holds for any K-coalgebra C such that $\dim_K \operatorname{Ext}^1_C(S, S')$ is finite for every pair S, S' of simple left C-comodules. In view of Theorem 6.11 and Proposition 6.14, the tame-wild comodule type dichotomy holds for this class of coalgebras.

EXAMPLE 6.17. Consider the cocommutative polynomial K-coalgebra $K[t]^\diamond = K[t]$ with comultiplication Δ and counity ε defined by the formulas $\Delta(t^m) = \sum_{r+s=m} t^r \otimes t^s$, $\varepsilon(1) = 1$ and $\varepsilon(t^s) = 0$ for $s \geq 1$. For any $m \geq 0$ the subspace $K[t]^\diamond_m = K \oplus Kt \oplus \ldots \oplus Kt^m$ of $K[t]^\diamond$ is a K-subcoalgebra of dimension $m+1$ and $K[t]^\diamond = \bigcup_{m=0}^\infty K[t]^\diamond_m$. Note that $\operatorname{soc} K[t]^\diamond = K[t]^\diamond_0 = K$ and the Grothendieck group $K_0(K[t]^\diamond)$ is infinite cyclic.

It is easy to see that the dual K-algebra $(K[t]^\diamond_m)^*$ is isomorphic to $K[[t]]/(t^m) \cong K[t]/(t^m)$ and the pseudocompact K-algebra (3.4) associated with $K[t]^\diamond$ is

$$(K[t]^\diamond)^* \cong \varprojlim_m (K[t]^\diamond_m)^* \cong \varprojlim_m K[t]/(t^m) \cong K[[t]]$$

with the Jacobson radical powers topology. It follows that under the identification $K[t]^\diamond$-Comod $= \operatorname{Dis}((K[t]^\diamond)^*)$ of (4.4) the left $K[t]^\diamond$-comodules are just the t-torsion $K[t]$-modules. It follows from [44, Proposition 14.12] that the category $K[t]^\diamond$-comod has almost split sequences, every indecomposable comodule in $K[t]^\diamond$-comod of dimension $m+1$ is isomorphic to the subcomodule $K[t]^\diamond_m$ of $K[t]^\diamond$, and the Auslander–Reiten quiver of $K[t]^\diamond$-comod is a rank one homogeneous tube of the form shown in [44, p. 289].

It is obvious that the coalgebra $K[t]^\diamond$ is of tame comodule type and for any $d \geq 1$ there is precisely one isoclass of indecomposable $K[t]^\diamond$-comodules of dimension d. In other words, the growth function $\boldsymbol{\mu}^1_{K[t]^\diamond} : \mathbb{Z} \to \mathbb{N}$ is zero, whereas $\boldsymbol{\mu}^0_{K[t]^\diamond}(d) = 1$ for all $d \geq 1$. ∎

EXAMPLE 6.18. Consider the cocommutative K-coalgebra

$$K[t_1, t_2]^\diamond = K[t_1, t_2]/(t_1 t_2) = K \oplus \bigoplus_{n=1}^\infty K t_1^n \oplus \bigoplus_{m=1}^\infty K t_2^m,$$

where the comultiplication $\Delta : K[t_1,t_2]^\diamond \to K[t_1,t_2]^\diamond \otimes K[t_1,t_2]^\diamond$ and counity $\varepsilon : K[t_1,t_2]^\diamond \to K$ are defined by $\Delta(t_j^m) = \sum_{r+s=m} t_j^r \otimes t_j^s$ for $j = 1,2$, $\varepsilon(1) = 1$ and $\varepsilon(t_j^s) = 0$ for $s \geq 1$ and $j = 1,2$. For any $m \geq 0$ the subspace

$$K[t_1,t_2]_m^\diamond = K \oplus Kt_1 \oplus Kt_1^2 \oplus \ldots \oplus Kt_1^m \oplus Kt_2 \oplus Kt_2^2 \oplus \ldots \oplus Kt_2^m$$

of $K[t_1,t_2]^\diamond$ is a K-subcoalgebra of dimension $2m+1$ and $K[t_1,t_2]^\diamond = \bigcup_{m=0}^\infty K[t_1,t_2]_m^\diamond$. Note that $\operatorname{soc} K[t_1,t_2]^\diamond = K[t_1,t_2]_0^\diamond = K$.

It is easy to see that the dual K-algebra $(K[t_1,t_2]_m^\diamond)^*$ is isomorphic to $\Lambda_m = K[t_1,t_2]/(t_1t_2, t_1^m, t_2^m)$ and the pseudocompact K-algebra (3.4) associated with $K[t_1,t_2]^\diamond$ is

$$(K[t_1,t_2]^\diamond)^* \cong K[[t_1,t_2]]/(t_1t_2) \cong \varprojlim_m (K[t_1,t_2]_m^\diamond)^* \cong \varprojlim_m \Lambda_m$$

with the Jacobson radical powers topology. It follows that the Grothendieck group $K_0(K[t_1,t_2]^\diamond)$ is infinite cyclic, and under the identification

$$K[t_1,t_2]^\diamond\text{-Comod} = \operatorname{Dis}(K[[t_1,t_2]]/(t_1t_2))$$

of (4.4) the finite-dimensional $K[t_1,t_2]^\diamond$-comodules are just the finite-dimensional modules over $K[t_1,t_2]/(t_1t_2)$ which are annihilated by some powers of $\overline{t}_1$ and $\overline{t}_2$. By applying a well known description of indecomposable Λ_m-modules (see [22] and [7]) we conclude that for any $d \geq 0$ the indecomposable modules of dimension d in $\operatorname{Dis}(K[[t_1,t_2]]/(t_1t_2))$ are Λ_d-modules. Hence we get a category isomorphism

$$K[t_1,t_2]^\diamond\text{-comod} \cong \operatorname{nilmod}^{\mathrm{lf}}(K[t_1,t_2]/(t_1t_2))$$

where $\operatorname{nilmod}^{\mathrm{lf}}(K[t_1,t_2]/(t_1t_2))$ consists of the modules of finite length that are nilpotent, that is, annihilated by some powers of $\overline{t}_1$ and $\overline{t}_2$.

We easily conclude that the indecomposable $K[t_1,t_2]^\diamond$-comodules of dimension d are in $K[t_1,t_2]_d^\diamond$-comod $\cong \operatorname{dis}(\Lambda_d) = \operatorname{mod}(\Lambda_d)$. It follows from Proposition 6.14 that the coalgebra $K[t_1,t_2]^\diamond$ is of tame comodule type, because according to [22] the algebra Λ_d is of tame representation type for any $d \geq 0$. If $d \geq 3$ the K-algebra Λ_d is of non-polynomial growth (see [51] and [53]). Thus the coalgebra $K[t_1,t_2]^\diamond$ is of tame comodule type and of non-polynomial growth. It follows from [7] that the category $K[t_1,t_2]^\diamond$-comod has no almost split sequences. ∎

By applying [36, Lemma 1.5] (or [46, Theorem 3.12(b)]) we get

Corollary 6.19. *Let K be an algebraically closed field and C a K-coalgebra. The coalgebra C is of fully wild comodule type if and only if there exists a pair of finite-dimensional left C-modules U_1, U_2 satisfying the following three conditions*:

(i) $\operatorname{End}_C(U_1) \cong \operatorname{End}_C(U_2) \cong K$,
(ii) $\operatorname{Hom}_C(U_1,U_2) = 0$ *and* $\operatorname{Hom}_C(U_2,U_1) = 0$,
(iii) $\dim_K \operatorname{Ext}_C^1(U_1,U_2) \geq 3$. ∎

7. Left pure semisimple K-coalgebras. A K-coalgebra C is said to be of *finite comodule type* if the number of the isomorphism classes of finite-dimensional indecomposable left C-comodules is finite. It follows from Corollary 4.6 that the definition is left-right symmetric.

Following [39, Section 7], [47] and [40] (see also [47] and [61]) we introduce the following definition.

DEFINITION 7.1. A K-coalgebra C is *left pure semisimple* if the Grothendieck category C-Comod of left C-comodules is pure semisimple, that is, every left C-comodule is a direct summand of a direct sum of finite-comodules.

It follows from Theorem 9.3 of Section 9 that the definition is not left-right symmetric.

The following characterisation is a consequence of [37], [38] and [40] applied to the category $\mathcal{A} = C$-Comod.

THEOREM 7.2. *For every K-coalgebra C the following conditions are equivalent.*

(a) *C is left pure semisimple.*

(b) *Every left C-comodule is a direct sum of finite-dimensional subcomodules.*

(c) *Every left C-comodule is algebraically compact* (*or pure-injective*) *in the sense of* [38, Section 4].

(d) *Every infinite sequence $N_1 \xrightarrow{f_1} N_2 \to \ldots \to N_m \xrightarrow{f_m} N_{m+1} \to \ldots$ of monomorphisms $f_1, f_2, \ldots$ between finite-dimensional indecomposable left C-comodules $N_1, N_2, \ldots$ terminates*, *that is*, *there exists $m_0 \geq 1$ such that f_j is bijective for all $j \geq m_0$.* ∎

PROPOSITION 7.3. *Assume that C is a left pure semisimple K-coalgebra.*

(a) *Every indecomposable projective object P of C*-comod *has a unique maximal subcomodule P_0 such that the inclusion $P_0 \hookrightarrow P$ is a minimal right almost split morphism.*

(b) *For every indecomposable non-projective comodule Z in C*-comod *there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in C*-comod.

Proof. We follow the proof of [42, Proposition 2.4(a)]. Denote by $L(C\text{-Comod})$ the Grothendieck category of all contravariant K-linear functors from C-comod to the category of finite-dimensional K-vector spaces. By applying [38, Theorem 6.3] and [41, Corollary 2.6] to the pure semisimple Grothendieck category $\mathcal{A} = C$-Comod we conclude that the category $L(C\text{-Comod})$ is perfect and locally noetherian. In particular, for every indecomposable comodule Z in C-comod the Yoneda functor $h_Z = \mathrm{Hom}_C(-, Z)$ is noetherian and the subfunctor $\mathrm{rad}_C(-, Z)$ of h_Z is finitely presented in the category $L(C\text{-Comod})$, where $\mathrm{rad}_C(U,Z)$ consists of all non-isomorphisms for every indecomposable comodule U in C-comod (see [2], [44, Section 11.1]).

