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ESTIMATION OF GREEN'S FUNCTION ON PIECEWISE DINI-SMOOTH BOUNDED JORDAN DOMAINS

BY

MOHAMED AMINE BEN BOUBAKER (Nabeul) and MOHAMED SELMI (Tunis)

Abstract. We establish inequalities for Green functions on general bounded piecewise Dini-smooth Jordan domains in $\mathbb{R}^2$. This enables us to prove a new version of the 3G Theorem which generalizes its previous version given in [M. Selmi, Potential Anal. 13 (2000)]. Using these results, we give a comparison theorem for the Green kernel of Δ and the Green kernel of $\Delta - \mu$, where μ is a nonnegative and exact Radon measure.

1. Introduction. A Jordan curve $\mathcal{C}$ is said to be *Dini-smooth* if it has a parametrization $\omega(t)$, $0 \le t \le 2\pi$, such that $\omega'(t)$ is Dini continuous and $\neq 0$. Let $\mathcal{D}$ be any simply connected domain in $\mathbb{C}$ with locally connected boundary. Let ϕ be a conformal mapping from $\mathcal{D}$ onto the unit disk D of $\mathbb{R}^2$. We say that $\partial\mathcal{D}$ has a *Dini-smooth corner* of opening angle π/α $(1/2 < \alpha < \infty)$ at $a = \phi^{-1}(e^{i\varphi})$ if there are closed arcs $A^+, A^- \subset C(0,1)$ ending at φ and lying on opposite sides of φ that are mapped by ϕ^{-1} onto Dini-smooth Jordan domain arcs C^+ and C^-, forming an angle of π/α at $a = \phi^{-1}(e^{i\varphi})$. That means

$$\arg(\phi^{-1}(e^{it}) - a) \to \begin{cases} \beta & \text{as } t \to \varphi^-, \\ \beta + \pi/\alpha & \text{as } t \to \varphi^+. \end{cases}$$

By $G_{\mathcal{D}}$, we denote the Green function for the Laplacian in a domain $\mathcal{D} \subset \mathbb{R}^n$ $(n \ge 2)$. For two positive functions f and g on a set $\mathcal{D}$, we say that f is *comparable* to g on $\mathcal{D}$, and we write $f \simeq g$, if there exists $C \ge 1$ such that for all $x \in \mathcal{D}$, $(1/C)g(x) \le f(x) \le Cg(x)$. We write $\delta(x) = d(x, \partial\mathcal{D})$ for the distance from $x \in \mathcal{D}$ to the Euclidean boundary $\partial\mathcal{D}$ of $\mathcal{D}$. For $a, b \in \mathbb{R}$, we denote $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

Bogdan [2] and Hansen [6] proved, for a bounded Lipschitz domain $\mathcal{D}$ in $\mathbb{R}^n$ $(n \ge 3)$, that if we fix $x_0 \in \mathcal{D}$ and let $g(x) = 1 \wedge G_{\mathcal{D}}(x, x_0)$, then

$$G_{\mathcal{D}}(x,y) \simeq \frac{g(x)g(y)}{g(b)^2} \frac{1}{|x-y|^{n-2}}, \quad \forall x, y \in \mathcal{D} \text{ and } b \in \mathcal{B}_0(x,y), \tag{1.1}$$

where $\mathcal{B}_0(x,y)$ is, roughly speaking, the set of points b in $\mathcal{D}$ that lie between

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x and y and satisfy $\delta(b) \simeq \max(\delta(x), \delta(y), |x-y|)$ (see [2, p. 328] for the precise definition). Such estimates play important roles; e.g. the following 3G inequality can be deduced from (1.1): there exists a constant $C > 0$ such that, for all $x, y, z \in \mathcal{D}$,

$$\frac{G_{\mathcal{D}}(x,y)G_{\mathcal{D}}(y,z)}{G_{\mathcal{D}}(x,z)} \le C\left(\frac{1}{|x-y|^{n-2}} + \frac{1}{|y-z|^{n-2}}\right). \tag{1.2}$$

Before the estimate (1.1), the 3G inequality was proved by Cranston, Fabes and Zhao [5] to study the conditional gauge theory for the Schrödinger operator. In [1], Aikawa and Lundh extended (1.2) to bounded uniformly John domains, and gave a counterexample to (1.2). In [4, Theorem 6.15, p. 175] and [4, Theorem 6.23, p. 182], Chung and Zhao established inequalities for the Green function on bounded C^2 domains $\mathcal{D}$ in $\mathbb{R}^2$. More precisely, they proved that

$$G_{\mathcal{D}}(x,y) \simeq \ln\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right), \qquad \forall x, y \in \mathcal{D},$$

and that there exists a constant $C > 0$ such that, for all $x, y, z \in \mathcal{D}$,

$$\frac{G_{\mathcal{D}}(x,y)G_{\mathcal{D}}(y,z)}{G_{\mathcal{D}}(x,z)} \le C\left(\left(\ln\left(\frac{1}{|x-y|}\right) \vee 1\right) + \left(\ln\left(\frac{1}{|y-z|}\right) \vee 1\right)\right).$$

In [13], M. Selmi generalized the results of [4]. He established inequalities for the Green function of a class of Dini-smooth Jordan domains in $\mathbb{R}^2$. He proved that if $\mathcal{D}$ is a bounded multiply connected Dini-smooth Jordan domain in $\mathbb{R}^2$, then

$$G_{\mathcal{D}}(x,y) \simeq \ln\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right), \qquad \forall x, y \in \mathcal{D},$$

and that there exists $C > 0$ depending only on $\mathcal{D}$ such that, for all $x, y, z \in \mathcal{D}$,

$$\frac{G_{\mathcal{D}}(x,y)G_{\mathcal{D}}(y,z)}{G_{\mathcal{D}}(x,z)} \le C\left(\frac{\delta(y)}{\delta(x)}G_{\mathcal{D}}(x,y) + \frac{\delta(y)}{\delta(z)}G_{\mathcal{D}}(y,z)\right) \tag{1.3}$$

$$\le C\left(\left(\ln\left(\frac{1}{|x-y|}\right) \vee 1\right) + \left(\ln\left(\frac{1}{|y-z|}\right) \vee 1\right)\right). \tag{1.4}$$

The aim of the present paper is to give estimates for the Green function in the case of a more general class of bounded domains in $\mathbb{R}^2$, called piecewise Dini-smooth Jordan domains. Our work generalizes all estimates given by K. Chung and Z. Zhao [4] and M. Selmi [13].

Main Theorem 1. *Let $\mathcal{D}$ be a bounded multiply connected piecewise Dini-smooth Jordan domain with $\partial\mathcal{D} = \bigcup_{k=0}^{m} \mathit{\Gamma}_k$, where $\mathit{\Gamma}_k$ $(0 \le k \le m)$ are disjoint closed piecewise Dini-smooth Jordan curves such that $\mathcal{D} \subset \operatorname{int} \mathit{\Gamma}_0$ and for all $0 \le k \le m$, $\mathit{\Gamma}_k$ has n_k Dini-smooth corners of opening angles*

$\pi/\alpha_1^k, \ldots, \pi/\alpha_{n_k}^k$ *at* $a_1^k, \ldots a_{n_k}^k$ ($\alpha_i^k \in]1/2, \infty[\setminus \{1\}$), *for all* $1 \leq i \leq n_k$ ($0 \leq k \leq m$). *Then, for all* $x, y \in \mathcal{D}$,

$$G_{\mathcal{D}}(x,y) \simeq \ln\left(1 + \prod_{\substack{1\leq i\leq n_k\\ 0\leq k\leq m}} \left(\frac{(|x-a_i^k| \wedge |y-a_i^k|)}{(|x-a_i^k| \vee |y-a_i^k|)}\right)^{\alpha_i^k-1} \frac{\delta(x)\delta(y)}{|x-y|^2}\right).$$

By using (1.3), we prove the 3G Theorem in the case of piecewise Dini-smooth Jordan domains.

Main Theorem 2. *Let* $\mathcal{D}$ *be as in Main Theorem* 1. *Then there exists a constant* $C > 0$ *such that, for all* $x, y, z \in \mathcal{D}$,

$$\frac{G_{\mathcal{D}}(x,z)G_{\mathcal{D}}(z,y)}{G_{\mathcal{D}}(x,y)} \leq C\left(\prod_{\substack{1\leq i\leq n_k\\ 0\leq k\leq m}} \left|\frac{z-a_i^k}{x-a_i^k}\right|^{\alpha_i^k-1} \frac{\delta(z)}{\delta(x)} G_{\mathcal{D}}(x,z) + \prod_{\substack{1\leq i\leq n_k\\ 0\leq k\leq m}} \left|\frac{z-a_i^k}{y-a_i^k}\right|^{\alpha_i^k-1} \frac{\delta(z)}{\delta(y)} G_{\mathcal{D}}(z,y)\right).$$

In addition, we exploit the preceding results to prove comparison theorems for Green kernels associated with Δ and $\Delta - \mu$, where μ is a nonnegative and exact Radon measure on a piecewise Dini-smooth Jordan domain.

Main Theorem 3. *Let* $\mathcal{D}$ *be as in Main Theorem* 1 *and* μ *be a nonnegative exact Radon measure on* $\mathcal{D}$ *which does not charge the polar sets of* $\mathcal{D}$. *Then the following two conditions are equivalent:*

(1) $G_{\mathcal{D}}$ *and* ${}^{\mu}G_{\mathcal{D}}$ *are comparable.*

(2) $x \mapsto \int_{\mathcal{D}} \prod_{\substack{1\leq i\leq n_k\\ 0\leq k\leq m}} \left|\frac{y-a_i^k}{x-a_i^k}\right|^{\alpha_i^k-1} \frac{\delta(y)}{\delta(x)} G_{\mathcal{D}}(x,y)\, d\mu(y)$ *is bounded on* $\mathcal{D}$.

