

EXPLICIT UPPER BOUNDS FOR $|L(1, \chi)|$ WHEN $\chi(3) = 0$

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Abstract. Let χ be a primitive Dirichlet character of conductor q and denote by $L(z, \chi)$ the associated L -series. We provide an explicit upper bound for $|L(1, \chi)|$ when 3 divides q .

1. Introduction and results. Let χ be a primitive Dirichlet character of conductor q and denote by $L(z, \chi)$ the associated L -series. Recall we describe χ as *even* when $\chi(-1) = 1$ and *odd* when $\chi(-1) = -1$. The upper bound for $|L(1, \chi)|$ has received considerable attention, especially in the past 25 years (see, for instance, [5], [14], [11], [6] and [4]). In [8] and [9], Louboutin used integral representations of Dirichlet L -functions to obtain the following upper bound for $|L(1, \chi)|$ when 3 divides q :

$$|L(1, \chi)| \leq \frac{1}{3} \log q + \begin{cases} 0.3816 & \text{for even } \chi, \\ 0.8436 & \text{for odd } \chi. \end{cases}$$

In 2001, Ramaré [13] described a different method (see Proposition 2.1 below). In this paper, we exploit Ramaré's method to improve on Louboutin's bound. Specifically, we show:

THEOREM 1.1. *Let χ be a primitive character of conductor q such that $3 \mid q$. Then*

$$|L(1, \chi)| \leq \frac{1}{3} \log q + \begin{cases} 0.368296 & \text{for even } \chi, \\ 0.838374 & \text{for odd } \chi. \end{cases}$$

In Section 4 we establish this result for conductor $q > 2 \cdot 10^6$. In Section 5 we describe a rigorous computation to confirm that the result also holds for conductor $q \in [3, 2 \cdot 10^6]$.

2. Lemmas from elsewhere. The proof of the following results can be found in [13]. We use $\tau(\chi)$ to denote the Gauss sum

$$\tau(\chi) := \sum_{a \bmod q} \chi(a) e(a/q)$$

2010 Mathematics Subject Classification: Primary 11M06; Secondary 11Y35.

Key words and phrases: Dirichlet characters, Dirichlet L -function, Gauss sums.

where, as usual

$$e(x) := \exp(2\pi ix).$$

The modulus of the Gauss sum is well known when χ is primitive and given by $|\tau(\chi)| = \sqrt{q}$.

PROPOSITION 2.1. *Set*

$$F_3(t) = \left(\frac{\sin(\pi t)}{\pi} \right)^2 \left(\frac{2}{t} + \sum_{m \in \mathbb{Z}} \frac{\operatorname{sgn}(m)}{(t-m)^2} \right), \quad F_4(t) = 1 - \left(\frac{\sin(\pi t)}{\pi t} \right)^2$$

and

$$j(t) = 2 \int_{|t|}^1 (\pi(1-u) \cot(\pi u) + 1) du.$$

Let χ be a primitive Dirichlet character of conductor q . Then

$$L(1, \chi)$$

$$= \begin{cases} \sum_{n \geq 1} \frac{(1 - F_3(\delta n))\chi(n)}{n} - \frac{\tau(\chi)}{q} \sum_{1 \leq m \leq \delta q} \bar{\chi}(m) j\left(\frac{m}{\delta q}\right) & \text{for even } \chi, \\ \sum_{n \geq 1} \frac{(1 - F_4(\delta n))\chi(n)}{n} + \frac{i\pi\tau(\chi)}{q} \sum_{1 \leq m \leq \delta q} \bar{\chi}(m)(1 - m\delta q)^2 & \text{for odd } \chi. \end{cases}$$

LEMMA 2.2. *Set*

$$j(t) = 2 \int_{|t|}^1 (\pi(1-u) \cot(\pi u) + 1) du.$$

Then

$$(2.1) \quad \int_0^1 j(t) dt = 1$$

and

$$(2.2) \quad -2 \log |t| - 2(\log(2\pi) - 1) \leq j(t) \leq -2 \log |t|.$$

LEMMA 2.3. *For $\delta \in]0, 1]$, we have*

$$\sum_{n \geq 1} \frac{1 - F_3(\delta n)}{n} = -\log \delta - 1 + \delta,$$

where F_3 is defined in Proposition 2.1.