(a) Assume that P is an indecomposable projective object in C-comod. By remarks above applied to $Z = P$, there exists a projective cover $h_v : h_V \to \mathrm{rad}_C(-,P)$ in the category $L(C\text{-Comod})$, where $v : V \to P$ is a homomorphism in C-comod. Since $\mathrm{rad}_C(-,P)$ is a proper subfunctor of $h_P = \mathrm{Hom}_C(-,P)$ and P is projective, v is not an epimorphism. It follows that the proper embedding $\mathrm{Im}\, v \hookrightarrow P$ admits a factorisation through v, and therefore v is injective, because otherwise $v : V \to \mathrm{Im}\, v$ is a splittable epimorphism, $\mathrm{Ker}\, v \neq 0$ and consequently the morphism $h_v : h_V \to \mathrm{rad}_C(-,P)$ is not a projective cover, a contradiction. Consequently, h_v is an isomorphism and $P_0 = \mathrm{Im}\, v$ is a unique maximal subcomodule of P such that the inclusion $P_0 \hookrightarrow P$ is a minimal right almost split morphism.

(b) Let Z be an non-projective indecomposable object in C-comod. It follows that there is a minimal projective presentation

$$0 \to h_X \xrightarrow{h_u} h_Y \xrightarrow{h_v} \mathrm{rad}_C(-,Z) \to 0$$

of the functor $\mathrm{rad}_C(-,Z)$ in the category $L(C\text{-Comod})$, where $u : X \to Y$ and $v : Y \to Z$ are homomorphisms in C-comod. Hence we easily conclude that $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is an almost split sequence in C-comod. This finishes the proof. ∎

The following theorem is an immediate consequence of [38, Theorem 7.1].

Theorem 7.4. *An indecomposable cocommutative coalgebra C is left pure semisimple if and only if* $\dim_K C$ *is finite and C is uniserial*, *that is*, *the subcoalgebras of C form a finite chain.* ∎

The following theorem shows that the study of coalgebras C of finite comodule type is equivalent to the study of algebras of finite representation type, because every such basic coalgebra C is finite-dimensional and there is a category isomorphism $C\text{-Comod} \cong \mathrm{Mod}(C^*)$.

Theorem 7.5. *Let K be a field and let C be a basic K-coalgebra with a decomposition* (6.1). *The following conditions are equivalent.*

(a) *C is of finite comodule type.*

(b) *C is left pure semisimple and* $\mathrm{soc}\,_C C$ *is finite-dimensional.*

(c) *C is left pure semisimple and* $\dim_K C$ *is finite.*

(d) *The Grothendieck group $K_0(C)$ is of finite rank and every indecomposable left C-comodule is finite-dimensional.*

(e) *The K-algebra $C^* = \mathrm{Hom}_K(C,K)$ is of finite representation type.*

If any of the conditions (a)–(e) *is satisfied*, *then C is finite-dimensional and there is a category isomorphism* $C\text{-Comod} \cong \mathrm{Mod}(C^*)$.

Proof. We recall that $\mathcal{A} = C$-Comod is a locally finite Grothendieck K-category and $\mathrm{fp}(\mathcal{A}) = C$-comod is the subcategory of $\mathcal{A}$ formed by the objects of finite length. Let $L(\mathcal{A}) = \mathrm{Add}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathcal{A}b)$ be the category of additive contravariant functors from $\mathrm{fp}(\mathcal{A})$ to the category $\mathcal{A}b$ of abelian groups. We know from [37] and [38, Section 6] that the functor

$$h_\bullet : \mathcal{A} \to L(\mathcal{A}), \tag{7.6}$$

$X \mapsto h_X = \mathrm{Hom}_{\mathcal{A}}(-, X)$, is a fully faithful embedding and defines an equivalence of $\mathcal{A}$ and the full subcategory $\mathrm{Fl}(\mathcal{A}^{\mathrm{op}}, \mathcal{A}b)$ of $L(\mathcal{A})$ formed by the flat functors. The category $\mathcal{A}$ is pure semisimple if and only if the category $L(\mathcal{A})$ is perfect.

(a)$\Rightarrow$(b). Assume that C is of finite comodule type and let $X = X_1 \oplus \ldots \oplus X_r$, where $X_1, \ldots, X_r$ is a complete set of representatives of the isomorphism classes of indecomposable comodules in C-comod. It follows that there is an equivalence of categories $L(\mathcal{A}) \cong \mathrm{Mod}(\Lambda_C)$, where $\Lambda_C = \mathrm{End}_C(X)$. Since Λ_C is a finite-dimensional K-algebra, according to the well known result of H. Bass the category $L(\mathcal{A}) \cong \mathrm{Mod}(\Lambda_C)$ is perfect. It follows that C is left pure semisimple. Since the coalgebra C is basic and of finite comodule type, we have $\mathrm{soc}\,{}_C C = \bigoplus_{j \in I_C} S(j)$, where I_C is finite. This finishes the proof of (a)$\Rightarrow$(b).

(b)$\Rightarrow$(c). Since $\mathrm{soc}\,{}_C C = \bigoplus_{j \in I_C} S(j)$, where I_C is finite, the injective left C-comodule C has a decomposition ${}_C C = \bigoplus_{j \in I_C} E(S(j))$, where $E(S(j))$ is the injective envelope of $S(j)$. Since C is left pure semisimple, the injective comodule $E(S(j))$ is finite-dimensional for each $j \in I_C$. Consequently, $\dim_K C$ is finite and (b)$\Rightarrow$(c) is proved.

The implication (c)$\Rightarrow$(b) is obvious.

(c)$\Leftrightarrow$(d). Since the coalgebra C is basic and $\mathrm{soc}\,{}_C C = \bigoplus_{j \in I_C} S(j)$, we have $K_0(C) \cong \mathbb{Z}^{(I_C)}$ and ${}_C C = \bigoplus_{j \in I_C} E(S(j))$, by Lemma 6.5. It follows that, if every indecomposable left C-comodule is finite-dimensional or C is left pure semisimple, then the group $K_0(C)$ is of finite rank if and only if $\dim_K C$ is finite. Then the equivalence of (c) and (d) is an immediate consequence of [41, Theorem 1.3].

(c)$\Rightarrow$(e). If $\dim_K C$ is finite, then according to Theorem 4.3 and Lemma 2.3 there is a category isomorphism C-Comod $\cong \mathrm{Dis}(C^*) = \mathrm{Mod}(C^*)$. Then the finite-dimensional K-algebra is right pure semisimple and the implication (c)$\Rightarrow$(e) is a consequence of a well known result of Auslander [1].

Since the implication (e)$\Rightarrow$(a) is a consequence of the category K-linear isomorphism C-Comod $\cong \mathrm{Dis}(C^*) = \mathrm{Mod}(C^*)$, the proof is complete. ∎

COROLLARY 7.7. *Let C be a left pure semisimple K-coalgebra with socle decomposition* $\mathrm{soc}\,{}_C C = \bigoplus_{j \in I_C} S(j)^{t_j}$.

(a) *For any finite set* $\mathcal{S}$ *of simple left* C*-comodules there are only a finite number of isomorphism classes of indecomposable left* C*-comodules in* C-comod *whose composition factors belong to* $\mathcal{S}$.

(b) C *is a directed union of finite-dimensional subcoalgebras of finite comodule type.*

(c) C *is domestic of tame comodule type and the growth function* $\boldsymbol{\mu}_C^1 : K_0(C) \to \mathbb{N}$ *is zero* (see (6.7)).

(d) *For any finite subset* T *of* I_C *the growth function* $\boldsymbol{\mu}_C^0 : K_0(C) \to \mathbb{N}$ *vanishes at all but a finite number of vectors* $v \in \mathbb{Z}^{(T)} \subseteq \mathbb{Z}^{(I_C)} \cong K_0(C)$.

Proof. According to Corollary 5.12, the left C-comodules in C-comod whose composition factors belong to $\mathcal{S}$ form an exact abelian subcategory $C\text{-comod}\|_{\mathcal{S}}$ of C-comod, and there exists a basic K-coalgebra $C_{\mathcal{S}}$ such that $C\text{-comod}\|_{\mathcal{S}} \cong C_{\mathcal{S}}$-comod and $\operatorname{soc} C_{\mathcal{S}}$ is finite-dimensional. Since C is left pure semisimple, so is $C_{\mathcal{S}}$ and according to Theorem 7.5 the coalgebra $C_{\mathcal{S}}$ is finite-dimensional of finite comodule type. Hence the corollary follows. ∎

REMARK 7.8. There are examples of indecomposable K-coalgebras of infinite comodule type which are left and right pure semisimple (see Example 8.5).

8. Quivers, relation ideals and path coalgebras. In order to formulate our main result we need some notation. A K-coalgebra C is called *hereditary* if the category C-Comod of left C-comodules is hereditary, that is, $\operatorname{Ext}_C^2(M,N) = 0$ for all M and N in C-Comod, or equivalently, epimorphic images of injective C-comodules are injective C-comodules. It was shown in [30] that the definition is left-right symmetric, and a coalgebra C is hereditary if $\operatorname{Ext}_C^2(M,N) = 0$ for all M and N in C-comod (see also [11]).

Following Gabriel [18], by a *quiver* $Q = (Q_0, Q_1)$ we mean an oriented graph (in general infinite) with the set Q_0 of vertices and the set Q_1 of arrows. We associate with Q the quiver $Q^* = (Q_0^*, Q_1^*)$ *opposite* to Q, where $Q_0^* = Q_0$ and Q_1^* consists of arrows of the form $\beta^* : j \to i$ corresponding to the arrows $\beta : i \to j$ in Q_1.

A *K-linear representation of the quiver* $Q = (Q_0, Q_1)$ is a system

$$X = (X_i, \varphi_\beta)_{i \in Q_0, \beta \in Q_1}$$

where X_i is a K-vector space and $\varphi_\beta : X_i \to X_j$ is a K-linear map for any $\beta : i \to j$. If X_j is a finite-dimensional K-vector space for any j and $X_j = 0$ for almost all j, then the representation X is of *finite length* (see also [2], [44, Chapter 14]).