In order to establish our results, without loss of generality we suppose throughout this paper that Ω is a bounded simply connected piecewise Dini-smooth Jordan domain in $\mathbb{R}^2$ having n Dini-smooth corners at $a_1, \ldots, a_n$ of opening angles respectively $\pi/\alpha_1, \ldots, \pi/\alpha_n$, $\alpha_i \in]1/2, \infty] \setminus \{1\}$. For all $0 \leq i \leq n$, we denote by $[a_i, a_{i+1}]$ the arc beginning at a_i and ending at a_{i+1}, with the convention $a_0 = a_n$ and $a_{n+1} = a_1$. The distance from $z \in \Omega$ to $[a_i, a_{i+1}]$ will be denoted $\delta_i(z)$. Our principal idea is to use conformal mappings. Note that, by the Riemann Theorem, there exists a conformal mapping from Ω onto the unit disk D. Consequently, for all $x, y \in \Omega$,

$$G_{\Omega}(x,y) = G_D(\phi(x), \phi(y)) \simeq \ln\left(1 + \frac{(1-|\phi(x)|^2)(1-|\phi(y)|^2)}{|\phi(x)-\phi(y)|^2}\right).$$

Thus, to give estimates for the Green function, it is sufficient to give estimates for $|\phi(x) - \phi(y)|$ and $1 - |\phi(x)|^2$ on Ω. The problem reduces to understanding the distortion introduced by ϕ, which is measured by $|\phi'|$. In Section 2, we prove Theorem 1.1 below, which gives estimates for $|\phi(x)-\phi(y)|$ on Ω.

THEOREM 1.1. *Let ϕ be a conformal mapping from Ω onto D. Then, for all $x, y \in \Omega$,*

$$\left|\frac{\phi(x)-\phi(y)}{x-y}\right|^2 \simeq |\phi'(x)|\,|\phi'(y)| \prod_{i=1}^{n}\left(\frac{|x-a_i|\vee|y-a_i|}{|x-a_i|\wedge|y-a_i|}\right)^{\alpha_i-1} \simeq \prod_{i=1}^{n}(|x-a_i|\vee|y-a_i|)^{2(\alpha_i-1)}. \tag{1.5}$$

In Section 3, we prove Lemma 1.1, which gives estimates for $1-|\phi(x)|^2$ on Ω.

LEMMA 1.1. *Let ϕ be a conformal mapping from Ω onto D. Then, for all $x \in \Omega$,*

$$\delta(\phi(x)) \simeq 1-|\phi(x)|^2 \simeq |\phi'(x)|\delta(x) \simeq \prod_{i=1}^{n}|x-a_i|^{\alpha_i-1}\delta(x).$$

By using Theorem 1.1, Proposition 3.1 and Lemma 1.1, we prove estimates for the Green function on Ω.

THEOREM 1.2. *For all $x, y \in \Omega$, we have*

$$G_\Omega(x,y) \simeq \ln\left(1+\prod_{k=1}^{n}\left(\frac{|x-a_k|\wedge|y-a_k|}{|x-a_k|\vee|y-a_k|}\right)^{\alpha_k-1}\frac{\delta(x)\delta(y)}{|x-y|^2}\right), \tag{1.6}$$

$$\simeq \ln\left(1+\frac{1}{|x-y|^2}\prod_{k=1}^{n}\left(\frac{|x-a_k|\wedge|y-a_k|}{|x-a_k|\vee|y-a_k|}\right)^{\alpha_k-2}\frac{\delta_k(x)\delta_k(y)}{(|x-a_k|\vee|y-a_k|)^2}\right). \tag{1.7}$$

In Section 4, by using again Proposition 3.1 and Lemma 1.1, we deduce the generalized version of the 3G Theorem on Ω. At the end of that section, we derive comparison theorems for Green kernels associated with Δ and $\Delta-\mu$, where μ is a nonnegative and exact Radon measure. In Section 5, we establish further generalizations by considering the case of a bounded multiply connected piecewise Dini-smooth Jordan domain. Finally, in Section 6, we study some interesting examples on which we verify the exactness of our main theorem.

The following notations will be adopted. For $z \in \mathbb{C}$, we denote $r=|z|$ and $\theta=\arg z$. For $\alpha>1/2$ and $0<\varepsilon<1$, we denote

$$\begin{aligned}
D_\alpha &= \{z \in \mathbb{C} : 0 < r < 1,\ 0 < \theta < \pi/\alpha\},\\
S_\alpha(D) &= \{z \in \mathbb{C} : 0 < r < 1,\ |\theta| < \pi/(2\alpha)\},\\
\Omega_{\alpha,\varepsilon} &= \{z \in \mathbb{C} : \varepsilon < r < 1,\ |\theta| < \pi/\alpha\},\\
\Delta_{\varepsilon,\alpha} &= \{z \in \mathbb{C} : \varepsilon < r < 1,\ \theta = \pi/\alpha\},\\
\Delta_{\varepsilon,-\alpha} &= \{z \in \mathbb{C} : \varepsilon < r < 1,\ \theta = -\pi/\alpha\},\\
C_{\varepsilon,\alpha} &= \{z \in \mathbb{C} : r = \varepsilon,\ |\theta| \le \pi/\alpha\}.
\end{aligned}$$

The letter C will denote a generic positive constant whose value is unimportant and may change from line to line.

2. Proof of Theorem 1.1

LEMMA 2.1. *Let* $\alpha>1$ *and* $\beta\in[(1+\alpha)/2,\alpha]$. *Then, for all* $\theta\in[0,\pi/(2\beta)]$,

$$\sin\alpha\theta \simeq \sin\theta.$$

Proof. It suffices to see that

$$\varphi_\alpha(\theta) = \begin{cases} \dfrac{\sin\alpha\theta}{\sin\theta} & \text{if } \theta \in\,]0, \frac{\pi}{2\beta}],\\ \alpha & \text{if } \theta = 0,\end{cases}$$

is continuous and does not vanish on $[0,\pi/(2\beta)]$. ∎

In the same way, we deduce the following lemma:

LEMMA 2.2. *Let* $\alpha \in\,]1/2,1[$ *and* $\beta \in [(1+2\alpha)/4,\alpha]$. *Then, for all* $\theta \in [0,\pi/(2\beta)]$,

$$\sin\alpha\theta \simeq \sin\theta.$$

LEMMA 2.3. *Let* $\alpha > 1$ *and* $\beta \in\,](1+\alpha)/2,\alpha[$. *Then, for all* $z \in S_{\beta/2}(D)$,

$$|1-z^\alpha| \simeq |1-z|.$$

Proof. Let $\varepsilon < 1/4$. The function

$$f_\alpha(z) = \begin{cases} \dfrac{1-z^\alpha}{1-z} & \text{if } z \neq 1,\\ \alpha & \text{if } z = 1,\end{cases}$$

is holomorphic on $\Omega_{\beta,\varepsilon}$, continuous and does not vanish on $\overline{\Omega_{\beta,\varepsilon}}$. Thus, by the maximum and minimum principles, it is sufficient to control f_α on $\partial\Omega_{\beta,\varepsilon}$. We have to discuss three cases.

CASE 1. If $z \in C_{1,\beta}$, then

$$\left|\frac{1-z^\alpha}{1-z}\right|^2 = \left(\frac{\sin\frac{\alpha\theta}{2}}{\sin\frac{\theta}{2}}\right)^2.$$

The result follows from Lemma 2.1.

Case 2. If $z \in C_{\varepsilon,\beta}$, then

$$\frac{3}{8} \leq \left|\frac{1-z^\alpha}{1-z}\right| \leq \frac{8}{3}.$$

Case 3. If $z \in \Delta_{\varepsilon,\beta} \cup \Delta_{\varepsilon,-\beta}$, then

$$\left|\frac{1-z^\alpha}{1-z}\right|^2 = \frac{1+r^{2\alpha}-2r^\alpha\cos\frac{\alpha\pi}{\beta}}{1+r^2-2r\cos\frac{\pi}{\beta}}.$$

It follows that

$$\frac{\sin^2\frac{\alpha\pi}{\beta}}{4} \leq \left|\frac{1-z^\alpha}{1-z}\right|^2 \leq \frac{4}{\sin^2\frac{\pi}{\beta}}.$$

Since the constants are independent of ε, we obtain, for all $z \in S_{\beta/2}(D)$,

$$|1-z^\alpha| \simeq |1-z|. \blacksquare$$

Lemma 2.4. *Let* $\alpha \in]1/2, 1[$ *and* $\beta \in](1+2\alpha)/4, \alpha[$. *Then, for all* $z \in D_\beta$,

$$|1-z^\alpha| \simeq |1-z|.$$

Proof. The proof is similar to that of Lemma 2.3. Let $\varepsilon < 1/4$. The function

$$g_\alpha(z) = \begin{cases} \dfrac{z^\alpha - e^{-i\frac{\alpha\pi}{2\beta}}}{z-e^{-i\frac{\pi}{2\beta}}} & \text{if } z \neq e^{-i\frac{\pi}{2\beta}}, \\ \alpha e^{-i\frac{\pi(\alpha-1)}{2\beta}} & \text{if } z = e^{-i\frac{\pi}{2\beta}}, \end{cases}$$

is holomorphic on $\Omega_{2\beta,\varepsilon}$, continuous and does not vanish on $\overline{\Omega_{2\beta,\varepsilon}}$. By the maximum and minimum principles, it is sufficient to control g_α on $\partial\Omega_{2\beta,\varepsilon}$. We have to discuss four cases.