LEMMA 2.4. *For $\delta > 0$, we have*

$$\sum_{n \geq 1} \frac{1 - F_4(\delta n)}{n} = -\log \delta + \frac{3}{2} - \log(2\pi) + 2 \int_0^1 (1-t) \log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| dt,$$

where F_4 is defined in Proposition 2.1.

LEMMA 2.5. Let k and $\beta > 0$ be two real numbers. Let f be a continuous, convex and non-increasing \mathbb{L}^1 -function on $]k - \theta, k + \beta]$. Then

$$f(k) \leq \frac{1}{\theta} \int_{k-\theta}^k f(t) dt - \theta \frac{f(k) - f(k + \beta)}{2\beta}.$$

LEMMA 2.6. For $0 < \delta \leq 1/2$, we have

$$2 \int_0^1 (1-t) \log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| dt \leq \frac{\pi^3 \delta^2}{12}.$$

3. Some auxiliary lemmas

LEMMA 3.1. Let $\alpha > 0$ and let a be 1 or 2. Let g be a continuous, convex, non-negative and non-increasing \mathbb{L}^1 -function on $]0, 1]$. Then

$$\begin{aligned} \sum_{\substack{a \leq n \leq \alpha \\ n=3k+a}} g\left(\frac{n}{\alpha}\right) &\leq \frac{1}{3} \int_0^\alpha g\left(\frac{u}{\alpha}\right) du \\ &\quad + \frac{1}{2} g(1) + \frac{2-a}{2} g\left(\frac{a}{\alpha}\right) + \frac{a}{6} g\left(\frac{2a}{\alpha}\right) - \frac{1}{2} g\left(\frac{a+3}{\alpha}\right). \end{aligned}$$

Proof. Let $3K + a$ be the largest integer less than or equal to α . Then

$$(3.1) \quad \sum_{\substack{a \leq n \leq 3K+a \\ n=3k+a}} g\left(\frac{n}{\alpha}\right) = g\left(\frac{a}{\alpha}\right) + \sum_{\substack{a+3 \leq n \leq 3K+a \\ n=3k+a}} g\left(\frac{n}{\alpha}\right).$$

We write the sum on the right-hand side of this equality as

$$\sum_{\substack{a+3 \leq n \leq 3K+a \\ n=3k+a}} g\left(\frac{n}{\alpha}\right) = \sum_{1 \leq k \leq K} g\left(\frac{3k+a}{\alpha}\right).$$

Using Lemma 2.5 with $\theta = \beta = 1$, we get

$$\begin{aligned} \sum_{k=1}^{K-1} g\left(\frac{3k+a}{\alpha}\right) &\leq \sum_{k=1}^{K-1} \int_{k-1}^k g\left(\frac{3t+a}{\alpha}\right) dt - \sum_{k=1}^{K-1} \frac{g\left(\frac{3k+a}{\alpha}\right) - g\left(\frac{3k+a+3}{\alpha}\right)}{2} \\ &\leq \int_0^{K-1} g\left(\frac{3t+a}{\alpha}\right) dt - \frac{g\left(\frac{a+3}{\alpha}\right) - g\left(\frac{3K+a}{\alpha}\right)}{2}. \end{aligned}$$

By the change of variable $3t + a = u$ in the integral above, we obtain

$$(3.2) \quad \sum_{k=1}^{K-1} g\left(\frac{3k+a}{\alpha}\right) \leq \frac{1}{3} \int_a^{3K+a-3} g\left(\frac{u}{\alpha}\right) du - \frac{g\left(\frac{a+3}{\alpha}\right) - g\left(\frac{3K+a}{\alpha}\right)}{2}.$$