A morphism $f : X \to X'$ of representations of Q is a system $f = (f_i)_{i \in Q_0}$ of K-linear maps $f_i : X_i \to X_i'$, $i \in Q_0$, such that $\varphi_\beta f_i = f_j \varphi_\beta$ for all

$\beta : i \to j$ in Q_1, that is, the following diagram is commutative:

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_\beta} & X_j \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_j} \\ X'_i & \xrightarrow{\varphi'_\beta} & X'_j \end{array}$$

We denote by $\mathrm{Hom}_Q(X, Y)$ the K-linear space of all morphisms from X to Y. We denote by $\mathrm{Rep}_K(Q)$ the Grothendieck K-category of K-linear representations of Q and by $\mathrm{Rep}^{\mathrm{lf}}_K(Q)$ the full Grothendieck K-subcategory of $\mathrm{Rep}_K(Q)$ formed by *locally finite-dimensional representations*, that is, directed unions of representations of finite length. Finally, we denote by $\mathrm{rep}_K(Q) \supseteq \mathrm{rep}^{\mathrm{lf}}_K(Q)$ the full subcategories of $\mathrm{Rep}_K(Q)$ formed by finitely generated objects (see [34]) and by locally finite-dimensional representations, or equivalently, by representations of finite length.

It follows that the category $\mathrm{Rep}^{\mathrm{lf}}_K(Q)$ is locally finite and $\mathrm{rep}^{\mathrm{lf}}_K(Q)$ consists of all objects of $\mathrm{Rep}^{\mathrm{lf}}_K(Q)$ of finite length.

Let $Q = (Q_0, Q_1)$ be a quiver. We recall that an *oriented path* in Q of length $m \geq 1$ starting from vertex $i = i_0$ and ending at vertex $j = i_m$ is a formal composition

$$(*) \qquad \omega = \beta_1\beta_2 \dots \beta_m \equiv (i_0 \xrightarrow{\beta_1} i_1 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_m} i_m)$$

of arrows $\beta_1, \dots, \beta_m$. To any vertex $i \in Q_0$ we attach a *stationary path* η_i starting and ending at i. The stationary path at j in the opposite quiver Q^* is also denoted by η_j. If $\omega = \beta_1\beta_2 \dots \beta_m$ is the path $(*)$ in Q we set

$$\omega^* = \beta^*_m\beta^*_{m-1} \dots \beta^*_1$$

and view it as a path in Q^*. Denote by Q_m the set of all oriented paths in Q of length $m \geq 0$.

Given a quiver $Q = (Q_0, Q_1)$ we denote by KQ the *path algebra* of Q with coefficients in K (see [2], [44, Chapter 14]), that is, the graded K-vector space

$$(**) \qquad KQ = KQ_0 \oplus KQ_1 \oplus \dots \oplus KQ_m \oplus \dots,$$

where $KQ_m = \bigoplus_{\omega \in Q_m} \omega K$, equipped with addition and multiplication defined as follows. Given $a = \sum_\omega \omega a_\omega$ and $b = \sum_{\omega'} \omega' b_{\omega'}$ in KQ we put

$$a + b = \sum_{\omega''} \omega''(a_{\omega''} + b_{\omega''}), \quad a \cdot b = \sum_{\omega,\omega'} \omega\omega' a_\omega b_{\omega'},$$

where $\omega = \beta_1 \dots \beta_m$, $\omega' = \beta'_1 \dots \beta'_t$, and we set $\omega\omega' = \beta_1 \dots \beta_m\beta'_1 \dots \beta'_t$ if the terminal vertex of β_m coincides with the source vertex of β'_1, and $\omega\omega' = 0$ otherwise. We set $\lambda\alpha = \alpha\lambda$ for $\lambda \in K$.

Given two vertices $a, b \in Q_0$ of Q we denote by $KQ(a, b)$ the subspace of KQ generated by all oriented paths from a to b, and by $KQ_m(a, b)$ the subspace of $KQ(a, b)$ generated by paths of length m.

It is clear that KQ endowed with the direct sum decomposition $(**)$ is a graded K-algebra, the stationary paths η_i, $i \in Q_0$, form a complete set of primitive orthogonal idempotents of KQ, and there is a right ideal decomposition

$$KQ = \bigoplus_{i \in Q_0} \eta_i KQ.$$

If Q_0 is finite then $\sum_{i \in Q_0} \eta_i$ is the identity of KQ. If Q_0 is infinite the algebra KQ has no identity element. It is clear that the dimension of KQ is finite if and only if Q is finite and has no oriented cycles. Note that for each $m \geq 1$ the K-vector space

$$(***) \qquad KQ_{\geq m} = \bigoplus_{j \geq m} KQ_j$$

is the two-sided ideal of KQ generated by all paths of length $\geq m$.

It is well known (see [44, Section 14.1]) that there are equivalences of categories $\mathrm{Rep}_K(Q) \cong \mathrm{Mod}(KQ)$ and $\mathrm{rep}_K(Q) \cong \mathrm{mod}(KQ)$.

The path K-algebra KQ endowed with the direct sum decomposition $(**)$ can be viewed as a graded K-coalgebra with comultiplication $\Delta : KQ \to KQ \otimes KQ$ and counity $\varepsilon : KQ \to K$ defined as follows (see [9] and [60]). Given the stationary path η_i at i we set $\Delta(\eta_i) = \eta_i \otimes \eta_i$ and $\varepsilon(\eta_i) = 1$. Given any path $\omega = \beta_1 \ldots \beta_m$ of length $m \geq 1$ starting from $i = i_0$ and ending at $j = i_m$ we set

$$\Delta(\omega) = \eta_i \otimes \omega + \omega \otimes \eta_j + \sum_{s=1}^{m-1} (\beta_1 \ldots \beta_s) \otimes (\beta_{s+1} \ldots \beta_m) = \sum_{uv = \omega} u \otimes v$$

and $\varepsilon(\omega) = 0$, where $\otimes = \otimes_K$. We call KQ together with the above coalgebra structure the *path K-coalgebra* of the quiver Q with coefficients in K.

It is clear that for each $m \geq 0$ the K-vector space

$$(****) \qquad KQ_{\leq m} = KQ_0 \oplus KQ_1 \oplus \ldots \oplus KQ_m$$

is a subcoalgebra of KQ.

Note that if Q is a finite quiver without oriented cycles, then the finite-dimensional K-algebra $(KQ^*)^* = \mathrm{Hom}_K(KQ^*, K)$, K-dual to the K-coalgebra KQ^*, is the path algebra KQ and KQ^*-comod $\cong \mathrm{mod}(KQ) \cong \mathrm{rep}_K(Q)$ (see [28], [56] and [44, Section 14.1]).

We define a K-linear representation X of Q to be *nilpotent* (or a small representation of Q, in the sense of Gabriel [19, Section 7.4]) if there exists an $m \geq 1$ such that the composite K-linear map

$$X_{i_0} \xrightarrow{\varphi_{\beta_1}} X_{i_1} \xrightarrow{\varphi_{\beta_2}} \ldots \xrightarrow{\varphi_{\beta_m}} X_{i_m}$$

is zero for any path $\beta_1 \ldots \beta_m$ in Q of length m. We denote by $\mathrm{nilrep}_K^{\mathrm{lf}}(Q)$ the full subcategory of $\mathrm{rep}_K(Q)$ formed by all nilpotent representations of finite length, and by $\mathrm{Rep}_K^{\mathrm{lnlf}}(Q)$ the full subcategory of $\mathrm{Rep}_K(Q)$ formed by all *locally nilpotent representations* that are locally finite, that is, directed unions of representations from $\mathrm{nilrep}^{\mathrm{lf}}_K(Q)$. It is easy to see that a K-linear representation X of Q is locally nilpotent if and only if for any $i_0 \in Q_0$ and any $x_0 \in X_{i_0}$ there exists an $m \geq 1$ such that $\varphi_{\beta_m} \dots \varphi_{\beta_1}(x_0) = 0$ for any path $\beta_1 \dots \beta_m$ in Q of length m (see also [8], [10], [60]).

We recall from [20, 4.2] that given an ideal Ω of the path algebra KQ contained in $KQ_{\geq 2}$ we define $\mathrm{rep}_K(Q, \Omega)$ to be the full subcategory of $\mathrm{rep}_K(Q)$ formed by all representations satisfying all relations in Ω (see also [44, Section 14.1]).

We start by collecting elementary properties of path K-coalgebras of arbitrary quivers (see also [9, Section 4], [60, Section 4]).

PROPOSITION 8.1. *Let Q be an arbitrary quiver, Q^* the quiver opposite to Q, and K a field. Given a vertex j in $Q_0 = Q_0^*$ we denote by η_j the stationary path at j.*

(a) *The path K-coalgebra $C = KQ^*$ is basic, the one-dimensional vector space $S(j) = K\eta_j$ is a simple left subcomodule of C and $KQ_0^* = \mathrm{soc}_C C = \bigoplus_{j \in Q_0} S(j)$. If $i \neq j$, then the left C-comodules $S(i)$ and $S(j)$ are not isomorphic. The subcoalgebras*

$$KQ_0^* \subseteq KQ_{\leq 1}^* \subseteq \dots \subseteq KQ_{\leq m}^* \subseteq \dots$$

give the coradical filtration of KQ^*.

(b) *For each $j \in Q_0$, the indecomposable left ideal $(KQ^*)\eta_j$ of the path K-algebra KQ^* generated by all oriented paths in Q^* ending at j is an indecomposable injective left coideal of the coalgebra KQ^*, $\mathrm{soc}(KQ^*)\eta_j = S(j)$, the injective envelope $E(S(j))$ of $S(j)$ is isomorphic to $(KQ^*)\eta_j$, and*

$$KQ^* = \bigoplus_{j \in Q_0^*} (KQ^*)\eta_j$$

is a left coideal decomposition.

(c) *For any finite subquiver Q^β of Q and any $m \geq 0$ there is a K-algebra surjection $\varepsilon_m^\beta : KQ \to KQ^\beta / KQ^\beta_{\geq m}$, the subspace $K(Q^\beta)^*_{\leq m}$ of KQ^* is a finite-dimensional subcoalgebra, the path coalgebra KQ^* is a directed union of the subcoalgebras $K(Q^\beta)^*_{\leq m}$ and the category $\mathrm{nilrep}^{\mathrm{lf}}_K(Q)$ has a directed union form*

$$\mathrm{nilrep}^{\mathrm{lf}}_K(Q) = \bigcup_{\beta} \bigcup_{m=1}^{\infty} \mathrm{rep}_K(Q^\beta, KQ^\beta_{\geq m}).$$

The two-sided ideals $U_m^\beta = \mathrm{Ker}\, \varepsilon_m^\beta$ define a linear topology on KQ (called the finite subquiver topology), the completion

$$\widehat{KQ} = \varprojlim_{\beta, m} KQ / U_m^\beta \tag{8.2}$$

of KQ is a pseudocompact K-algebra containing KQ (a complete tensor algebra [19, 7.5]) *and the topological dual coalgebra* $(\widehat{KQ})^\circ = \mathrm{hom}_K(\widehat{KQ}, K)$ *is isomorphic to KQ^*.*

(d) *There exist K-linear category isomorphisms*

$$KQ^*\text{-comod} \cong \mathrm{nilrep}_K^{\mathrm{lf}}(Q) \cong \text{comod-}KQ. \tag{8.3}$$

Proof. (a) Let H be a simple subcoalgebra of KQ^*. It follows from the multiplication formula in KQ^* that H contains the stationary path η_j for some $j \in Q_0$. Since the vector space $K\eta_j$ is obviously a subcoalgebra (and a subcomodule) of H and H is simple, we have $H = \eta_j K$. In view of Lemmas 5.1 and 5.3, the coalgebra KQ^* is basic and the remaining statements in (a) follow easily.