Case 1. If $z \in C_{1,2\beta}$, then

$$|g_\alpha(z)|^2 = \frac{1-\cos(\alpha(\theta+\frac{\pi}{2\beta}))}{1-\cos(\theta+\frac{\pi}{2\beta})} = \left(\frac{\sin\alpha\nu}{\sin\nu}\right)^2,$$

where $\nu = \frac{\theta+\frac{\pi}{2\beta}}{2} = \frac{\frac{\pi}{2\beta}+\theta}{2} \in [0, \pi/(2\beta)]$. The result follows from Lemma 2.2.

Case 2. If $z \in C_{\varepsilon,2\beta}$, then

$$\frac{1}{4} < \frac{1-\varepsilon^\alpha}{1+\varepsilon} \leq |g_\alpha(z)| \leq \frac{1+\varepsilon^\alpha}{1-\varepsilon} < \frac{8}{3}.$$

Case 3. If $z \in \Delta_{\varepsilon,2\beta}$, then

$$\frac{\sin^2\frac{\alpha\pi}{\beta}}{4} \leq |g_\alpha(z)|^2 = \frac{r^{2\alpha}+1-2r^\alpha\cos\frac{\alpha\pi}{\beta}}{r^2+1-2r\cos\frac{\pi}{\beta}} \leq \frac{4}{\sin^2\frac{\pi}{\beta}}.$$

CASE 4. If $z \in \Delta_{\varepsilon,-2\beta}$, then

$$\frac{1}{2} \le |g_\alpha(z)| = \frac{1-r^\alpha}{1-r} = \frac{\int_r^1 \alpha t^{\alpha-1}\,dt}{\int_r^1 dt} \le 1+\alpha.$$

As the constants are independent of ε, we obtain, for all $z \in S_\beta(D)$,

$$\left|z^\alpha - e^{-i\frac{\alpha\pi}{2\beta}}\right| \simeq \left|z - e^{-i\frac{\pi}{2\beta}}\right|.$$

Now let $z \in D_\beta$; then $e^{-i\frac{\pi}{2\beta}} z \in S_\beta(D)$. By using the previous relation, we get, for all $z \in D_\beta$,

$$|z^\alpha - 1| \simeq |z-1|. \ ■$$

By using the conjugate expression in the last lemma, we obtain

LEMMA 2.5. *Let $\alpha \in]1/2,1[$ and $\beta \in](1+2\alpha)/4,\alpha[$. Then, for all $z \in D \setminus D_{\beta/(2\beta-1)}$,*

$$|1-z^\alpha| \simeq |1-z|.$$

LEMMA 2.6. *Let $\alpha > 1$ and $\beta \in](1+\alpha)/2,\alpha[$. Then, for all $x, y \in S_\beta$,*

$$|x^\alpha - y^\alpha| \simeq |x-y|(|x| \vee |y|)^{\alpha-1}.$$

Proof. Assume that $|x| > |y|$ and put $z = y/x$. Then $|z| < 1$ and $|\arg z| \le \pi/\beta$. Since

$$\left|\frac{x^\alpha - y^\alpha}{x-y}\right| = |x^{\alpha-1}|\left|\frac{1-z^\alpha}{1-z}\right|,$$

the result follows by Lemma 2.3. ■

LEMMA 2.7. *Let $\alpha \in]1/2,1[$ and $\beta \in](1+2\alpha)/4,\alpha[$. Then, for all $x, y \in S_\beta$,*

$$|x^\alpha - y^\alpha| \simeq |x-y|(|x| \vee |y|)^{\alpha-1}.$$

Proof. Assume that $|x| > |y|$ and put $z = y/x$. Then $|z| < 1$ and $\arg z \in [0, \pi/\beta]$, or $\arg z \in \left[\frac{\pi}{\beta/(2\beta-1)}, 2\pi\right]$. Since

$$\left|\frac{x^\alpha - y^\alpha}{x-y}\right| = |x^{\alpha-1}|\left|\frac{1-z^\alpha}{1-z}\right|,$$

the result follows by Lemmas 2.4 and 2.5. ■

LEMMA 2.8. *If $\partial\Omega$ has a Dini-smooth corner of opening angle π/α, $1/2 < \alpha < \infty$, at a and ϕ is a conformal mapping from Ω onto D, then there exists $r > 0$ such that for all $x, y \in \overline{\Omega} \cap D(a,r)$,*

$$|\phi(x) - \phi(y)| \simeq |(x-a)^\alpha - (y-a)^\alpha|.$$

Proof. Using [10, Theorem 3.9], we get

$$\lim_{z\to a} \frac{\phi(z)-\phi(a)}{(z-a)^\alpha} = b \quad (b \in \mathbb{C}^*) \quad \text{and} \quad \lim_{z\to a} \frac{\phi'(z)}{(z-a)^{\alpha-1}} = d \quad (d \in \mathbb{C}^*).$$

Then

$$\lim_{x\to a}\left(\lim_{y\to x}\frac{\phi(x)-\phi(y)}{(x-a)^{\alpha}-(y-a)^{\alpha}}\right)=\lim_{x\to a}\frac{\phi'(x)}{\alpha(x-a)^{\alpha-1}}=d.$$

This implies the existence of $r>0$ and $c>1$ such that, for all $x,y\in\overline{\Omega}\cap D(a,r)$,

$$\frac{1}{C}\le\left|\frac{\phi(x)-\phi(y)}{(x-a)^{\alpha}-(y-a)^{\alpha}}\right|\le C.\ \blacksquare$$

REMARK 2.1. Let ϕ be a conformal mapping from Ω onto D. Then, for all $z\in\Omega$,

$$|\phi'(z)|\simeq\prod_{i=1}^{n}|(z-a_i)|^{\alpha_i-1}.$$

This follows from [10, Theorem 3.9].

Proof of Theorem 1.1. It is clear that (1.5) follows immediately from Remark 2.1. We have to discuss four cases.

Let $r>0$ be sufficiently small so that the conclusion of Lemma 2.8 is true and put, for all $1\le i\le n$,

$$r_i=\tfrac{1}{2}\min_{\substack{i\neq j\\ i-1\neq j}}|a_i-a_j|.$$

CASE 1. If $x,y\in\Omega\cap D(a_k,r_k)$, then Lemma 2.8 implies that

$$\left|\frac{\phi(x)-\phi(y)}{x-y}\right|^2\simeq\left|\frac{(x-a_k)^{\alpha_k}-(y-a_k)^{\alpha_k}}{x-y}\right|^2.$$

Let $S(a_k)$ be the sector of vertex a_k and opening angle π/β_k that contains $\Omega\cap D(a_k,r_k)$, where $\beta_k\in](\alpha_k+1)/2,\alpha_k[$ if $\alpha_k>1$ and $\beta_k\in](1+2\alpha_k)/4,\alpha_k[$ if $\alpha_k\in]1/2,1[$. Then, by Lemmas 2.6 and 2.7, we deduce that, for all $x,y\in\Omega\cap D(a_k,r_k)$,

$$\left|\frac{(x-a_k)^{\alpha_k}-(y-a_k)^{\alpha_k}}{x-y}\right|^2\simeq(|x-a_k|\vee|y-a_k|)^{2\alpha_k-2}.$$

Moreover,

$$r\le|x-a_i|\vee|y-a_i|\le\delta,\quad\forall i\neq k.$$

This implies that, for all $x,y\in\Omega\cap D(a_k,r_k)$,

$$\prod_{i=1}^{n}(|x-a_i|\vee|y-a_i|)^{2\alpha_i-2}\simeq(|x-a_k|\vee|y-a_k|)^{2\alpha_k-2}.$$

CASE 2. If $x\in\Omega\cap D(a_k,r_k)$, $y\in\Omega\cap D(a_l,r_l)$, $1\le k\neq l\le n$, then the function

$$(x,y)\mapsto\frac{\phi(x)-\phi(y)}{x-y}$$

is continuous and does not vanish on $\overline{\Omega \cap D(a_k, r) \times \Omega \cap D(a_l, r)}$. Hence, for all $(x, y) \in \Omega \cap D(a_k, r_k) \times \Omega \cap D(a_l, r_l)$,

$$|\phi(x) - \phi(y)| \simeq |x - y|.$$

As $r \le |y - a_k| \le \delta$ and $r \le |x - a_l| \le \delta$, we have

$$|x - a_l| \vee |y - a_l| \simeq |x - a_k| \vee |y - a_k| \simeq 1.$$

Moreover,

$$r \le |x - a_i| \vee |y - a_i| \le \delta, \quad \forall i \notin \{k, l\}.$$

This implies that

$$\prod_{i=1}^{n} (|x - a_i| \vee |y - a_i|)^{2\alpha_i - 2} \simeq 1.$$

Case 3. If $x \in \Omega \cap D(a_k, r_k/2)$, $y \in \Omega_r = \{z \in \Omega : |z - a_i| > r, 1 \le i \le n\}$, then $|x - y| \ge |y - a_k| - |x - a_k| \ge r/2$, and so the function $(x, y) \mapsto \frac{\phi(x) - \phi(y)}{x - y}$ is continuous and does not vanish on $\overline{\Omega \cap D(a_k, r/2)} \times \overline{\Omega_r}$. Hence,

$$\left| \frac{\phi(x) - \phi(y)}{x - y} \right| \simeq 1.$$

Moreover,

$$r \le |y - a_i| \le \delta, \quad 1 \le i \le n.$$

Consequently,

$$\prod_{i=1}^{n} (|x - a_i| \vee |y - a_i|)^{2\alpha_i - 2} \simeq 1.$$

Case 4. If $x, y \in \Omega_r$, we consider a Dini-smooth Jordan domain Ω' without corners such that $\Omega_r \subset \Omega' \subset \Omega$. By [10, Theorem 3.5], the functions ϕ, ϕ' can be extended to $\overline{\Omega'}$. As a result, there exists a constant $C > 1$ such that

$$\frac{1}{C} \le \left| \frac{\phi(x) - \phi(y)}{x - y} \right| \le C \quad (x, y \in \overline{\Omega'}) \quad \text{and} \quad \frac{1}{C} \le |\phi'(z)| \le C \quad (z \in \overline{\Omega'}).$$