We again use Lemma 2.5 but this time with $\theta = 3$ and $\beta = \alpha - (3K + a)$. When $\beta = 0$ the proof is complete. Otherwise $\beta \leq 3$ and we get

$$g\left(\frac{3K+a}{\alpha}\right) \leq \frac{1}{3} \int_{3K+a-3}^{3K+a} g\left(\frac{u}{\alpha}\right) du - \frac{3(g(\frac{3K+a}{\alpha}) - g(1))}{2\beta}.$$

Substituting this in (3.2) with $\beta \leq 3$, we find that

$$\sum_{k=1}^K g\left(\frac{3k+a}{\alpha}\right) \leq \frac{1}{3} \int_a^{3K+a} g\left(\frac{u}{\alpha}\right) du - \frac{g(\frac{a+3}{\alpha}) - g(1)}{2}.$$

Then (3.1) becomes

$$(3.3) \quad \sum_{\substack{a \leq n \leq 3K+a \\ n=3k+a}} g\left(\frac{n}{\alpha}\right) \leq \frac{1}{3} \int_a^{3K+a} g\left(\frac{u}{\alpha}\right) du - \frac{g(\frac{a+3}{\alpha}) - g(1)}{2} + g\left(\frac{a}{\alpha}\right).$$

Now, we apply Lemma 2.5 to $g(a/\alpha)$ with $\theta = \beta = a$, to get

$$g\left(\frac{a}{\alpha}\right) \leq \frac{1}{a} \int_0^a g\left(\frac{u}{\alpha}\right) du - \frac{g(a/\alpha) - g(2a/\alpha)}{2};$$

consequently,

$$\frac{3}{2} g\left(\frac{a}{\alpha}\right) \leq \frac{1}{a} \int_0^a g\left(\frac{u}{\alpha}\right) du + \frac{g(2a/\alpha)}{2}.$$

Multiplying this with $a/3$, we find that

$$\frac{a}{2} g\left(\frac{a}{\alpha}\right) \leq \frac{1}{3} \int_0^a g\left(\frac{u}{\alpha}\right) du + \frac{a}{6} g\left(\frac{2a}{\alpha}\right).$$

Here, we have to distinguish two cases.

When $a = 1$, we write

$$\frac{g(1/\alpha)}{2} \leq \frac{1}{3} \int_0^1 g\left(\frac{u}{\alpha}\right) du + \frac{g(2/\alpha)}{6}.$$

Then (3.3) becomes

$$(3.4) \quad \sum_{\substack{1 \leq n \leq 3K+1 \\ n=3k+a}} g\left(\frac{n}{\alpha}\right) \leq \frac{1}{3} \int_0^{3K+1} g\left(\frac{u}{\alpha}\right) du + \frac{g(1)}{2} + \frac{g(1/\alpha)}{2} + \frac{g(2/\alpha)}{6} - \frac{g(4/\alpha)}{2}.$$

When $a = 2$, we rewrite

$$g\left(\frac{2}{\alpha}\right) \leq \frac{1}{3} \int_0^2 g\left(\frac{u}{\alpha}\right) du + \frac{g(4/\alpha)}{3}.$$

Then (3.3) becomes

$$(3.5) \quad \sum_{\substack{2 \leq n \leq 3K+2 \\ n=3k+a}} g\left(\frac{n}{\alpha}\right) \leq \frac{1}{3} \int_0^{3K+2} g\left(\frac{u}{\alpha}\right) du + \frac{g(1)}{2} + \frac{g(4/\alpha)}{3} - \frac{g(5/\alpha)}{2}$$

From (3.4) and (3.5), we conclude that

$$\begin{aligned} \sum_{\substack{a \leq n \leq \alpha \\ n=3k+a}} g\left(\frac{n}{\alpha}\right) &\leq \frac{1}{3} \int_0^{3K+a} g\left(\frac{u}{\alpha}\right) du \\ &\quad + \frac{1}{2} g(1) + \frac{2-a}{2} g\left(\frac{a}{\alpha}\right) + \frac{a}{6} g\left(\frac{2a}{\alpha}\right) - \frac{1}{2} g\left(\frac{a+3}{\alpha}\right). \end{aligned}$$