(b) It is clear that $(KQ^*)\eta_j \subseteq KQ^*$ is a left coideal generated by all oriented paths in Q^* with terminus at $j \in Q_0^*$. Since $KQ_0^* = \mathrm{soc}_C C = \bigoplus_{j\in Q_0} S(j)$, obviously $\mathrm{soc}(KQ^*)\eta_j = S(j)$ and therefore the left coideal $(KQ^*)\eta_j$ is indecomposable. Since $(KQ^*)\eta_j$ is a direct summand of the injective left comodule KQ^*, it is an injective comodule and (b) follows.

(c) The first part follows immediately by applying definitions. To prove the second part we recall that KQ^* is a directed union of the finite-dimensional subcoalgebras $H_m^\beta = K(Q^\beta)^*_{\le m} \subseteq K(Q^\beta)^*$ and therefore the dual pseudocompact algebra $(KQ^*)^* = \mathrm{Hom}_K(KQ^*, K)$ has the form $(KQ^*)^* \cong \varprojlim_{\beta,m} H_m^\beta$. It is clear that for any integer $m \ge 1$ and any finite subquiver Q^β of Q there is an isomorphism of finite-dimensional K-algebras $(H_m^\beta)^* \cong KQ^\beta/KQ^\beta_{\ge m}$ and the restriction $\varepsilon_m^\beta : KQ \to (H_m^\beta)^* \cong KQ^\beta/KQ^\beta_{\ge m}$ of the canonical algebra surjection $(KQ^*)^* \to (H_m^\beta)^*$ to KQ is a surjection. It follows that the ideals $U_m^\beta = \mathrm{Ker}\,\varepsilon_m^\beta$ define a linear topology on KQ and there is an isomorphism of pseudocompact algebras

$$(KQ^*)^* \cong \varprojlim_{\beta,m} KQ/U_m^\beta = \widehat{KQ}.$$

By Theorem 3.6, the coalgebra $(\widehat{KQ})^\circ$ is isomorphic to KQ^*. This finishes the proof of (c).

(d) Since KQ^* is a directed union of finite-dimensional subcoalgebras of the form $K(Q^\beta)^*_{\le m}$, where Q^β runs through all finite subquivers of Q and $m \ge 1$, according to Theorem 4.3(b) and Lemma 2.3(a) any comodule M in KQ^*-comod lies in $K(Q^\beta)^*_{\le m}$-comod $\subseteq KQ^*$-comod for some finite subquiver Q^β of Q and $m \ge 1$. Since there is a K-algebra isomorphism $(K(Q^\beta)^*_{\le m})^* \cong KQ^\beta/KQ^\beta_{\ge m+1}$, the required category isomorphism $KQ^*\text{-comod} \cong \mathrm{nilrep}_K^{\mathrm{lf}}(Q)$ is defined by the composition

$$\begin{aligned} K(Q^\beta)^*_{\le m}\text{-comod} &\cong \mathrm{dis}(KQ^\beta/KQ^\beta_{\ge m+1}) = \mathrm{mod}(KQ^\beta/KQ^\beta_{\ge m+1}) \\ &\cong \mathrm{rep}_K(Q^\beta, KQ^\beta_{\ge m+1}) \subseteq \mathrm{nilrep}_K^{\mathrm{lf}}(Q) \end{aligned}$$

(apply (c), (4.4) and [44, Section 14.1]). The K-linear category isomorphism KQ^*-comod $\cong$ comod-KQ follows from the fact that the coalgebra $(KQ)^{\mathrm{op}}$ is isomorphic to KQ^*. ■

One can give an alternative proof of (d) following an idea of Gabriel [19, Section 7.5] (see also [60, 4.11]).

As an immediate consequence of Proposition 8.1 we get

COROLLARY 8.4. *If Q is a locally finite quiver without oriented cycles, then there exist category isomorphisms* KQ^*-comod $\cong \mathrm{rep}_K^{\mathrm{lf}}(Q) \cong$ comod-KQ. ■

Following Gabriel [20, 4.2], by a *quiver with relations* we mean a pair (Q,Ω), where Q is a quiver (in general infinite) and Ω is a two-sided ideal of the path K-algebra KQ contained in $KQ_{\geq 2}$ and such that $\Omega = \bigoplus_{a,b\in Q_0}\Omega(a,b)$, where $\Omega(a,b) = \Omega\cap KQ(a,b)$ (compare with [44, Section 14.1]). We call such an ideal Ω an *ideal of relations*. If (Q,Ω) is a quiver with relations we define $\mathrm{rep}_K(Q,\Omega)$ to be the full subcategory of $\mathrm{rep}_K(Q)$ formed by the representations of Q satisfying all relations in Ω.

With any quiver with relations (Q,Ω) we associate the path subcoalgebra

$$C(Q,\Omega) = \{x\in KQ;\ \langle x,\Omega\rangle = 0\}$$

of KQ, where $\langle -,-\rangle : KQ\times KQ\to K$ is the standard K-bilinear form on KQ. One can show that the path subcoalgebra $C(Q^*,\Omega^*)$ of KQ^* is just the left path coalgebra of (Q,Ω) defined in [48]. It then follows from [48] that (8.3) induces K-linear category isomorphisms

$$\text{comod-}C(Q,\Omega)\cong \mathrm{nilrep}_K^{\mathrm{lf}}(Q,\Omega),\qquad \text{Comod-}C(Q,\Omega)\cong \mathrm{Rep}_K^{\mathrm{lnlf}}(Q,\Omega).$$

We expect that any subcoalgebra H of KQ containing $KQ_{\leq 1}$ is of the form $C(Q,\Omega)$. This together with [19, Section 7], [9, Theorem 4.3] and [60, Theorem 4.13] would imply that any basic K-coalgebra C over an algebraically closed field K is isomorphic to a coalgebra $C(Q,\Omega)$.

Following [10], one can prove that any hereditary basic coalgebra over an algebraically closed field is isomorphic to the path coalgebra of a quiver. These problems will be discussed in a subsequent paper [49].

EXAMPLE 8.5. Let Q be the infinite quiver

$$\ldots \xrightarrow{\beta_{-2}} \mathbf{-1} \xrightarrow{\beta_{-1}} \mathbf{0} \xrightarrow{\beta_0} \mathbf{1} \xrightarrow{\beta_1} \mathbf{2} \xrightarrow{\beta_2} \mathbf{3} \xrightarrow{\beta_3} \ldots$$

and let Ω be the ideal of the path K-algebra KQ generated by the zero-relations $\beta_j\beta_{j+1}$ for all $j\in\mathbb{Z}$. Let $C = C(Q,\Omega)$ be the subcoalgebra $KQ_0\oplus KQ_1$ of KQ. One can easily show that the category C-comod $\cong$ comod-C is equivalent to the category $\mathrm{rep}_K(Q,\Omega)$ of K-linear representations of Q satisfying $\beta_j\beta_{j+1} = 0$ for all $j\in\mathbb{Z}$ (see Proposition 8.1 and Corollary 8.4). It follows that the coalgebra $C = C(Q,\Omega)$ is of infinite comodule type and is both left and right pure semisimple. ■

In [19, Section 7] Gabriel has associated a quiver (and even a species) with any length K-category. By applying his construction to the category C-comod we introduce the following definition.

DEFINITION 8.6. Assume that K is an algebraically closed field and C is a basic K-coalgebra with a decomposition $\operatorname{soc}_C C = \bigoplus_{j\in I_C} S(j)$. We associate with C the (left) *Gabriel quiver* ${}_CQ$ as follows. The vertices of ${}_CQ$ are just the elements j of I_C (identified with the simple left C-comodules $S(j)$). The arrows from i to j are elements of a fixed basis of the K-vector space ${}_jE_i = \operatorname{Ext}^1_C(S(i), S(j))$.

The Gabriel quiver ${}_CQ$ is also known as the Ext quiver of C-comod (see [29] and [9, p. 45]). It is shown in [29, Theorem 1.7] that the quiver ${}_CQ$ is isomorphic to a link quiver Γ_C, up to multiple arrows. Then the following useful fact is a consequence of [29, Corollary 2.2].

COROLLARY 8.7. *Let C be a coalgebra and ${}_CQ$ its Gabriel quiver. Then ${}_CQ$ is connected if and only if the coalgebra C is indecomposable.* ■

Here a K-coalgebra C is said to be *indecomposable* if C is not a direct sum of two subcoalgebras, or equivalently, if the category C-Comod is not a direct sum of two non-stationary subcategories.

Following [19, 7.3] we give an alternative definition of the Gabriel quiver ${}_CQ$ by means of the pseudocompact K-algebra C^* of (3.4) as follows.