In particular, for all $x, y, z \in \Omega_r$, we obtain the result. ∎

3. Estimates for the Green function of a bounded simply connected piecewise Dini-smooth Jordan domain. Set $S_\alpha = \{z \in \mathbb{C} : |\theta| < \frac{\pi}{2\alpha}\}$ and

$$\Delta_1 = \left\{ z \in \mathbb{C} : r \ge 0,\ \theta = \frac{\pi}{2\alpha} \right\}, \qquad \Delta_0 = \left\{ z \in \mathbb{C} : r \ge 0,\ \theta = -\frac{\pi}{2\alpha} \right\}.$$

For $z \in S_\alpha$, we denote by $d_0(z) = d(z, \Delta_0)$ (respectively $d_1(z) = d(z, \Delta_1)$) the distance from z to Δ_0 (respectively from z to Δ_1).

Lemma 3.1. *Let $\alpha \in]1/2, \infty] \setminus \{1\}$. Then, for all $z \in S_\alpha$,*
$$|z| \cos \alpha \arg z \simeq \delta(z) \quad \textit{and} \quad d_0(z) \vee d_1(z) \simeq |z|.$$

Proof. First, let us remark that
$$d_1(z) = \begin{cases} r & \text{if } \frac{\pi}{2\alpha} - \theta > \frac{\pi}{2}, \\ r \sin\left(\dfrac{\pi}{2\alpha} - \theta\right) & \text{if } \frac{\pi}{2\alpha} - \theta < \frac{\pi}{2}, \end{cases}$$
$$d_0(z) = \begin{cases} r & \text{if } \frac{\pi}{2\alpha} + \theta > \frac{\pi}{2}, \\ r \sin\left(\dfrac{\pi}{2\alpha} + \theta\right) & \text{if } \frac{\pi}{2\alpha} + \theta < \frac{\pi}{2}. \end{cases}$$
We have to discuss two cases:

Case 1. If $\alpha > 1$, put
$$S_1 = \left\{z \in \mathbb{C} : 0 \le \theta \le \frac{\pi}{2\alpha}\right\}, \qquad S_2 = \left\{z \in \mathbb{C} : -\frac{\pi}{2\alpha} \le \theta \le 0\right\}.$$
If $z \in S_1$, then $\delta(z) = d_1(z) = r \sin(\frac{\pi}{2\alpha} - \theta)$. Moreover, since $\frac{\pi}{2\alpha} - \theta \in [0, \frac{\pi}{2\alpha}]$, it follows, by Lemma 2.1, that
$$r \cos \alpha\theta = r \sin\left(\alpha\left(\frac{\pi}{2\alpha} - \theta\right)\right) \simeq r \sin\left(\frac{\pi}{2\alpha} - \theta\right) = \delta(z).$$
On the other hand, since $\theta \in [0, \frac{\pi}{2\alpha}]$, we have $\sin \frac{\pi}{2\alpha} \le \sin(\theta + \frac{\pi}{2\alpha}) \le \sin \frac{\pi}{\alpha}$. This gives $d_0(z) \simeq r$.

If $z \in S_2$, then $\delta(z) = d_0(z) = r \sin(\theta + \frac{\pi}{2\alpha})$. Moreover, since $\theta + \frac{\pi}{2\alpha} \in [0, \frac{\pi}{2\alpha}]$, Lemma 2.1 yields
$$r \cos \alpha\theta = r \sin\left(\alpha\left(\theta + \frac{\pi}{2\alpha}\right)\right) \simeq r \sin\left(\theta + \frac{\pi}{2\alpha}\right) = \delta(z).$$
On the other hand, since $\theta \in [-\frac{\pi}{2\alpha}, 0]$, we have $\sin \frac{\pi}{2\alpha} \le \sin(\frac{\pi}{2\alpha} - \theta) \le \sin \frac{\pi}{\alpha}$. This gives $d_1(z) \simeq r$.

Case 2. If $1/2 < \alpha < 1$, put
$$S_3 = \left\{z \in \mathbb{C} : \frac{\pi}{2\alpha} - \frac{\pi}{2} \le \theta \le \frac{\pi}{2\alpha}\right\},$$
$$S_4 = \left\{z \in \mathbb{C} : -\frac{\pi}{2\alpha} \le \theta \le -\frac{\pi}{2\alpha} + \frac{\pi}{2}\right\},$$
$$S_5 = \left\{z \in \mathbb{C} : -\frac{\pi}{2\alpha} + \frac{\pi}{2} \le \theta \le \frac{\pi}{2\alpha} - \frac{\pi}{2}\right\}.$$
If $z \in S_3$, then $d_0(z) = r$ and $\delta(z) = d_1(z) = r \sin(\frac{\pi}{2\alpha} - \theta)$. Moreover, since $\frac{\pi}{2\alpha} - \theta \in [0, \frac{\pi}{2}]$ Lemma 2.1 implies that
$$r \cos \alpha\theta = r \sin\left(\alpha\left(\frac{\pi}{2\alpha} - \theta\right)\right) \simeq r \sin\left(\frac{\pi}{2\alpha} - \theta\right) = \delta(z).$$

If $z \in S_4$, then $d_1(z) = r$ and $\delta(z) = d_0(z) = r\sin(\theta + \frac{\pi}{2\alpha})$. Moreover, since $\theta + \frac{\pi}{2\alpha} \in [0, \frac{\pi}{2}]$, Lemma 2.1 yields

$$r\cos\alpha\theta = r\sin\Big(\alpha\Big(\theta + \frac{\pi}{2\alpha}\Big)\Big) \simeq r\sin\Big(\theta + \frac{\pi}{2\alpha}\Big) = \delta(z).$$

If $z \in S_5$, then $\delta(z) = d_0(z) = d_1(z) = r$. Moreover, we have

$$\cos\Big((1-\alpha)\frac{\pi}{2}\Big) \le \cos\alpha\theta \le 1, \quad \forall z \in S_5.$$

Hence,

$$r\cos\alpha\theta \simeq r \simeq \delta(z) \simeq d_0(z) \simeq d_1(z). \ \blacksquare$$

From the last lemma, we deduce

LEMMA 3.2.

$$d_0(z) \vee d_1(z) \simeq |z| \quad (z \in \mathbb{C}).$$

PROPOSITION 3.1. *For all $z \in \Omega$, we have*

$$\delta_i(z) \vee \delta_{i-1}(z) \simeq |z - a_i| \quad (1 \le i \le n,\ n \ge 3). \tag{3.1}$$

$$\delta_1(z) \vee \delta_2(z) \simeq |z - a_1| \wedge |z - a_2| \simeq |z - a_1|\,|z - a_2| \quad (n = 2). \tag{3.2}$$

$$\delta(z) \simeq \prod_{i=1}^{n} \frac{\delta_i(z)}{|z - a_i|} \quad (n \ge 2). \tag{3.3}$$

Proof. We choose r_i sufficiently small such that

$$r_i \le \tfrac{1}{2} \min_{\substack{i \ne j \\ j \ne i-1}} \delta_j(a_i),$$

and set $r = \min_{1 \le i \le n} r_i$. Then, for all $z \in \Omega \cap D(a_i, r)$, we have

$$\delta_i(z) \vee \delta_{i-1}(z) \le r_i \le \min_{\substack{j \ne i \\ j \ne i-1}} \delta_j(z).$$

For (3.1), it is sufficient to study the situation on $\Omega \cap D(a_i, r)$. Let $\beta > 0$, so that the sector $S_\beta(a_i, \Delta_i, \Delta_{i-1})$ of opening angle π/β at the vertex a_i and boundaries Δ_i, Δ_{i-1} is included in $\Omega \cap D(a_i, r)$. It is clear that

$$\delta_i(z) \vee \delta_{i-1}(z) \le |z - a_i|, \quad \forall z \in \Omega \cap D(a_i, r). \tag{3.4}$$

Conversely, we remark that

$$\delta_i(z) \vee \delta_{i-1}(z) \ge d(z, \Delta_i) \vee d(z, \Delta_{i-1}), \quad \forall z \in \Omega \cap D(a_i, r).$$

This implies, by using Lemma 3.2 and (3.4), that

$$\delta_i(z) \vee \delta_{i-1}(z) \simeq |z - a_i|, \quad \forall z \in \Omega \cap D(a_i, r). \tag{3.5}$$

Now, since $\delta_i(z) + \delta_{i-1}(z)$ and $|z - a_i|$ do not vanish on $\overline{\Omega \setminus D(a_i, r)}$, by compactness and continuity arguments, the result follows.