We complete the integral from $3K+a$ to α by using the non-negativity of g and get

$$\begin{aligned} \sum_{\substack{a \leq n \leq 3K+a \\ n=3k+a}} g\left(\frac{n}{\alpha}\right) &\leq \frac{1}{3} \int_0^{\alpha} g\left(\frac{u}{\alpha}\right) du \\ &\quad + \frac{1}{2} g(1) + \frac{2-a}{2} g\left(\frac{a}{\alpha}\right) + \frac{a}{6} g\left(\frac{2a}{\alpha}\right) - \frac{1}{2} g\left(\frac{a+3}{\alpha}\right). \end{aligned}$$

This completes the proof of Lemma 3.1. ■

LEMMA 3.2. *For $0 < \delta \leq 1/2$, we have*

$$\int_0^1 (1-t) \left(\log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| - \frac{1}{3} \log \left| \frac{3\pi \delta t}{\sin(3\pi \delta t)} \right| \right) dt \leq \frac{\pi^3 \delta^2}{36} + \frac{\pi^2 \delta^2}{27}.$$

Proof. We begin with

$$\log \left| \frac{3\pi \delta t}{\sin(3\pi \delta t)} \right| = \log 3 + \log(\pi \delta t) - \log |\sin(3\pi \delta t)|.$$

Since

$$\sin(3x) = -4 \sin^3 x + 3 \sin x,$$

it follows that

$$\begin{aligned} \log \left| \frac{3\pi \delta t}{\sin(3\pi \delta t)} \right| &= \log 3 + \log(\pi \delta t) - \log |\sin(\pi \delta t)| - \log |4 \sin^2(\pi \delta t) - 3| \\ &= \log \frac{\pi \delta t}{t |\sin(\pi \delta t)|} - \log \left| \frac{4}{3} \sin^2(\pi \delta t) - 1 \right|. \end{aligned}$$

Then

$$\begin{aligned} (3.6) \quad \int_0^1 (1-t) \left(\log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| - \frac{1}{3} \log \left| \frac{3\pi \delta t}{\sin(3\pi \delta t)} \right| \right) dt \\ &= \frac{2}{3} \int_0^1 (1-t) \log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| dt + \frac{1}{3} \int_0^1 (1-t) \log \left| \frac{4}{3} \sin^2(\pi \delta t) - 1 \right| dt. \end{aligned}$$

Since $\log x \leq x - 1$ and $\sin x \leq x$ if $0 \leq x \leq \pi/2$, we see that

$$\log \left| \frac{4}{3} \sin^2(\pi\delta t) - 1 \right| \leq \left| \frac{4}{3} \sin^2(\pi\delta t) - 1 \right| - 1 \leq \frac{4}{3} \sin^2(\pi\delta t) \leq \frac{4}{3} (\pi\delta t)^2.$$

Hence, we can write (3.6) as

$$\begin{aligned} & \int_0^1 (1-t) \left(\log \left| \frac{\pi\delta t}{\sin(\pi\delta t)} \right| - \frac{1}{3} \log \left| \frac{3\pi\delta t}{\sin(3\pi\delta t)} \right| \right) dt \\ & \leq \frac{2}{3} \int_0^1 (1-t) \log \left| \frac{\pi\delta t}{\sin(\pi\delta t)} \right| dt + \frac{4(\pi\delta)^2}{9} \int_0^1 (1-t)t^2 dt. \end{aligned}$$

Using Lemma 2.6, we infer

$$\int_0^1 (1-t) \left(\log \left| \frac{\pi\delta t}{\sin(\pi\delta t)} \right| - \frac{1}{3} \log \left| \frac{3\pi\delta t}{\sin(3\pi\delta t)} \right| \right) dt \leq \frac{\pi^3 \delta^2}{36} + \frac{\pi^2 \delta^2}{27}$$

and this completes the proof of Lemma 3.2. ■

4. Proof of Theorem 1.1. We are now ready to prove our upper bound for $|L(1, \chi)|$ when 3 divides the conductor q . We break the proof into two cases:

- For even characters, we have

$$\sum_{\substack{n \geq 1 \\ (n, 3)=1}} \frac{1 - F_3(\delta n)}{n} = \sum_{n \geq 1} \frac{1 - F_3(\delta n)}{n} - \sum_{n \geq 1} \frac{1 - F_3(3\delta n)}{3n},$$

where F_3 is defined in Proposition 2.1. Thanks to Lemma 2.3, we can write the sums on the right-hand side above as

$$\sum_{n \geq 1} \frac{1 - F_3(\delta n)}{n} = -\log \delta - 1 + \delta, \quad \sum_{n \geq 1} \frac{1 - F_3(3\delta n)}{n} = -\log(3\delta) - 1 + 3\delta.$$

Again using Proposition 2.1 and recalling $0 < \delta \leq 1$, we get

$$(4.1) \quad |L(1, \chi)| \leq -\frac{2}{3} \log \delta + \frac{1}{3} \log 3 - \frac{2}{3} + \frac{1}{\sqrt{q}} R(q),$$

where

$$R(q) = \sum_{\substack{1 \leq m \leq \delta q \\ (m, 3)=1}} j\left(\frac{m}{\delta q}\right),$$

and $j(t)$ is defined in Lemma 2.2. Now, we apply Lemmas 2.2 and 3.1 to the

sum of $j(t)$ given in (4.1) to obtain

$$\begin{aligned} R(q) &= \sum_{\substack{1 \leq m \leq \delta q \\ m=3k+1}} j\left(\frac{m}{\delta q}\right) + \sum_{\substack{2 \leq m \leq \delta q \\ m=3k+2}} j\left(\frac{m}{\delta q}\right) \\ &\leq \frac{2\delta q}{3} + j(1) + \frac{1}{2} j\left(\frac{1}{\delta q}\right) + \frac{1}{6} j\left(\frac{2}{\delta q}\right) - \frac{1}{6} j\left(\frac{4}{\delta q}\right) - \frac{1}{2} j\left(\frac{5}{\delta q}\right). \end{aligned}$$

Using Lemma 2.2 again, we find that

$$R(q) \leq \frac{2\delta q}{3} + \frac{5}{3} \log 2 + \log 5 + \frac{4}{3}(\log \pi - 1).$$

Then (4.1) becomes simply

$$\begin{aligned} |L(1, \chi)| &\leq -\frac{2}{3} \log \delta + \frac{1}{3} \log 3 - \frac{2}{3} + \frac{2\delta\sqrt{q}}{3} \\ &\quad + \frac{1}{\sqrt{q}} \left(\frac{5}{3} \log 2 + \log 5 + \frac{4}{3} \log \pi - \frac{4}{3} \right). \end{aligned}$$

The best possible choice for δ is $1/\sqrt{q}$. This yields

$$(4.2) \quad |L(1, \chi)| \leq \frac{1}{3} \log q + \frac{1}{3} \log 3 + \frac{1}{\sqrt{q}} \left(\frac{5}{3} \log 2 + \log 5 + \frac{4}{3} \log \pi - \frac{4}{3} \right),$$

where $\frac{1}{3} \log 3 = 0.3662\dots$ and the error term depends only on q . Setting

$$(4.3) \quad C_{\text{even}}(q) = \frac{1}{3} \log 3 + \frac{1}{\sqrt{q}} \left(\frac{5}{3} \log 2 + \log 5 + \frac{4}{3} \log \pi - \frac{4}{3} \right),$$