Assume that the field K is algebraically closed and C is a basic K-coalgebra. Then the pseudocompact K-algebra C^* is basic and there is a direct product decomposition (see Corollary 5.5)

$$F = C^*/J(C^*) = \prod_{j\in I_C} K_j, \tag{8.8}$$

where K_j is a field isomorphic to K. The field K_j can be identified with the one-dimensional simple comodule $S(j)$ and with the endomorphism algebra of $S(j)$. Consider the closure $\overline{J(C^*)^2}$ of $J(C^*)^2$ in C^* and view the F-F-bimodule

$$M = J(C^*)/\overline{J(C^*)^2}$$

as a right pseudocompact C^*-module, or as a pseudocompact K-module. This implies a decomposition of M into a topological product

$$M = \prod_{i,j\in I_C} {}_iM_j, \tag{8.9}$$

where ${}_iM_j = K_iMK_j$ is viewed as a K_i-K_j-bimodule. This bimodule is related to the K_j-K_i-bimodule ${}_jE_i = \operatorname{Ext}^1_C(S(i), S(j))$ by the formulas

$${}_iM_j \cong \operatorname{Hom}_{K_j}({}_jE_i, K_j) \quad \text{and} \quad {}_jE_i \cong \operatorname{hom}_{K_j}({}_iM_j, K_j). \tag{8.10}$$

Let Q be a quiver such that the number m_{ij} of arrows from i to j in Q is finite for all $i, j \in Q_0$. The *bilinear form* of Q is the integral $\mathbb{Z}$-bilinear form

$$b_Q : \mathbb{Z}^{(Q_0)} \times \mathbb{Z}^{(Q_0)} \to \mathbb{Z} \tag{8.11}$$

assigning to any pair of vectors $v = (v_j)_{j \in Q_0}$, $w = (w_j)_{j \in Q_0}$ (with finitely many non-zero integral coordinates) the integer

$$b_Q(v, w) = \sum_{j \in Q_0} v_j w_j - \sum_{i,j \in Q_0} m_{ij} v_i w_j.$$

The *quadratic form* of Q,

$$q_Q : \mathbb{Z}^{(Q_0)} \to \mathbb{Z}, \tag{8.12}$$

is defined by the formula $q_Q(v) = b_Q(v, v)$. Here $\mathbb{Z}^{(Q_0)}$ is the direct sum of Q_0 copies of the free abelian group $\mathbb{Z}$.

Proposition 8.13. *Assume that Q is an arbitrary quiver, K a field and $C = KQ$ is the path K-coalgebra of Q with* $\operatorname{soc}_C C = \bigoplus_{j \in Q_0} S(j)$, *where $S(j) = K\eta_j$. Given two vertices i and j in Q_0 denote by m_{ij} the number of arrows from i to j in Q.*

(a) *For any simple left C-comodule $S(j) = K\eta_j$ there exists an exact sequence*

$$0 \to S(j) \to E(S(j)) \to \bigoplus_{a \in Q_0} E(S(a))^{(m_{aj})} \to 0$$

*in C-*Comod, *where $E(S(j)) = (KQ)\eta_j$ is the injective envelope of $S(j)$* (see 8.1) *and $U^{(m)}$ denotes the direct sum of m copies of U for any cardinal number m.*

(b) *The path K-coalgebra $C = KQ$ is hereditary,*

$$\dim_K \operatorname{Ext}^1_C(S(i), S(j)) \;=\; m_{ij} \tag{8.14}$$

and $\operatorname{Hom}_C(E(S(i)), E(S(j))) \cong \eta_j \widehat{KQ^*} \eta_i$ (see (8.2)) *for all $i, j \in Q_0$, in C-*comod $\cong$ comod-KQ^*. *The Gabriel quiver ${}_CQ$ is isomorphic to Q.*

(c) *Given $i, j \in Q_0$, m_{ij} is finite if and only if* $\dim_K \operatorname{Ext}^1_C(S(i), S(j))$ *is finite.*

(d) *Assume m_{ij} is finite for all $i, j \in Q_0$. Let b_Q be the bilinear form* (8.11)*. Then ${}_CQ = Q$, $\dim_K \operatorname{Ext}^1_C(M, N)$ is finite and*

$$b_Q(\mathbf{length}\, M, \mathbf{length}\, N) = \dim_K \operatorname{Hom}_C(M, N) - \dim_K \operatorname{Ext}^1_C(M, N) \tag{8.15}$$

*for any comodules M, N in C-*comod $\cong$ -comod-KQ^*.

Proof. (a) It follows from Proposition 8.1 that C is basic, $\operatorname{soc}_C C = \bigoplus_{j \in Q_0} S(j)$ and $E(S(j)) = (KQ)\eta_j$, where $S(j) = \eta_j K$ is a simple left subcomodule of C. If $i \neq j$, the left C-comodules $S(i)$ and $S(j)$ are not isomorphic. In particular, this shows that Q_0 is the set of vertices of the Gabriel quiver ${}_CQ$.

Consider the exact sequence

$$(*)\qquad 0\to S(j)\xrightarrow{v(j)} E(S(j))\to \operatorname{Coker} v(j)\to 0$$

in C-Comod, where $S(j)=\eta_j K$, $E(S(j))=(KQ)\eta_j$ and $v(j)$ is the natural embedding. Let $Q(\to j)$ be the set of all non-stationary oriented paths in Q ending at $j\in Q_0$, $\mathcal{S}(j)$ the set of all source points of arrows ending at j, and given $a\in\mathcal{S}(j)$ we denote by $Q(a,j)$ the set of arrows from a to j in Q.

To prove (8.14) we note that given $\beta\in Q(a,j)$ the map $u\mapsto u\beta$ defines a C-comodule isomorphism $E(S(a))=(KQ^*)\eta_a\xrightarrow{\sim}(KQ^*)\eta_a\beta=E(S(a))\beta$. A straightforward analysis yields

$$\begin{aligned}(**)\qquad \operatorname{Coker} v(j) &= \bigoplus_{\omega\in Q(\to j)} K\omega=\bigoplus_{a\in\mathcal{S}(j)}\bigoplus_{\beta\in Q(a,j)}\bigoplus_{u} Ku\beta\\ &=\bigoplus_{a\in\mathcal{S}(j)}\bigoplus_{\beta\in Q(a,j)} E(S(a))\beta\cong\bigoplus_{a\in Q_0}E(S(a))^{(m_{aj})},\end{aligned}$$

where u runs through all oriented paths ending at a. This proves (a).

(b) The isomorphism $\operatorname{Hom}_C(E(S(i)),E(S(j)))\cong \eta_j\widehat{KQ^*}\eta_i$ is a consequence of (8.2) and the duality (4.5), because $C^*\cong\widehat{KQ^*}$, $\widetilde{D}_1(E(S(t)))\cong\widehat{KQ^*}\eta_t$ and therefore we get the isomorphisms

$$\operatorname{Hom}_C(E(S(i)),E(S(j)))\cong \hom_{\widehat{KQ^*}}(\widehat{KQ^*}\eta_j,\widehat{KQ^*}\eta_i)\cong\eta_j\widehat{KQ^*}\eta_i.$$

To prove that C is hereditary it is sufficient to show that $\operatorname{Ext}^2_C=0$, or equivalently, $\operatorname{Ext}^2_C(S(i),S(j))=0$ for all $i,j\in Q_0$ (see [11], [30]). But this is an immediate consequence of (a).

Now we note that $(*)$ yields the exact sequence

$$\operatorname{Hom}_C(S(i),E(S(j)))\xrightarrow{\ell_{ij}}\operatorname{Hom}_C(S(i),\operatorname{Coker} v(j))\xrightarrow{\partial_{ij}}\operatorname{Ext}^1_C(S(i),S(j))\to 0$$

for any $i,j\in Q_0$.

Assume that $j=i$. It is easy to see that $\ell_{ii}=0$ and therefore ∂_{ii} is an isomorphism. In view of $(**)$ we get isomorphisms

$$\begin{aligned}\operatorname{Ext}^1_C(S(i),S(i))&\cong\operatorname{Hom}_C(S(i),\operatorname{Coker} v(i))\\ &\cong\bigoplus_{a\in Q_0}\operatorname{Hom}_C(S(i),E(S(a))^{(m_{ai})})\\ &\cong\operatorname{Hom}_C(S(i),E(S(i))^{(m_{ii})})\oplus\bigoplus_{a\neq i}\operatorname{Hom}_C(S(i),E(S(a))^{(m_{ai})})\\ &\cong\operatorname{Hom}_C(S(i),E(S(i))^{(m_{ii})})\cong K^{(m_{ii})},\end{aligned}$$

because $\operatorname{Hom}_C(S(i),E(S(t)))=0$ for all $t\neq i$ and $\operatorname{Hom}_C(S(i),E(S(i)))\cong\operatorname{Hom}_C(S(i),S(i))\cong K$. Consequently, $\operatorname{Ext}^1_C(S(i),S(i))\cong K^{(m_{ii})}$.

Assume that $j \neq i$. Then $\mathrm{Hom}_C(S(i), E(S(j))) = 0$ and ∂_{ij} is a K-linear isomorphism. In view of $(**)$ we get K-linear isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_C(S(i), S(j)) &\cong \mathrm{Hom}_C(S(i), \mathrm{Coker}\, v(j))\\ &\cong \bigoplus_{a \in Q_0} \mathrm{Hom}_C(S(i), E(S(a))^{(m_{aj})}) \cong K^{(m_{ij})}.\end{aligned}$$

By (8.14), there are precisely m_{ij} arrows from i to j in Q and in ${}_CQ$ for all $i, j \in Q_0 = {}_CQ_0$. Consequently, the quivers ${}_CQ$ and Q are isomorphic. This finishes the proof of (b).

The statement (c) follows easily from (8.14).

(d) If m_{ij} is finite for all $i, j \in Q_0$, then it follows from (8.14), by an obvious induction on the length of comodules M and N in C-comod, that $\dim_K \mathrm{Ext}^1_C(M, N)$ is finite.

Now we note that from (8.14) and the definition of b_Q it follows easily that (8.15) holds for $M = S(i)$ and $N = S(j)$ with arbitrary $i, j \in Q_0$. The general case reduces to the above one by an obvious induction on the length of M and N in C-comod, because both sides of (8.15) are additive functions on C-comod with respect to each of the variables M and N. ∎

The formula (8.15) extends that of Ringel [36] established for hereditary algebras.

Lemma 8.16. *Let K be a field, $\mathcal{L}$ be the quiver consisting of one vertex and two loops, and let $\mathcal{W}$ be the path K-coalgebra $K\mathcal{L}$.*

(a) *$\mathcal{W}$ is hereditary of fully wild comodule type and there are K-linear equivalences of categories* $\mathcal{W}$-comod $\cong \mathrm{nilrep}^{\mathrm{lf}}_K(\mathcal{L}) \cong \mathrm{nilmod}^{\mathrm{lf}}(K\langle t_1, t_2\rangle)$.

(b) *If H is an arbitrary K-subcoalgebra of the path K-coalgebra KQ of a finite quiver Q, then there exists a fully faithful exact K-linear functor* H-comod $\to \mathcal{W}$-comod.