We now prove (3.2). If Ω is a bounded simply connected piecewise Dini-smooth Jordan domain having two Dini-smooth corners of vertices a_1, a_2, then $\delta_1(z)+\delta_2(z)$ vanishes at a_1 and a_2. Let $r=|a_1-a_2|/2$. Then it follows from (3.5) that for all $z\in\Omega\cap(D(a_1,r)\cup D(a_2,r))$,

$$\delta_1(z)\vee\delta_2(z)\simeq|z-a_1|\wedge|z-a_2|.$$

On the other hand, the function $z\mapsto|z-a_1|+|z-a_2|$ is continuous and does not vanish on Ω. Hence, for all $z\in\Omega\cap(D(a_1,r)\cup D(a_2,r))$,

$$\delta_1(z)\vee\delta_2(z)\simeq|z-a_1|\wedge|z-a_2|\simeq|z-a_1|\,|z-a_2|.$$

Now, for $z\in\overline{\Omega\setminus(D(a_1,r)\cup D(a_2,r))}$, the result is obtained by compactness and continuity arguments.

Finally, we prove (3.3). If $n=2$, then

$$\delta(z)=\delta_1(z)\wedge\delta_2(z)=\frac{\delta_1(z)\delta_2(z)}{\delta_1(z)\vee\delta_2(z)}$$

and the result follows from (3.2).

If $n\geq 3$, put $\Omega_r=\{z\in\Omega:|z-a_i|>r,\ 1\leq i\leq n\}$ and assume that $\delta(z)=\delta_n(z)=\min_{1\leq i\leq n}\delta_i(z)$. The function $z\mapsto\prod_{i=1}^{n-1}\delta_i(z)$ is continuous and does not vanish on Ω_r. Moreover, for all $z\in\Omega_r$,

$$r^n\leq\prod_{i=1}^{n}|z-a_i|\leq\delta^n.$$

It follows that, for all $z\in\Omega_r$,

$$\delta(z)\simeq\prod_{i=1}^{n}\frac{\delta_i(z)}{|z-a_i|}.$$

Now, for all $z\in\Omega\setminus\Omega_r$, it is sufficient the study $\Omega\cap D(a_i,r)$. It is clear that $\delta(z)=\delta_i(z)\wedge\delta_{i-1}(z)$. Moreover, we can see that if $j\neq i$ and $j\neq i-1$, then

$$r\leq\delta_j(z)\leq|z-a_j|\leq\delta\quad\text{and}\quad r\leq|z-a_j|\leq\delta.$$

Thus, the result follows from (3.5). ■

Proof of Lemma 1.1. The function $\psi=\phi^{-1}$ is a conformal mapping from D onto Ω. By [10, Corollary 1.4], for all $y\in D$ we have

$$d(\psi(y),\partial\Omega)\simeq(1-|y|^2)|\psi'(y)|.$$

If we replace y by $\phi(z)$, we obtain, for all $z\in\Omega$,

$$d(z,\partial\Omega)\simeq(1-|\phi(z)|^2)|\psi'(\phi(z))|.$$

Thus, for all $z\in\Omega$,

$$\delta(z)\simeq(1-|\phi(z)|^2)\frac{1}{|\phi'(z)|},$$

and so, for all $z\in\Omega$,

$$|\phi'(z)|\delta(z)\simeq 1-|\phi(z)|^2.$$

On the other hand, for all $z \in \Omega$,

$$d(\phi(z), \partial D) = 1 - |\phi(z)| \le 1 - |\phi(z)|^2 \le 2(1 - |\phi(z)|). \ ■$$

REMARK 3.1. Let us remark that if Ω is a Dini-smooth Jordan domain then, by [10, Theorem 3.5], the function ϕ' is bounded. So, we obtain the classical result: $1 - |\phi(z)|^2 \simeq \delta(z)$ (see [13]).

Proof of Theorem 1.2. Let ϕ be a conformal mapping from Ω onto D. Then

$$G_\Omega(x,y) = \frac{1}{2}\ln\biggl(1 + \frac{(1-|\phi(x)|^2)(1-|\phi(y)|^2)}{|\phi(x)-\phi(y)|^2}\biggr), \qquad \forall x, y \in \Omega.$$

Thus (1.6) follows from Lemma 1.1 and Theorem 1.1. Moreover, by using (3.3) and (1.6), we deduce (1.7). ■

4. Comparison theorem for Green kernels and 3G inequalities

THEOREM 4.1 (3G Theorem). *Let ϕ be a conformal mapping from Ω onto D. Then there exists a constant $C > 0$ such that, for all $x, y, z \in \Omega$,*

$$\begin{aligned}\frac{G_\Omega(x,z)G_\Omega(z,y)}{G_\Omega(x,y)} &\le C\biggl(\frac{\delta(z)}{\delta(x)}\frac{|\phi'(z)|}{|\phi'(x)|}G_\Omega(x,z) + \frac{\delta(z)}{\delta(y)}\frac{|\phi'(z)|}{|\phi'(y)|}G_\Omega(z,y)\biggr)\\ &\le C\biggl(\frac{\delta(z)}{\delta(x)}\prod_{k=1}^{n}\biggl|\frac{z-a_k}{x-a_k}\biggr|^{\alpha_k-1}G_\Omega(x,z) + \frac{\delta(z)}{\delta(y)}\prod_{k=1}^{n}\biggl|\frac{z-a_k}{y-a_k}\biggr|^{\alpha_k-1}G_\Omega(z,y)\biggr)\\ &\le C\biggl(\prod_{k=1}^{n}\frac{\delta_k(z)}{\delta_k(x)}\biggl|\frac{z-a_k}{x-a_k}\biggr|^{\alpha_k-2}G_\Omega(x,z) + \prod_{k=1}^{n}\frac{\delta_k(z)}{\delta_k(y)}\biggl|\frac{z-a_k}{y-a_k}\biggr|^{\alpha_k-2}G_\Omega(z,y)\biggr).\end{aligned}$$

Proof. Since ϕ is a conformal mapping from Ω onto D, we have

$$G_\Omega(x,y) = G_D(\phi(x), \phi(y)), \qquad \forall x, y \in \Omega.$$

On the other hand, the 3G Theorem on D (see [12]) states that, for all $x', y', z' \in D$,

$$\frac{G_D(x',z')G_D(z',y')}{G_D(x',y')} \le C\biggl(\frac{1-|z'|^2}{1-|x'|^2}G_D(x',z') + \frac{1-|z'|^2}{1-|y'|^2}G_D(y',z')\biggr).$$

Thus, the result follows by Lemma 1.1, Remark 2.1 and (3.3). ■

For a nonnegative Radon measure μ which does not charge the polar sets of Ω (see [7]–[9]), we denote by ${}^\mu G_\Omega$ the Green function associated with $\Delta - \mu$ on Ω. We have, for all $x \in \Omega$,

$$\begin{aligned}\int_\Omega G_\Omega(x,y)f(y)\,dy = &\int_\Omega {}^\mu G_\Omega f(y)\,dy\\ &+ \int_\Omega G_\Omega(x,y)\Bigl(\int_\Omega {}^\mu G_\Omega(z,y)f(y)\,dy\Bigr)\,d\mu(z),\end{aligned}$$

for any nonnegative measurable function f on Ω. The function ${}^{\mu}G_{\Omega}$ is called the *perturbation* of G_{Ω} by μ. It is clear that ${}^{\mu}G_{\Omega} \leq G_{\Omega}$. We recall the following theorem (see [9]):

THEOREM 4.2. *G_{Ω} and ${}^{\mu}G_{\Omega}$ are comparable on Ω if, and only if, there exists a constant $k > 0$ such that, for all x, y in Ω,*

$$\int_{\Omega} G_{\Omega}(x,z) G_{\Omega}(z,y)\, d\mu(z) \leq k G_{\Omega}(x,y).$$

We denote by K the set of nonnegative exact Radon measures μ such that $G_{\Omega} \simeq {}^{\mu}G_{\Omega}$ on Ω. It is a convex cone in the space of Radon measures on Ω (see [8]).

THEOREM 4.3. *Let μ be a nonnegative exact Radon measure on Ω which does not charge the polar sets of Ω, and ϕ a conformal mapping from Ω onto D. Then the following three conditions are equivalent:*

(1) *G_{Ω} and ${}^{\mu}G_{\Omega}$ are comparable.*

(2) $x \mapsto \int_{\Omega} \frac{|\phi'(y)|}{|\phi'(x)|} G_{\Omega}(x,y)\, d\mu(y)$ *is bounded on Ω.*

(3) $x \mapsto \int_{\Omega} \prod_{i=1}^{n} \left| \frac{y - a_i}{x - a_i} \right|^{\alpha_i - 1} \frac{\delta(y)}{\delta(x)} G_{\Omega}(x,y)\, d\mu(y)$ *is bounded on Ω.*

Proof. (1)⇒(2). From [8], [9] and [11], if G_{Ω} and ${}^{\mu}G_{\Omega}$ are comparable, then there exists a constant $C > 0$ such that, for all superharmonic and nonnegative functions s on Ω, we have

$$\int_{\Omega} \frac{s(y)}{s(x)} G_{\Omega}(x,y) d\mu(y) \leq C, \qquad \forall x \in \Omega.$$

In particular, $s(z) = 1 - |\phi(z)|^2$ is superharmonic on Ω. This implies the result since, by Proposition 3.1, $1 - |\phi(z)|^2 \simeq |\phi'(z)|\delta(z)$.

(2)⇒(3). We use the fact that $|\phi'(z)|\delta(z) \simeq \prod_{k=1}^{n} |z - a_k|^{\alpha_k - 1} \delta(z)$.