it follows that

$$(4.4) \quad |L(1, \chi)| \leq \frac{1}{3} \log q + C_{\text{even}}(q).$$

- In the case of odd characters, we have

$$\sum_{\substack{n \geq 1 \\ (n, 3)=1}} \frac{1 - F_4(\delta n)}{n} = \sum_{n \geq 1} \frac{1 - F_4(\delta n)}{n} - \sum_{n \geq 1} \frac{1 - F_4(3\delta n)}{3n},$$

where F_4 is defined in Proposition 2.1. Thanks to Lemma 2.4, we get

$$\begin{aligned} \sum_{\substack{n \geq 1 \\ (n, 3)=1}} \frac{1 - F_4(\delta n)}{n} &= -\log \delta - \log(2\pi) + \frac{3}{2} + 2 \int_0^1 (1-t) \log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| dt \\ &\quad + \frac{1}{3} \log \delta + \frac{1}{3} \log(2\pi) - \frac{1}{2} + \frac{1}{3} \log 3 - \frac{2}{3} \int_0^1 (1-t) \log \left| \frac{3\pi \delta t}{\sin(3\pi \delta t)} \right| dt. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{\substack{n \geq 1 \\ (n,3)=1}} \frac{1 - F_4(\delta n)}{n} &= -\frac{2}{3} \log \delta - \frac{2}{3} \log(2\pi) + 1 + \frac{1}{3} \log 3 \\ &\quad + 2 \int_0^1 (1-t) \left(\log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| - \frac{1}{3} \log \left| \frac{3\pi \delta t}{\sin(3\pi \delta t)} \right| \right) dt. \end{aligned}$$

Now, we use Proposition 2.1 and Lemma 3.2 to obtain

$$(4.5) \quad |L(1, \chi)| \leq -\frac{2}{3} \log \delta - \frac{2}{3} \log(2\pi) + 1 + \frac{1}{3} \log 3 + \frac{\pi^3 \delta^2}{18} + \frac{2\pi^2 \delta^2}{27} + \frac{\pi}{\sqrt{q}} S(q),$$

where

$$S(q) = \sum_{\substack{1 \leq m \leq \delta q \\ (m,3)=1}} \left(\frac{m}{\delta q} - 1 \right)^2.$$

For the last sum above, we use Lemma 3.1 to find that

$$S(q) = \sum_{\substack{m=3k+1 \\ 1 \leq m \leq \delta q}} \left(\frac{m}{\delta q} - 1 \right)^2 + \sum_{\substack{m=3k+2 \\ 2 \leq m \leq \delta q}} \left(\frac{m}{\delta q} - 1 \right)^2 \leq \frac{2\delta q}{9} - \frac{14}{\delta^2 q^2} + \frac{14}{3\delta q}.$$

Then (4.5) becomes

$$\begin{aligned} |L(1, \chi)| &\leq -\frac{2}{3} \log \delta - \frac{2}{3} \log(2\pi) + 1 + \frac{1}{3} \log 3 \\ &\quad + \frac{\pi^3 \delta^2}{18} + \frac{2\pi^2 \delta^2}{27} + \frac{2\pi \delta \sqrt{q}}{9} + \frac{\pi}{\delta q \sqrt{q}} \left(\frac{14}{3} - \frac{14}{\delta q} \right). \end{aligned}$$

The choice $\delta = 3/(\pi \sqrt{q})$ yields

$$(4.6) \quad |L(1, \chi)| \leq \frac{1}{3} \log q - \frac{1}{3} \log 12 + \frac{5}{3} + \frac{1}{q} \left(\frac{\pi}{2} + \frac{2}{3} + \frac{14\pi^2}{9} - \frac{14\pi^3}{9\sqrt{q}} \right),$$

with $\frac{5}{3} - \frac{1}{3} \log 12 = 0.8383 \dots$. Setting

$$(4.7) \quad C_{\text{odd}}(q) = \frac{5}{3} - \frac{1}{3} \log 12 + \frac{1}{q} \left(\frac{\pi}{2} + \frac{2}{3} + \frac{14\pi^2}{9} - \frac{14\pi^3}{9\sqrt{q}} \right),$$

it follows that

$$(4.8) \quad |L(1, \chi)| \leq \frac{1}{3} \log q + C_{\text{odd}}(q).$$

We list below some values of $C_{\text{even}}(q)$ and $C_{\text{odd}}(q)$.