Proof. (a) Since $\mathcal{L}^* = \mathcal{L}$, the first equivalence is a consequence of Proposition 8.1, and the second one follows from [44, 14.6]. By Proposition 8.13, $\mathcal{W}$ is hereditary. By [4] (see also [44, Proposition 14.10] and its proof) there is a fully faithful exact K-linear functor

$$\mathrm{mod}\, \Gamma_3(K) \to \mathrm{nilmod}^{\mathrm{lf}}(K\langle t_1, t_2\rangle) \cong \mathrm{nilrep}^{\mathrm{lf}}_K(\mathcal{L}).$$

Since according to Proposition 8.1 there is an equivalence $\mathrm{nilrep}^{\mathrm{lf}}_K(\mathcal{L}) \cong$ $\mathcal{W}$-comod we are done.

(b) There is a K-linear fully faithful embedding functor H-comod $\to$ KQ-comod and, by Proposition 8.1(d), there are K-linear category isomorphisms KQ-comod $\cong \mathrm{nilrep}^{\mathrm{lf}}_K(Q^*) \cong \mathrm{nilmod}^{\mathrm{lf}}(KQ^*)$. On the other hand, KQ^* is finitely generated (as a K-algebra), because Q is a finite quiver. It follows from [4] (see also [44, Proposition 14.10] and its proof) that there exists a K-linear fully faithful embedding functor $\mathrm{nilmod}^{\mathrm{lf}}(KQ^*) \hookrightarrow \mathrm{nilmod}^{\mathrm{lf}}(\mathcal{W})$.

Since, by [20] and Proposition 8.1(d), there are K-linear category isomorphisms $\mathrm{nilmod}^{\mathrm{lf}}(\mathcal{W}) \cong \mathrm{nilrep}^{\mathrm{lf}}_K(\mathcal{L}) \cong \mathcal{W}$-comod, the statement (b) follows and the proof is complete. ∎

Now we are able to prove a coalgebra analogue of a well known characterisation of wild algebras (see [13] and [44, Corollary 14.11]), which indicates a wild behaviour of wild coalgebras.

Corollary 8.17. *Let C be a K-coalgebra and $\mathcal{W} = K\mathcal{L}$, where K is a field and $\mathcal{L}$ is the quiver consisting of one vertex and two loops α and β. The following conditions are equivalent.*

(a) *C is of wild comodule type* (*resp. of fully wild comodule type*).

(b) *There is a K-linear representation embedding functor $\mathcal{W}$*-comod $\to$ *C*-comod (*resp. fully faithful exact functor $\mathcal{W}$*-comod $\to C$-comod).

(c) *If H is an arbitrary K-subcoalgebra of the path K-coalgebra KQ of a finite quiver Q, then there exists a K-linear representation embedding functor H*-comod $\to C$-comod (*resp. fully faithful exact functor H*-comod $\to$ *C*-comod).

Proof. (a)⇒(b). In view of Proposition 8.1, there exists an equivalence of K-categories $\mathcal{W}$-comod $\cong \mathrm{nilrep}^{\mathrm{lf}}_K(\mathcal{L})$. On the other hand there is a fully faithful exact K-linear functor $\mathrm{nilrep}^{\mathrm{lf}}_K(\mathcal{L}) \to \mathrm{rep}_K(\circ \overset{\to}{\underset{\to}{\to}} \circ) \cong \mathrm{mod}\, \Gamma_3(K)$ defined by $(X, \varphi_\alpha, \varphi_\beta) \mapsto (X_1, X_2, \mathrm{id}_X, \varphi_\alpha, \varphi_\beta)$ with $X_1 = X_2 = X$ (see [44, Example 13, pp. 286–287]). This shows that (a) implies (b).

(b)⇒(c). Apply Lemma 8.16(b).

(c)⇒(a). Apply (c) to the K-coalgebra $H = KQ$, where Q is the three arrows quiver $\circ \overset{\to}{\underset{\to}{\to}} \circ$. Note that $Q^* = Q$ and there is a K-algebra isomorphism $(KQ)^* \cong \Gamma_3(K)$. It then follows from Corollary 8.4 that KQ-comod $\cong \mathrm{mod}\,(KQ)^* \cong \mathrm{mod}\, \Gamma_3(K)$ and therefore (c) implies (a). This finishes the proof. ∎

Corollary 8.18. *Let K be a field, $s \geq 0$ and let $\mathcal{T}_s$ be the quiver*

$$\mathcal{T}_s : \quad \begin{tikzcd} \overset{0}{\bullet} \arrow[r, "\gamma_0"] & \overset{1}{\bullet} \arrow[r, "\gamma_1"] & \overset{2}{\bullet} \arrow[r, "\gamma_2"] & \cdots \arrow[r, "\gamma_{s-1}"] & \overset{s}{\bullet} \arrow[dl, "\gamma_s"] \\ & & & \underset{s+1}{\bullet} \arrow[ull, "\gamma_{s+1}"] & \end{tikzcd} \tag{8.19}$$

The path coalgebras $K\mathcal{T}_s$ and $K\mathcal{T}_s^*$ are hereditary of fully wild comodule type.

Proof. In view of Lemma 6.9 and the K-linear equivalence $K\mathcal{T}_s$-comod $\cong$ comod-$K\mathcal{T}_s^*$ of (8.3), it is sufficient to prove that $K\mathcal{T}_s^*$ is of fully wild comodule type. By Proposition 8.1(d), there exist K-linear equivalences

$K\mathcal{L}$-comod $\cong \mathrm{nilrep}_K^{\mathrm{lf}}(\mathcal{L})$ and $K\mathcal{T}^*$-comod $\cong \mathrm{nilrep}_K^{\mathrm{lf}}(\mathcal{T}_s)$. Then it is sufficient to construct a K-linear fully faithful exact functor (see [44, p. 286])

$$T : \mathrm{nilrep}_K^{\mathrm{lf}}(\mathcal{L}) \to \mathrm{nilrep}_K^{\mathrm{lf}}(\mathcal{T}_s).$$

Given a representation $(M, \varphi_\alpha, \varphi_\beta)$ in $\mathrm{nilrep}_K^{\mathrm{lf}}(\mathcal{L})$, with $\varphi_\alpha, \varphi_\beta : M \to M$, we set

$$T(M, \varphi_\alpha, \varphi_\beta) = (M_0, M_1, \ldots, M_{s+1}; \varphi_{\gamma_0}, \varphi_{\gamma_1}, \ldots, \varphi_{\gamma_{s+1}}),$$

where $M_0 = M$, $M_1 = \ldots = M_{s+1} = M \oplus M$, $\varphi_{\gamma_0}(m) = (m, 0)$ for all $m \in M$, $\varphi_{\gamma_1} = \ldots = \varphi_{\gamma_s} = \mathrm{id}_{M \oplus M}$ and

$$\varphi_{\gamma_{s+1}} = \begin{pmatrix} 0 & \varphi_\alpha \\ \mathrm{id}_M & \varphi_\beta \end{pmatrix}.$$

Given a morphism $f : (M, \varphi_\alpha, \varphi_\beta) \to (M', \varphi'_\alpha, \varphi'_\beta)$ in $\mathrm{rep}_K^{\mathrm{lf}}(\mathcal{W})$ we define a morphism $T(f) : T(M, \varphi_\alpha, \varphi_\beta) \to T(M', \varphi'_\alpha, \varphi'_\beta)$ in $\mathrm{rep}_K^{\mathrm{lf}}(\mathcal{T}_s)$ by setting $T(f) = (f_0, f_1, \ldots, f_{s+1})$, where $f_0 = f$ and $f_1 = \ldots = f_{s+1} = (f, f)$. It is easy to see that the representation $T(M, \varphi_\alpha, \varphi_\beta)$ is nilpotent if $(M, \varphi_\alpha, \varphi_\beta)$ is, and $T(f)$ is a morphism in $\mathrm{rep}_K^{\mathrm{lf}}(\mathcal{T}_s)$ for any morphism f in $\mathrm{rep}_K^{\mathrm{lf}}(\mathcal{L})$. The proof that T is a K-linear fully faithful exact functor is straightforward. ∎

Let us finish this section by answering the question when the category C-comod, where $C = KQ^*$ has enough almost split sequences, that is, for every indecomposable non-injective comodule X in C-comod there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in C-comod, and for every indecomposable non-projective comodule Z in C-comod there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in C-comod.

THEOREM 8.20. *Let Q be a connected quiver and K a field. The category* comod-KQ *of right finite-dimensional KQ-comodules has enough almost split sequences if and only if the quiver Q is of one of the following types*:

(a) *Q is of extended Dynkin type $\widetilde{\mathbb{A}}_n$ with cyclic orientation.*
(b) *Q is finite and has no oriented cycles*,
(c) *Q is infinite and contains no infinite oriented path*,
(d) *Q or Q^* is infinite of the form $\mathbb{A}_\infty^{(0)}$* : $\bullet\!\to\!\bullet\!\to\!\bullet\!\to \ldots \to\!\bullet\!\to\!\bullet\!\to \ldots$,
(e) *Q is infinite of the form* $\ldots \to\!\bullet\!\to\!\bullet\!\to\!\bullet\!\to \ldots \to\!\bullet\!\to\!\bullet\!\to \ldots$

Proof. Proposition 8.1(d) yields comod-$KQ \cong \mathrm{nilrep}_K^{\mathrm{lf}}(Q)$. It is easy to see that $\mathrm{nilrep}_K^{\mathrm{lf}}(Q)$ coincides with the category $\mathrm{fd}_0(Q, K)$ defined in [55] and formed by all finite-dimensional K-linear representations of Q having composition factors only among discrete simples. Then our theorem is a consequence of [55, Theorem 1] (see also [32, Theorem 3.1]). ∎

The existence of almost split sequences in C-comod is also studied in [8]. In relation with Proposition 8.13 the following question arrises.

QUESTION 8.21. Is the category $\mathrm{Rep}_K(Q)$ hereditary for any quiver Q?

This is the case if Q is a finite quiver without oriented cycles, or Q is an infinite star quiver (see Section 9). On the other hand, by a well known result of P. M. Cohn, this is the case if Q consists of one vertex with s loops, because the path coalgebra is then isomorphic to the free K-algebra $K\langle t_1, \ldots, t_s\rangle$ of polynomials in non-commuting indeterminates $t_1, \ldots, t_s$. See also *Note added in proof*.