(3)⇒(1). Using the fact that K is a cone, the 3G Theorem and the estimates

$$1 - |\phi(z)|^2 \simeq |\phi'(z)|\delta(z) \simeq \prod_{k=1}^{n} |z - a_k|^{\alpha_k - 1} \delta(z),$$

we obtain the result. ■

REMARK 4.1. If G and ${}^{\mu}G$ are comparable, then $p(z) = \int_{\Omega} G(x,z)\, d\mu(x)$ is bounded but the converse is not true (see [13, Remark 5]).

5. Generalization

Proof of Main Theorem 1. By the Riemann Theorem, $\operatorname{int} \Gamma_0$ can be mapped conformally onto the unit disk D by a conformal mapping ϕ. Using this mapping, $\mathcal{D}$ is transformed into a new domain where the images of $\Gamma_1, \ldots, \Gamma_m$ and $C = \{z \in \mathbb{C} : |z| = 1\}$ constitute its boundary, such that $\phi(\mathcal{D}) \subset D$. From [10], this mapping has a continuous extension derivative from $\overline{\operatorname{int} \Gamma_0}$ onto its image. Moreover, as Γ_0 has n_0 Dini-smooth corners at $a_1^0, a_2^0, \ldots, a_{n_0}^0$ of opening angles π/α_i^0 ($\alpha_i^0 \in]1/2, \infty[\setminus 1$), for all $z \in \mathcal{D}$ we have

$$|\phi'(z)| \simeq \Big| \prod_{i=1}^{n_0} (z - a_i^0)^{\alpha_i^0 - 1} \Big|. \tag{5.1}$$

In addition, for all $x, y \in \mathcal{D}$,

$$\begin{aligned} \left| \frac{\phi(x) - \phi(y)}{x - y} \right|^2 &\simeq |\phi'(x)|\,|\phi'(y)| \prod_{i=1}^{n_0} \left(\frac{(|x - a_i^0| \vee |y - a_i^0|)}{(|x - a_i^0| \wedge |y - a_i^0|)} \right)^{\alpha_i^0 - 1} \\ &\simeq \prod_{i=1}^{n_0} (|x - a_i^0| \vee |y - a_i^0|)^{2(\alpha_i^0 - 1)}. \end{aligned} \tag{5.2}$$

Let $z_0 \in \operatorname{int} \phi(\Gamma_1)$ and $\rho > 0$ be such that $\operatorname{int} \phi(\Gamma_1) \subset B(z_0, \rho)$, $B(z_0, \rho) \cap \operatorname{int} \phi(\Gamma_i) = \emptyset$ for all $i \in \{2, \ldots, n\}$ and $\partial B(z_0, \rho) \cap C = \emptyset$. Let $\Psi(z) = z_0 + \rho^2 \frac{1}{z - z_0}$ for $z \in \phi(\mathcal{D})$. Then $\phi_0 = \Psi \circ \phi$ is a conformal mapping from $\mathcal{D}$ onto $\phi_0(\mathcal{D})$, with $\phi_0(\mathcal{D}) \subset \operatorname{int} \phi_0(\Gamma_1)$. Moreover, the set $\phi_0(\Gamma_0)$ is a closed analytic curve. By using (5.1) and (5.2), we deduce that, for all $z \in \mathcal{D}$,

$$|\phi_o'(z)| \simeq \Big| \prod_{i=1}^{n_0} (z - a_i^0)^{\alpha_i^0 - 1} \Big| \tag{5.3}$$

and

$$\begin{aligned} \left| \frac{\phi_0(x) - \phi_0(y)}{x - y} \right|^2 &\simeq |\phi_0'(x)|\,|\phi_0'(y)| \prod_{i=1}^{n_0} \left(\frac{(|x - a_i^0| \vee |y - a_i^0|)}{(|x - a_i^0| \wedge |y - a_i^0|)} \right)^{\alpha_i^0 - 1} \\ &\simeq \prod_{i=1}^{n_0} (|x - a_i^0| \vee |y - a_i^0|)^{2(\alpha_i^0 - 1)} \quad (x, y \in \mathcal{D}). \end{aligned} \tag{5.4}$$

Now, consider $\operatorname{int} \phi_0(\Gamma_1)$. In the same way, there exists a conformal mapping ϕ_1 such that $\phi_1(\operatorname{int} \phi_0(\Gamma_1)) \subset \operatorname{int} \phi_1 \circ \phi_0(\Gamma_2)$ and $\phi_1 \circ \phi_0(\Gamma_0)$, $\phi_1 \circ \phi_0(\Gamma_1)$ are closed analytic curves. By [10], since $\operatorname{int} \phi_0(\Gamma_1)$ has n_1 Dini-smooth corners of opening angles $\pi/\alpha_1^1, \ldots, \pi/\alpha_{n_1}^1$ at $\phi_0(a_1^1), \ldots, \phi_0(a_{n_1}^1)$ respectively, for all $z \in \mathcal{D}$ we have

$$|\phi_1'(\phi_0(z))| \simeq \prod_{i=1}^{n_1} |\phi_0(z) - \phi_0(a_i^1)|^{\alpha_i^1 - 1}.$$

This implies by using (5.4) that for all $z \in \mathcal{D}$,

$$|(\phi_1 \circ \phi_0)'(z)| \simeq \prod_{\substack{1 \le i \le n_k \\ 0 \le k \le 1}} |(z - a_i^k)|^{\alpha_i^k - 1}.$$

In addition, for all $x, y \in \mathcal{D}$,

$$\left| \frac{\phi_1 \circ \phi_0(x) - \phi_1 \circ \phi_0(y)}{x - y} \right|$$
$$\simeq |(\phi_1 \circ \phi_0)'(x)| \, |(\phi_1 \circ \phi_0)'(y)| \prod_{\substack{1 \le i \le n_k \\ 0 \le k \le 1}} \left(\frac{(|x - a_i^k| \vee |y - a_i^k|)}{(|x - a_i^k| \wedge |y - a_i^k|)} \right)^{\alpha_i^k - 1}.$$

The process can be repeated until we end up with a domain $\widetilde{\mathcal{D}}$ which is bounded by analytic curves that are conformally equivalent to $\mathcal{D}$ and $\Phi = \phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_0$ is a conformal mapping from $\mathcal{D}$ onto $\widetilde{\mathcal{D}}$ satisfying, for all $z \in \mathcal{D}$,

$$|\Phi'(z)| \simeq \prod_{\substack{1 \le i \le n_k \\ 0 \le k \le m}} |(z - a_i^k)|^{\alpha_i^k - 1} \tag{5.5}$$

and

$$\left| \frac{\Phi(x) - \Phi(y)}{x - y} \right| \tag{5.6}$$
$$\simeq |\Phi'(x)| \, |\Phi'(y)| \prod_{\substack{1 \le i \le n_k \\ 0 \le k \le m}} \left(\frac{(|x - a_i^k| \vee |y - a_i^k|)}{(|x - a_i^k| \wedge |y - a_i^k|)} \right)^{\alpha_i^k - 1} \quad (\forall x, y \in \mathcal{D}).$$

So, the Green functions $G_{\mathcal{D}}$ and $G_{\tilde{\mathcal{D}}}$ are related, for all $x, y \in \mathcal{D}$, by

$$G_{\mathcal{D}}(x, y) = G_{\tilde{\mathcal{D}}}(\Phi(x), \Phi(y)) \simeq \ln\left(1 + \frac{\delta(\Phi(x))\delta(\Phi(y))}{|\Phi(x) - \Phi(y)|^2} \right)$$
$$\simeq \ln\left(1 + \frac{|\Phi'(x)|\delta(x)|\Phi'(y)|\delta(y)}{|\Phi(x) - \Phi(y)|^2} \right).$$

By using (5.5) and (5.6), we obtain, for all $x, y \in \mathcal{D}$,

$$G_{\mathcal{D}}(x, y) \simeq \ln\left(1 + \prod_{\substack{1 \le i \le n_k \\ 0 \le k \le m}} \left(\frac{(|x - a_i^k| \wedge |y - a_i^k|)}{(|x - a_i^k| \vee |y - a_i^k|)} \right)^{\alpha_i^k - 1} \frac{\delta(x)\delta(y)}{|x - y|^2} \right). \quad \blacksquare$$

Proof of Main Theorem 2. Let Φ be the conformal mapping from $\mathcal{D}$ onto $\tilde{\mathcal{D}}$, defined in the last proof. By (1.3), there exists $C > 0$ such that, for all $x, y, z \in \tilde{\mathcal{D}}$,

$$\frac{G_{\tilde{\mathcal{D}}}(x, z) G_{\tilde{\mathcal{D}}}(z, y)}{G_{\tilde{\mathcal{D}}}(x, y)} \le C \left(\frac{\delta(z)}{\delta(x)} G_{\tilde{\mathcal{D}}}(x, z) + \frac{\delta(z)}{\delta(y)} G_{\tilde{\mathcal{D}}}(z, y) \right).$$

Hence, for all $x, y, z \in \tilde{\mathcal{D}}$,

$$\begin{aligned}\frac{G_{\mathcal{D}}(x,z)G_{\mathcal{D}}(z,y)}{G_{\mathcal{D}}(x,y)} &\le C\bigg(\frac{\delta(\Phi(z))}{\delta(\Phi(x))}G_{\mathcal{D}}(x,z) + \frac{\delta(\Phi(z))}{\delta(\Phi(y))}G_{\mathcal{D}}(z,y)\bigg)\\ &\le C\bigg(\prod_{\substack{1\le i\le n_k\\ 0\le k\le m}}\left|\frac{z-a_i^k}{x-a_i^k}\right|^{\alpha_i^k-1}\frac{\delta(z)}{\delta(x)}\,G_{\mathcal{D}}(x,z)\\ &\qquad + \prod_{\substack{1\le i\le n_k\\ 0\le k\le m}}\left|\frac{z-a_i^k}{y-a_i^k}\right|^{\alpha_i^k-1}\frac{\delta(z)}{\delta(y)}G_{\mathcal{D}}(z,y)\bigg). \quad \blacksquare\end{aligned}$$

Similarly to the proof of Theorem 4.3, we can deduce Main Theorem 3.