q	10^4	10^5	10^6	$2 \cdot 10^6$	∞
$C_{\text{even}} \leq$	0.395781	0.375558	0.369162	0.368296	0.366205
$C_{\text{odd}} \leq$	0.840076	0.838539	0.838382	0.838374	0.838365

Thus, for $q > 2 \cdot 10^6$, we have proved that

$$|L(1, \chi)| \leq \frac{1}{3} \log q + \begin{cases} 0.368296 & \text{when } \chi(-1) = 1, \\ 0.838374 & \text{when } \chi(-1) = -1. \end{cases}$$

In the next section, we check that our result is valid for $2 \leq q \leq 2 \cdot 10^6$ using a rigorous and efficient algorithm for computing $L(1, \chi)$ for all primitive χ . This completes the proof of Theorem 1.1. ■

5. Numerical verification

5.1. The algorithm. In [12], the first author describes two efficient and rigorous algorithms for computing values of Dirichlet L -functions. We adapt one of those to our current purpose and for convenience we restate the key lemma here.

LEMMA 5.1. *For $q \in \mathbb{Z}_{\geq 3}$ and given $\varphi(q)$ complex values $a(n)$ for $n \in [1, q-1]$ and $(n, q) \neq 0$, we can compute*

$$\sum_{n=1}^{q-1} a(n)\chi(n)$$

for the $\varphi(q)$ characters χ in $\mathcal{O}(\varphi(q) \log(q))$ time and $\mathcal{O}(\varphi(q))$ space.

Proof. We construct the sum via a series of Discrete Fourier Transforms and we refer the reader to [12] for the details. The existence of Fast Fourier Transform algorithms for arbitrary length inputs then gives us the claimed time complexity. ■

To exploit Lemma 5.1 we use the following:

LEMMA 5.2. *Let χ be any non-principal Dirichlet character of conductor $q \in \mathbb{Z}_{>0}$, $L(z, \chi)$ be its associated L -function and $\psi(\alpha)$ be the digamma function. Then*

$$L(1, \chi) = \frac{-1}{q} \sum_{a=1}^q \chi(a) \psi\left(\frac{a}{q}\right).$$

Proof. For $z \neq 1$ we have the identity (see for example [3, §12])

$$(5.1) \quad L(z, \chi) = q^{-z} \sum_{a=1}^q \chi(a) \zeta\left(z, \frac{a}{q}\right).$$

In addition, for any non-principal character χ we have

$$\sum_{a \bmod q} \chi(a) = 0.$$

Thus for χ non-principal and for any complex constant C , we can replace equation (5.1) with

$$L(z, \chi) = q^{-z} \sum_{a=1}^q \chi(a) \left(\zeta\left(z, \frac{a}{q}\right) - C \right).$$

In particular, again for χ non-principal, we have

$$\begin{aligned} L(1, \chi) &= \lim_{z \downarrow 1} \sum_{a=1}^q q^{-z} \chi(a) \left(\zeta\left(z, \frac{a}{q}\right) - \zeta(z) \right) \\ &= \sum_{a=1}^q \frac{1}{q} \chi(a) \sum_{n=1}^{\infty} \frac{1 - a/q}{n(n + a/q - 1)}. \end{aligned}$$

Finally we have the series representation of the digamma function (see [1, 6.3.16]) valid for $z \neq -1, -2, \dots$,

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)},$$

and the result follows. ■

5.2. The computation. We implemented the above algorithm using the C++ programming language. To avoid potential problems with the propagation of rounding and truncation errors, we used the first author's double precision interval arithmetic package throughout. This in turn exploits the work of Lambov [7] and the CRLIBM package [10].