9. Path coalgebras of tame comodule type and left pure semisimple hereditary coalgebras. Recall that a *homogeneous Dynkin diagram* is any of the diagrams presented below.

$$\mathbb{A}_n: \quad \bullet - \bullet - \cdots - \bullet - \bullet \qquad (n \text{ vertices}, \ n \geq 1);$$

$$\begin{array}{ll} & \phantom{\bullet - {}}\bullet \\ & \phantom{\bullet - {}}| \\ \mathbb{D}_n: & \bullet - \bullet - \bullet - \cdots - \bullet - \bullet \qquad (n \text{ vertices}, \ n \geq 4); \end{array}$$

$$\begin{array}{ll} & \phantom{\bullet - \bullet - {}}\bullet \\ & \phantom{\bullet - \bullet - {}}| \\ \mathbb{E}_6: & \bullet - \bullet - \bullet - \bullet - \bullet \end{array}$$

$$\begin{array}{ll} & \phantom{\bullet - \bullet - {}}\bullet \\ & \phantom{\bullet - \bullet - {}}| \\ \mathbb{E}_7: & \bullet - \bullet - \bullet - \bullet - \bullet - \bullet \end{array}$$

$$\begin{array}{ll} & \phantom{\bullet - \bullet - {}}\bullet \\ & \phantom{\bullet - \bullet - {}}| \\ \mathbb{E}_8: & \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet \end{array}$$

By a *Dynkin quiver* (resp. an *extended Dynkin quiver*) we mean a finite quiver whose underlying non-oriented graph is a homogeneous Dynkin diagram (resp. an extended Dynkin diagram $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$).

DEFINITION 9.1 ([32]). (a) A quiver Q is said to be a *locally Dynkin quiver* if Q is locally finite and every finite subquiver Q' of Q is a homogeneous Dynkin quiver.

(b) Q is said to be a *pure semisimple locally Dynkin quiver* if it is either a Dynkin quiver or any of the infinite quivers presented in Table 9.2 below.

Table 9.2. Infinite pure semisimple locally Dynkin quivers

$$\mathbb{A}_\infty^{(s)}: \mathbf{0} - \mathbf{1} - \mathbf{2} - \ldots - \mathbf{s-1} \leftarrow \mathbf{s} \rightarrow \mathbf{s+1} \rightarrow \ldots$$

$${}_\infty\mathbb{A}_\infty^{(s)}: \ldots \leftarrow \mathbf{-2} \leftarrow \mathbf{-1} \leftarrow \mathbf{0} - \mathbf{1} - \mathbf{2} - \ldots - \mathbf{s-1} \leftarrow \mathbf{s} \rightarrow \mathbf{s+1} \rightarrow \ldots$$

$$\begin{array}{ll} & \phantom{\mathbf{0} - {}}\mathbf{-1} \\ & \phantom{\mathbf{0} - {}}| \\ \mathbb{D}_\infty^{(s)}: & \mathbf{0} - \mathbf{1} - \mathbf{2} - \ldots - \mathbf{s-1} \leftarrow \mathbf{s} \rightarrow \mathbf{s+1} \rightarrow \ldots \end{array}$$

Here $0 \leq s < \infty$ and $\mathbf{t} - \mathbf{r}$ means $\mathbf{t} \leftarrow \mathbf{r}$ or $\mathbf{t} \rightarrow \mathbf{r}$.

It is easy to see that a connected quiver Q is a locally Dynkin quiver if and only if it is one of the homogeneous Dynkin quivers or one of the quivers

$\mathbb{A}_\infty^{(s)}$, ${}_\infty\mathbb{A}_\infty^{(s)}$, $\mathbb{D}_\infty^{(s)}$ of Table 9.2, up to orientation. This happens if and only if the quadratic form q_Q is positive definite.

Note that any locally Dynkin quiver is a *star*, that is, a union of a locally finite subquiver Q' having no infinite paths and a subquiver which is a disjoint union of finitely many infinite chains of the form (see [35])

$$\mathbf{0} - \mathbf{1} - \mathbf{2} - \ldots - \mathbf{m} - \mathbf{m+1} - \ldots$$

where $\mathbf{m} - \mathbf{m+1}$ means either $\mathbf{m} \to \mathbf{m+1}$ or $\mathbf{m} \leftarrow \mathbf{m+1}$.

It was proved in [14, Theorem 1] that given a connected quiver Q the category $\mathrm{Rep}_K(Q)$ is pure semisimple if and only if Q is a pure semisimple locally Dynkin quiver.

Now we are able to present a coalgebra analogue of Gabriel's well known characterisation [18] of hereditary K-algebras of finite representation type (see also [32]).

THEOREM 9.3. *Let K be an algebraically closed field*, *C an indecomposable basic K-coalgebra*, *and $Q = {}_CQ$ the Gabriel quiver of C. The following conditions are equivalent.*

(a) *The coalgebra C is hereditary and left pure semisimple.*

(b) *The quiver Q^* opposite to $Q = {}_CQ$ is either one of the Dynkin quivers* $\mathbb{A}_n$, $n \geq 1$, $\mathbb{D}_n$, $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, *or one of the infinite locally Dynkin quivers* $\mathbb{A}_\infty^{(s)}$, ${}_\infty\mathbb{A}_\infty^{(s)}$, $\mathbb{D}_\infty^{(s)}$, *with $s \geq 0$*, *and the K-coalgebra C is isomorphic to the path K-coalgebra KQ.*

(c) *Q is locally finite*, *does not have infinitely many source vertices and does not contain infinite chains of the form $\bullet \to \bullet \to \bullet \to \ldots \to \bullet \to \bullet \to \ldots$*, *the quadratic form $q_Q : \mathbb{Z}^{(Q_0)} \to \mathbb{Z}$ is positive definite and there is a K-coalgebra isomorphism $C \cong KQ$.*

If C is hereditary and left pure semisimple, *then the map*

$$\mathbf{length} : C\text{-comod} \to \mathbb{Z}^{(Q_0)}$$

defines a bijection between the isomorphism classes of indecomposable left C-comodules and the positive roots of q_Q, *that is*, *the vectors $v \in \mathbb{N}^{(Q_0)}$ such that $q_Q(v) = 1$. Moreover*, *the growth function $\boldsymbol{\mu}_C^1 : K_0(C) \to \mathbb{N}$ is zero* (see 6.7), *the function $\boldsymbol{\mu}_C^0 : K_0(C) \to \mathbb{N}$ vanishes at all vectors $v \in \mathbb{Z}^{(Q_0)} \cong K_0(C)$ which are not positive roots of q_Q and $\boldsymbol{\mu}_C^0(v) = 1$ if v is a positive root of q_Q.*

Proof. (a)$\Rightarrow$(b). By applying [32, Theorem 2.2] to the hereditary category $\mathcal{A} = C$-Comod we get an equivalence of K-categories C-Comod $\cong \mathrm{Rep}_K^{\mathrm{lf}}(Q')$, where Q' is a pure semisimple locally Dynkin quiver. On the other hand, by Proposition 8.1, the coalgebra KQ'^* is basic and there is an equivalence of K-categories KQ'^*-Comod $\cong \mathrm{Rep}_K^{\mathrm{lf}}(Q') \cong C$-Comod. Since C is basic by assumption, according to Proposition 5.6 there is a coalgebra isomorphism $C \cong KQ'^*$. Since pure semisimple locally Dynkin quivers are obviously star quivers, by Proposition 8.13 the Gabriel quiver of KQ'^* is isomorphic to Q'^* and to $Q = {}_CQ$. This shows that $C \cong KQ$ and finishes the proof of (a)⇒(b).

The implication (b)⇒(a) follows from [14, Theorem 1], because obviously $Q = {}_CQ$ is a star quiver, there is an equivalence of categories KQ^*-Comod $\cong \mathrm{Rep}^{\mathrm{lf}}_K(Q)$ (see (8.3)) and according to Proposition 8.13 the category $\mathrm{Rep}^{\mathrm{lf}}_K(Q)$ is hereditary.

The equivalence (b)⇔(c) and the final statement in the theorem are easily verified, because they reduce to finite subquivers Q' of ${}_CQ$ and then, in view of KQ'^*-comod $\cong \mathrm{rep}^{\mathrm{lf}}_K(Q') \cong \mathrm{rep}_K(Q')$ (Proposition 8.1), to Gabriel's theorem [18] (see also [2, VIII. 5–6] and [21, Chapter VII]). ■

The following theorem is a coalgebra analogue of the well known characterisation of finite quivers of tame representation type due to Nazarova [31].

THEOREM 9.4. *Let Q be a connected quiver and let KQ^* be the path K-coalgebra of Q^*. The following conditions are equivalent.*

(a) *KQ^* is domestic of tame comodule type.*
(a′) *KQ^* is of tame comodule type.*
(b) *Q is either a locally Dynkin quiver, or an extended Dynkin quiver.*
(c) *Q is locally finite and q_Q is positive definite or positive semidefinite.*
(d) *KQ^* is not of wild comodule type.*
(e) *KQ^* is not of fully wild comodule type.*

Proof. The implication (a)⇒(a′) is obvious. The equivalence of (b) and (c) is well known and easy to prove by straightforward combinatorial arguments (consult [2, VIII. 5–6] and [21, Chapter VII]).

(a′)⇒(b). Assume that KQ^* is of tame comodule type. If Q has no oriented cycles, then for any finite connected subquiver Q' of Q the path K-coalgebra KQ'^* is a finite-dimensional subcoalgebra of KQ^* and according to Theorem 6.11 the coalgebra KQ'^* is of tame comodule type. Proposition 8.1 yields KQ'^*-comod $\cong \mathrm{rep}^{\mathrm{lf}}_K(Q') \cong \mathrm{rep}_K(Q')$ and therefore Q' is of tame representation type. It follows from the well known theorems of Gabriel [18] and Nazarova [31] that Q' is either a Dynkin quiver or one of the extended Dynkin quivers $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$.

Assume now that Q has an oriented cycle $\mathcal{S}$, and choose a minimal one. It follows that $\mathcal{S}$ is of type $\widetilde{\mathbb{A}}_s$ with a cyclic orientation, where $s \geq 0$, and $K\mathcal{S}^* \subseteq KQ^*$ is a subcoalgebra of KQ^*. Assume to the contrary that Q is not of extended Dynkin type. Then $\mathcal{S}$ is a proper subquiver of Q and therefore Q contains a subquiver $\widehat{\mathcal{S}}$ containing $\mathcal{S}$ such that $\widehat{\mathcal{S}}$ or $\widehat{\mathcal{S}}^*$ is of the form $\mathcal{T}_s$ shown in (8.19).