6. Examples and applications

6.1. Estimates for the Green function on S_α

THEOREM 6.1. *Let $\alpha \in]1/2, \infty[\setminus \{1\}$. Then, for all $x, y \in S_\alpha$,*

$$\begin{aligned}G_{S_\alpha}(x,y) &\simeq \ln\bigg(1 + \bigg(\frac{|x|\wedge|y|}{|x|\vee|y|}\bigg)^{\alpha-2}\frac{d_1(x)d_1(y)d_2(x)d_2(y)}{|x-y|^2(|x|\vee|y|)^2}\bigg)\\ &\simeq \ln\bigg(1 + \bigg(\frac{|x|\wedge|y|}{|x|\vee|y|}\bigg)^{\alpha-1}\frac{\delta(x)\delta(y)}{|x-y|^2}\bigg).\end{aligned}$$

We present two proofs:

First proof. If $\alpha = 1$, then for all $x, y \in S_1$,

$$G_{S_1}(x,y) = \ln\left|\frac{x+\overline{y}}{x-y}\right| = \frac{1}{2}\ln\bigg(1 + \frac{4\Re e(x)\Re e(y)}{|x-y|^2}\bigg).$$

As, for $\alpha \ne 1$, the function $z \mapsto z^\alpha$ is a conformal mapping from S_α onto S_1, we have

$$G_{S_\alpha}(x,y) = \frac{1}{2}\ln\bigg(1 + \frac{|x|^\alpha\cos(\alpha\arg x)|y|^\alpha\cos(\alpha\arg y)}{|x^\alpha - y^\alpha|}\bigg), \quad \forall x, y \in S_\alpha.$$

Thus, the result follows from Lemmas 2.6, 2.7 and 3.1. $\blacksquare$

Second proof. For $\alpha \ne 1$ and the conformal mapping $\phi_\alpha(z) = \frac{z^\alpha - 1}{z^\alpha + 1}$ from S_α onto D, 0 is the only singular point of order $\alpha - 1$ and $\phi_\alpha(z) \simeq \phi_\alpha(0) + 2z^\alpha$ in a neighborhood of 0. Moreover, for $x, y \in S_\alpha$,

$$\begin{aligned}G_{S_\alpha}(x,y) &= G_D(\phi_\alpha(x), \phi_\alpha(y))\\ &\simeq \ln\bigg(1 + \frac{(1-|\phi_\alpha(x)|^2)(1-|\phi_\alpha(y)|^2)}{|\phi_\alpha(x)-\phi_\alpha(y)|^2}\bigg).\end{aligned}$$

We observe that

$$\begin{aligned}|\phi_\alpha(x)-\phi_\alpha(y)|^2 &= \frac{4|x^\alpha-y^\alpha|^2}{|(x^\alpha+1)(y^\alpha+1)|^2}\\ &\simeq 4|x-y|^2\frac{(|x|\vee|y|)^{2\alpha-2}}{|x^\alpha+1|^2|y^\alpha+1|^2}\end{aligned}$$

and for all $z\in S_\alpha$,

$$1-|\phi_\alpha(z)|^2=\frac{4|z|^{\alpha-1}|z|\cos(\alpha\arg z)}{|z^\alpha+1|^2}\simeq\delta(z)|\phi'_\alpha(z)|.$$

Consequently,

$$|\phi_\alpha(x)-\phi_\alpha(y)|^2\simeq 4|x-y|^2|\phi'_\alpha(x)|\,|\phi'_\alpha(y)|\frac{(|x|\vee|y|)^{2\alpha-2}}{|x|^{\alpha-1}|y|^{\alpha-1}}.$$

This implies that for all $x,y\in S_\alpha$,

$$G_{S_\alpha}(x,y)\simeq\ln\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\left(\frac{|x|\wedge|y|}{|x|\vee|y|}\right)^{\alpha-1}\right).\ \blacksquare$$

6.2. Estimates for the Green function on $S_\alpha(D)$

PROPOSITION 6.1. *Let $\alpha>1/2$. Then, for all $x,y\in S_\alpha(D)$,*

$$|1+x^\alpha y^\alpha|\simeq(|x-e^{i\frac{\pi}{2\alpha}}|\vee|y-e^{i\frac{\pi}{2\alpha}}|)(|x-e^{-i\frac{\pi}{2\alpha}}|\vee|y-e^{-i\frac{\pi}{2\alpha}}|).$$

Proof. We equip $\mathbb{C}\times\mathbb{C}$ with the norm $|(x,y)|=\sqrt{|x|^2+|y|^2}$, $(x,y)\in\mathbb{C}\times\mathbb{C}$. We have

$$|x|\vee|y|\le|(x,y)|\le\sqrt{2}(|x|\vee|y|).$$

Let $\alpha>1/2$. The function

$$f_\alpha:(\mathbb{C}\setminus\,]-\infty,0])^2\to\mathbb{C},\qquad(x,y)\mapsto x^\alpha y^\alpha,$$

is continuous, differentiable at the two points $(e^{i\frac{\pi}{2\alpha}},e^{i\frac{\pi}{2\alpha}}),(e^{-i\frac{\pi}{2\alpha}},e^{-i\frac{\pi}{2\alpha}})$ and

$$\begin{aligned}&\lim_{(x,y)\to(e^{i\frac{\pi}{2\alpha}},e^{i\frac{\pi}{2\alpha}})}\frac{|f_\alpha(x,y)-f_\alpha(e^{i\frac{\pi}{2\alpha}},e^{i\frac{\pi}{2\alpha}})|}{|(x,y)-(e^{i\frac{\pi}{2\alpha}},e^{i\frac{\pi}{2\alpha}})|}\\ &\qquad=\lim_{(x,y)\to(e^{-i\frac{\pi}{2\alpha}},e^{-i\frac{\pi}{2\alpha}})}\frac{|f_\alpha(x,y)-f_\alpha(e^{-i\frac{\pi}{2\alpha}},e^{-i\frac{\pi}{2\alpha}})|}{|(x,y)-(e^{-i\frac{\pi}{2\alpha}},e^{-i\frac{\pi}{2\alpha}})|}=\alpha\sqrt{2}.\end{aligned}$$

It follows that

$$\begin{aligned}&\lim_{(x,y)\to(e^{i\frac{\pi}{2\alpha}},e^{i\frac{\pi}{2\alpha}})}\frac{|1+x^\alpha y^\alpha|}{|(x,y)-(e^{i\frac{\pi}{2\alpha}},e^{i\frac{\pi}{2\alpha}})|}\\ &\qquad=\lim_{(x,y)\to(e^{-i\frac{\pi}{2\alpha}},e^{-i\frac{\pi}{2\alpha}})}\frac{|1+x^\alpha y^\alpha|}{|(x,y)-(e^{-i\frac{\pi}{2\alpha}},e^{-i\frac{\pi}{2\alpha}})|}=\alpha\sqrt{2}.\end{aligned}$$

Hence, the function

$$g_\alpha(x,y)=\frac{|1+x^\alpha y^\alpha|}{|(x,y)-(e^{i\frac{\pi}{2\alpha}},e^{i\frac{\pi}{2\alpha}})|\,|(x,y)-(e^{-i\frac{\pi}{2\alpha}},e^{-i\frac{\pi}{2\alpha}})|}$$

defined on $\overline{S_\alpha(D)\times S_\alpha(D)}$ is continuous, with

$$g_\alpha(e^{i\frac{\pi}{2\alpha}},e^{i\frac{\pi}{2\alpha}})=g_\alpha(e^{-i\frac{\pi}{2\alpha}},e^{-i\frac{\pi}{2\alpha}})=\frac{\alpha}{2\sin\frac{\pi}{2\alpha}}\neq 0.$$

Moreover, the numerator and the denominator of g_α do not vanish on the set $\overline{S_\alpha(D)\times S_\alpha(D)}\setminus(e^{i\frac{\pi}{2\alpha}},e^{i\frac{\pi}{2\alpha}}),(e^{-i\frac{\pi}{2\alpha}},e^{-i\frac{\pi}{2\alpha}})$. In fact, $1+x^\alpha y^\alpha=0$ implies $x^\alpha y^\alpha=-1$, so $|x|=|y|=1$ and $\alpha(\arg x+\arg y)=\pm\pi$. Consequently,

$$\arg x=\arg y=\frac{\pi}{2\alpha}\quad\text{or}\quad\arg x=\arg y=-\frac{\pi}{2\alpha}.$$

For the denominator the argument is clear. Finally, g_α is continuous on the compact subset $\overline{S_\alpha(D)\times S_\alpha(D)}$ and does not vanish. Consequently, $g_\alpha\simeq 1$ on $\overline{S_\alpha(D)\times S_\alpha(D)}$. ∎