Suppose that $\widehat{\mathcal{S}} = \mathcal{T}_s$. Note that

$$H := K\widehat{\mathcal{S}}^*_{\leq 6} = K\mathcal{S}^*_0 \oplus K\mathcal{S}^*_1 \oplus \ldots \oplus K\mathcal{S}^*_6$$

(see $(****)$) is a finite-dimensional K-subcoalgebra of $K\widehat{\mathcal{S}}^*$ and therefore it is of tame comodule type, by Theorem 6.11. On the other hand, since $\dim_K H$ is finite and there is a K-algebra isomorphism $H^* \cong K\widehat{\mathcal{S}}/K\widehat{\mathcal{S}}_{\geq 7}$ (see $(***)$), in view of Theorem 4.3 (a) and [44, Corollary 14.7] we have

$$H\text{-comod} \cong \operatorname{mod}(H^*) \cong \operatorname{mod}(K\widehat{\mathcal{S}}/K\widehat{\mathcal{S}}_{\geq 7}) \cong \operatorname{rep}_K(\widehat{\mathcal{S}}, K\widehat{\mathcal{S}}_{\geq 7}).$$

In other words, H-comod is equivalent to the category of finite-dimensional K-linear representations of $\widehat{\mathcal{S}}$ satisfying all zero-relations of length ≥ 7 (see [44, Section 14.1]). By looking at the Galois covering of the bound quiver $(\mathcal{S}, K\mathcal{S}_{\geq 7})$ and applying [12, Theorem 3.6] we easily conclude that the finite-dimensional K-algebra H^* is representation-wild. Therefore H^* is not representation-tame and consequently H is not of tame comodule type, a contradiction. This shows that $Q = \mathcal{S}$ is of type $\widetilde{\mathbb{A}}_s$ with a cyclic orientation, and finishes the proof of (a)⇒(b).

(b)⇒(a). By Proposition 8.1(d), there exists a K-linear equivalence of categories $KQ^*\text{-comod} \cong \operatorname{nilrep}^{\mathrm{lf}}_K(Q)$. If Q is any of $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$, then according to Nazarova's theorem [31] the category $\operatorname{nilrep}^{\mathrm{lf}}_K(Q)$ is domestic of tame representation type and therefore the coalgebra KQ^* is domestic of tame comodule type.

Assume that Q is a locally Dynkin quiver. If Q is finite, it is a Dynkin quiver and in view of the equivalence $KQ^*\text{-comod} \cong \operatorname{rep}^{\mathrm{lf}}_K(Q) \cong \operatorname{rep}_K(Q)$ the coalgebra KQ^* is of finite comodule type, by the theorem of Gabriel [18]. If Q is infinite, it is easy to see that Q is one of the quivers $\mathbb{A}^{(s)}_\infty$, ${}_\infty\mathbb{A}^{(s)}_\infty$, $\mathbb{D}^{(s)}_\infty$ of Table 9.2, up to orientation. In particular, Q has no oriented cycles and $KQ^*\text{-comod} \cong \operatorname{rep}^{\mathrm{lf}}_K(Q)$, by Corollary 8.4.

From the description of the indecomposable representations of the Dynkin quivers given in [18] it follows that for each vector $v \in K_0(KQ^*) \cong \mathbb{Z}^{(Q_0)}$ the number of indecomposable representations M in $\operatorname{rep}^{\mathrm{lf}}_K(Q)$ with $\mathbf{length}\, M = v$ is finite, up to isomorphism (see also [2, VIII. 5–6] and [21, Chapter VII]). It follows that KQ^* is domestic of tame comodule type, because $\boldsymbol{\mu}^1_{KQ^*}(v) = 0$ for all $v \in K_0(KQ^*)$.

(b)⇒(d). By Proposition 8.1(d), $KQ^*\text{-comod} \cong \operatorname{nilrep}^{\mathrm{lf}}_K(Q)$. If Q is any of $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$, then according to Nazarova's theorem [31] the category $\operatorname{nilrep}^{\mathrm{lf}}_K(Q)$ is not of wild representation type and therefore KQ^* is not of wild comodule type.

Assume that Q is infinite. Then Q is a locally Dynkin quiver, has no oriented cycles and $KQ^*\text{-comod} \cong \operatorname{rep}^{\mathrm{lf}}_K(Q)$, by Corollary 8.4. Assume to the contrary that KQ^* is of wild comodule type. It follows that there exists a K-linear representation embedding exact functor $F : \operatorname{mod} \mathit{\Gamma}_3(K) \to \operatorname{rep}^{\mathrm{lf}}_K(Q)$.

Let S_1 (resp. S_2) be the unique simple injective (resp. projective) right $\Gamma_3(K)$-module, up to isomorphism. Then for every $\Gamma_3(K)$-module X there exist integers $n_1, n_2 \geq 0$ and an exact sequence $0 \to S_2^{n_2} \to X \to S_1^{n_1} \to 0$ in $\operatorname{mod} \Gamma_3(K)$. Since the functor F is exact, the induced sequence $0 \to U_2^{n_2} \to F(X) \to U_1^{n_1} \to 0$ is exact in $\operatorname{rep}_K^{\mathrm{lf}}(Q)$. It follows that there exists a finite full subquiver Q' of Q such that $F(X)$ is in $\operatorname{rep}_K^{\ell f}(Q') \subseteq \operatorname{rep}_K^{\mathrm{lf}}(Q)$, that is, F has a factorisation $F' : \operatorname{mod} \Gamma_3(K) \to \operatorname{rep}_K^{\mathrm{lf}}(Q')$ through $\operatorname{rep}_K^{\mathrm{lf}}(Q')$. It follows that $\operatorname{rep}_K^{\mathrm{lf}}(Q')$ is of infinite representation type, and we get a contradiction with the theorem of Gabriel [18], because Q' is a Dynkin quiver by assumption. Consequently, KQ^* is not of wild comodule type and (d) follows.

The implication (d)⇒(e) is obvious.

(e)⇒(b). Assume that KQ^* is not of fully wild comodule type. It follows from Corollary 8.18 that Q does not contain a finite subquiver of type $\mathcal{T}_s$ (see (8.19)). Moreover, Q has no finite subquiver Q' which is a one-arrow extension of any of $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$ without oriented cycles, because $KQ^*\text{-comod} \cong \operatorname{nilrep}_K^{\mathrm{lf}}(Q)$ (Corollary 8.4), and according to [4, Proposition 2] and [31] for every such Q' the category $\operatorname{rep}_K(Q') = \operatorname{nilrep}_K^{\mathrm{lf}}(Q') \subseteq \operatorname{nilrep}_K^{\mathrm{lf}}(Q)$ is of fully wild representation type (see also [26, Lemma 3.8]). ∎

10. Concluding remarks. It follows from Proposition 5.9 and Corollary 5.10 that the study of comodule categories over K-coalgebras and their representation types includes the study of arbitrary abelian length K-categories, and in particular the study of module categories over finite-dimensional K-algebras, the categories of finite-dimensional modules over infinite-dimensional K-algebras and categories of locally finite rational representations of nice algebraic groups.

Let us end the paper by stating some open questions and suggestions for further investigation and development of representation theory of coalgebras. In principle, they grow up from the representation theory of finite-dimensional K-algebras.

A LIST OF OPEN PROBLEMS. Assume that K is an algebraically closed field.

(10.1) Prove that the tame-wild dichotomy remains valid for all K-coalgebras (cf. Theorem 6.11 and Proposition 6.14).

(10.2) Give simple criteria for tame comodule type and for the polynomial growth of K-coalgebras C with locally finite Gabriel quiver ${}_CQ$ in terms of ${}_CQ$, of the Auslander–Reiten quiver $\Gamma(C\text{-comod})$ of the category C-comod, of the Euler form (or Tits form) associated with C, or in terms of the transfinite sequence of powers of the infinite radical $\operatorname{rad}^\infty(C\text{-comod})$ of C-comod (see [32, Section 3] and [50]).

(10.3) Describe the structure of connected components of $\Gamma(C\text{-comod})$ if C is of tame comodule type.

(10.4) Give necessary and sufficient conditions for a K-coalgebra C to be left pure semisimple in terms of $\Gamma(C\text{-comod})$ (cf. [32, Theorem 3.1]).

Example 6.17 shows that the connectedness of $\Gamma(C\text{-comod})$ does not imply the left pure semisimplicity of C, because according to Theorem 7.4 the indecomposable cocommutative coalgebra $K[t]^\diamond$ is not pure semisimple and the Auslander–Reiten quiver of $K[t]^\diamond$-comod is connected. Recall that $\Gamma(K[t]^\diamond\text{-comod})$ is a rank one homogeneous tube of the form shown in [44, p. 289].

(10.5) Develop a (co)tilting theory for comodule categories.

(10.6) Show that every basic K-coalgebra C is isomorphic to the path K-coalgebra $C(Q,\Omega)$ of a quiver with relations (Q,Ω).

(10.7) Develop a covering technique for K-coalgebras (see [12]).

(10.8) Develop a comodule variety technique for K-coalgebras (see [44, Section 14.5]).

(10.9) Following Theorem 9.13 give a characterisation of left pure semisimple hereditary K'-coalgebras C over any non-algebraically closed field K' by means of a valued Gabriel quiver $({}_CQ,\mathbf{d})$ of C and locally Dynkin diagrams defined in a suitable way (see [2] and [44]). ■

Note added in proof. 1° It is useful to note that, by [48, Corollary 5.5] and Theorem 9.4, an indecomposable hereditary basic K-coalgebra C over an algebraically closed field K is of tame comodule type if and only if the Gabriel quiver of C is a locally Dynkin quiver or an extended Dynkin quiver.

2° Assume that C is a basic K-coalgebra over an algebraically closed field K. One can prove the following two statements:

(a) If $\dim_K \operatorname{Ext}^1_C(S(i),S(j)) \geq 3$ for some $i,j \in I_C$, then C contains a wild subcoalgebra of dimension 4 or 5.

(b) If C is of tame comodule type, then $\dim_K \operatorname{Ext}^1_C(S(i),S(j)) \leq 2$ for all $i,j \in I_C$.

It follows that Theorems 6.10, 6.11(c) and Proposition 6.14(c) remain valid without the assumption that $\dim_K \operatorname{Ext}^1_C(S(i),S(j))$ is finite for all $i,j \in I_C$. In particular, if K is algebraically closed then there is no wild comodule type K-coalgebra which is of tame comodule type.

3° Michael Butler has pointed out to me that the answer to Question 8.21 is affirmative if Q is a quiver with finite vertex set. This is a consequence of the Appendix in the paper of M. C. R. Butler and A. D. King, *Minimal resolutions of algebras*, J. Algebra 212 (1999), 323–362.

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