THEOREM 6.2. *Let $\alpha\in\,]1/2,\infty[\,\setminus\{1\}$. Then, for all $x,y\in S_\alpha(D)$,*

$$G_{S_\alpha(D)}\simeq$$

$$\ln\Bigg(1+\bigg(\frac{|x-e^{i\frac{\pi}{2\alpha}}|\wedge|y-e^{i\frac{\pi}{2\alpha}}|}{|x-e^{i\frac{\pi}{2\alpha}}|\vee|y-e^{i\frac{\pi}{2\alpha}}|}\bigg)\bigg(\frac{|x-e^{-i\frac{\pi}{2\alpha}}|\wedge|y-e^{-i\frac{\pi}{2\alpha}}|}{|x-e^{-i\frac{\pi}{2\alpha}}|\vee|y-e^{-i\frac{\pi}{2\alpha}}|}\bigg)\bigg(\frac{|x|\wedge|y|}{(|x|\vee|y|}\bigg)^{\alpha-1}\frac{\delta(x)\delta(y)}{|x-y|^2}\Bigg).$$

First proof. Note that $S_\alpha(D)$ is a simply connected piecewise Dini-smooth Jordan domain having three Dini-smooth corners at $a_1=0$, $a_2=e^{-i\pi/(2\alpha)}$, $a_3=e^{i\pi/(2\alpha)}$ of respective opening angles π/α, $\pi/2$, $\pi/2$. The Green function G_D of D is given, for all $x,y\in D$, by

$$G_D(x,y)=\ln\left|\frac{1-x\overline{y}}{x-y}\right|=\frac{1}{2}\ln\bigg(1+\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}\bigg).$$

Hence, the Green function of $S_1(D)$ is

$$\begin{aligned}G_{S_1(D)}(x,y)&=\ln\left|\frac{1-x\overline{y}}{x-y}\right|-\ln\left|\frac{1+xy}{x+\overline{y}}\right|=\ln\left|\frac{(1-x\overline{y})(x+\overline{y})}{(x-y)(1+xy)}\right|\\&=\frac{1}{2}\ln\bigg(1+\frac{4x_1y_1(1-|x|^2)(1-|y|^2)}{|(x-y)|^2|(1+xy)|^2}\bigg).\end{aligned}$$

Since $z\mapsto z^\alpha$ is a conformal mapping from $S_\alpha(D)$ onto $S_1(D)$, we have

$$\begin{aligned}G_{S_\alpha(D)}(x,y)&=G_{S_1(D)}(x^\alpha,y^\alpha)\\&=\frac{1}{2}\ln\bigg(1+\frac{4\Re e(x^\alpha)\cdot\Re e(y^\alpha)(1-|x|^{2\alpha})(1-|y|^{2\alpha})}{|x^\alpha-y^\alpha|^2|1+x^\alpha y^\alpha|^2}\bigg)\end{aligned}$$

for all $x,y\in S_\alpha(D)$. On the other hand, if $t\in[0,1]$, then $1\le\frac{1-t^{2\alpha}}{1-t}\le 2\alpha$. It follows that

$$1-|z|^{2\alpha}\simeq 1-|z|=\delta_2(z),\qquad\forall z\in S_\alpha(D).$$

By Lemma 3.1, for all $z \in S_\alpha(D)$,

$$\Re e(z^\alpha) = |z|^\alpha \cos(\alpha \arg z) \simeq |z|^{\alpha-2}\delta_1(z)\delta_3(z).$$

Therefore, the result follows from (3.3), Lemma 2.6, Proposition 6.1 and Lemma 2.7. ∎

Second proof. The function

$$z \mapsto \phi_1(z) = \frac{z^2+2z-1}{z^2-2z-1} = 1 + \frac{4z}{z^2-2z-1} \qquad (z \in S_1(D))$$

is a conformal mapping from $S_1(D)$ onto D. Consequently, for all $x, y \in S_1(D)$,

$$G_{S_1(D)} = G_D(\phi_1(x), \phi_1(y)) = \frac{1}{2}\ln\left(1 + \frac{(1-|\phi_1(x)|^2)(1-|\phi_1(y)|^2)}{|\phi_1(x)-\phi_1(y)|^2}\right).$$

As $\phi_1'(z) = 4\frac{z^2+1}{(z^2-2z-1)^2}$ has two simple singularities i and $-i$, it follows that, for all $x, y \in S_1(D)$,

$$\begin{aligned}
|\phi_1(x)-\phi_1(y)|^2 &= \frac{|(y-x)(1+xy)|^2}{|(x^2-2x-1)(y^2-2y-1)|^2} \\
&\simeq \frac{|x-y|^2(|x-i|\vee|y-i|)^2(|x+i|\vee|y+i|)^2}{|x^2-2x-1|^2|y^2-2y-1|^2} \\
&\simeq \frac{|x-y|^2(|x-i|\vee|y-i|)^2(|x+i|\vee|y+i|)^2|\phi_1'(x)|\,|\phi_1'(x)|}{|x^2+1|\,|y^2+1|} \\
&\simeq |x-y|^2(|x-i|\vee|y-i|)^2(|x+i|\vee|y+i|)^2.
\end{aligned}$$

We also have

$$1-|\phi_1(z)|^2 \simeq \frac{8|z|\cos(\arg z)(1-|z|^2)}{|z^2-2z-1|^2} \simeq |z|\cos(\arg z)(1-|z|^2), \quad \forall z \in S_1(D).$$

By using (3.3), we get

$$\begin{aligned}
1-|\phi_1(z)|^2 &\simeq \delta(z)|z-i|\,|z+i| \\
&\simeq |\phi_1'(z)|\delta(z), \qquad \forall z \in S_1(D).
\end{aligned}$$

Thus,

$$G_{S_1(D)} \simeq \ln\left(1 + \left(\frac{|x-i|\wedge|y-i|}{|x-i|\vee|y-i|}\right)\left(\frac{|x+i|\wedge|y+i|}{|x+i|\vee|y+i|}\right)\frac{\delta(x)\delta(y)}{|x-y|^2}\right)$$

for all $x, y \in S_1(D)$.

Now, if $\alpha \in]1/2, \infty[\backslash\{1\}$, the function $z \mapsto \phi_2(z) = \phi_1(z^\alpha)$ $(z \in S_\alpha(D))$ is a conformal mapping from $S_\alpha(D)$ onto D. Then, for all $x, y \in S_\alpha(D)$,

$$G_{S_\alpha(D)} = G_D(\phi_2(x), \phi_2(y)) = \frac{1}{2}\ln\left(1 + \frac{(1-|\phi_2(x)|^2)(1-|\phi_2(y)|^2)}{|\phi_2(x)-\phi_2(y)|^2}\right).$$

Let us remark that the function $\phi_2'(z) = \alpha z^{\alpha-1}\phi_1'(z^\alpha)$ has three singularities: 0 of order $\alpha - 1$ and $e^{i\frac{\pi}{2\alpha}}, e^{-i\frac{\pi}{2\alpha}}$ which are simple singularities. Moreover,

$$|x^\alpha - i| = \left|x^\alpha - \left(e^{i\frac{\pi}{2\alpha}}\right)^\alpha\right| \simeq \left|x - e^{i\frac{\pi}{2\alpha}}\right|(|x| \vee 1)^{\alpha-1} = \left|x - e^{i\frac{\pi}{2\alpha}}\right|.$$

Consequently, for all $x, y \in S_\alpha(D)$,

$$\begin{aligned}
&|\phi_2(x) - \phi_2(y)|^2 \\
&\simeq |x-y|^2 \frac{(|x| \vee |y|)^{2\alpha-2}(|x-e^{i\frac{\pi}{2\alpha}}| \vee |y-e^{i\frac{\pi}{2\alpha}}|)^2(|x-e^{-i\frac{\pi}{2\alpha}}| \vee |y-e^{-i\frac{\pi}{2\alpha}}|)^2|\phi_2'(x)||\phi_2'(y)|}{(|x||y|)^{\alpha-1}|x^{2\alpha}+1||y^{2\alpha}+1|} \\
&\simeq |x-y|^2(|x| \vee |y|)^{2\alpha-2}(|x-e^{i\frac{\pi}{2\alpha}}| \vee |y-e^{i\frac{\pi}{2\alpha}}|)^2(|x-e^{-i\frac{\pi}{2\alpha}}| \vee |y-e^{-i\frac{\pi}{2\alpha}}|)^2.
\end{aligned}$$

Moreover, for all $z \in S_\alpha(D)$,

$$\begin{aligned}
1 - |\phi_2(z)|^2 = 1 - |\phi_1(z^\alpha)|^2 &\simeq \delta(z^\alpha)|z^\alpha - i|\,|z^\alpha + i| \\
&\simeq |z|^\alpha \cos(\alpha \arg z)\left|z - e^{i\frac{\pi}{2\alpha}}\right|\left|z - e^{-i\frac{\pi}{2\alpha}}\right| \\
&\simeq |z|^{\alpha-1}\left|z - e^{i\frac{\pi}{2\alpha}}\right|\left|z - e^{-i\frac{\pi}{2\alpha}}\right| \simeq |\phi_2'(z)|\delta(z).
\end{aligned}$$

This implies the result. ∎

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Mohamed Amine Ben Boubaker
Institut Preparatoire aux Études d'Ingénieur de Nabeul
Université de Carthage
Campus Universitaire Merezka
8000, Nabeul, Tunisia
E-mail: MohamedAmine.BenBoubaker@ipein.rnu.tn

Mohamed Selmi
Faculty of Sciences of Tunis
Department of Mathematics
University of Tunis el Manar
Campus Universitaire
2092, Tunis, Tunisia
E-mail: Mohamed.Selmi@fst.rnu.tn

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