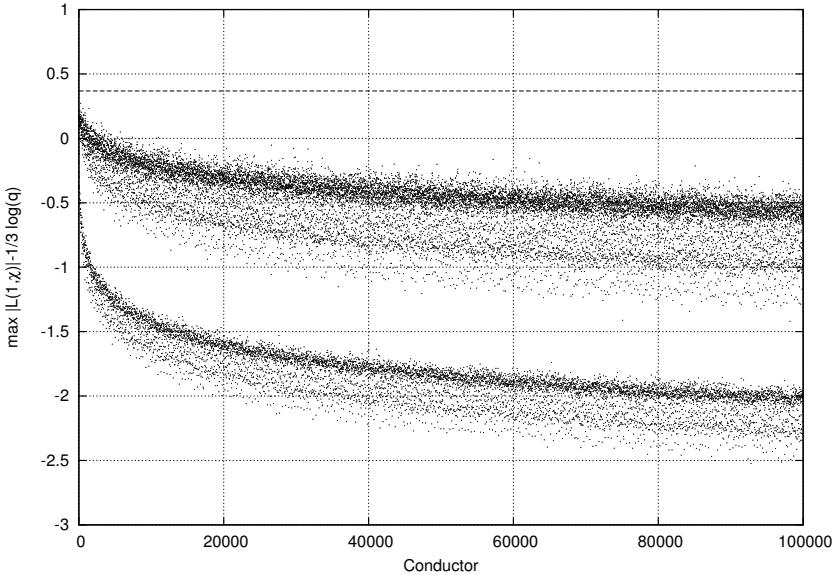


Fig. 1

We ran the code on a single node of the University of Bristol’s BlueCrystal cluster [2] using all 8 cores. The elapsed time was 69 hours and there were no exceptions to Theorem 1.1 over the 115 492 010 081 primitive L -functions checked.

In Figure 1 we plot the maximum value of $|L(1, \chi)| - \frac{1}{3} \log q$ achieved over all even characters for each conductor $q \equiv 0 \pmod{3}$ with $q \in [3, 100000]$. Figure 2 shows the same information for odd characters. The reference lines indicate the bounds of Theorem 1.1.

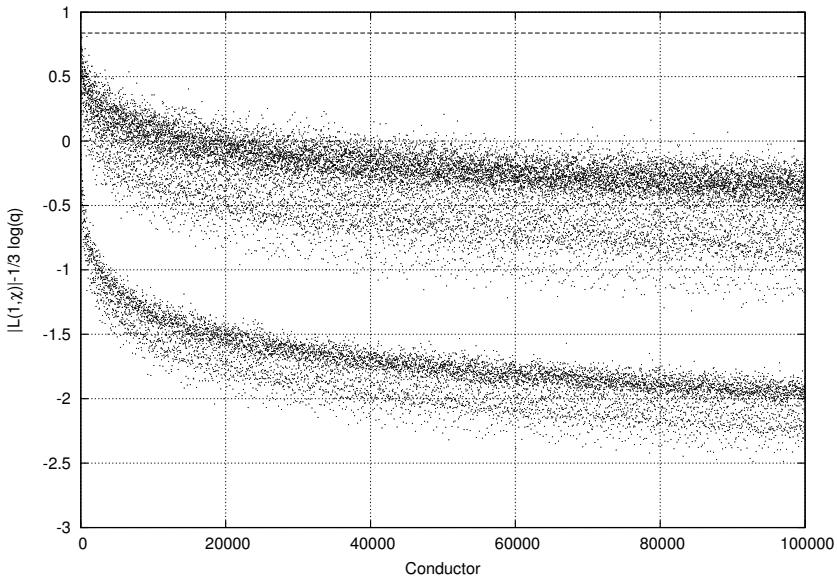


Fig. 2

The “banding” observed appears to be driven by the divisibility properties of the conductor. For example, the lower of the two main bands seen in each figure comprises those q divisible by 12. The largest value seen for even characters was at $q = 249$ where $|L(1, \chi)| - \frac{1}{3} \log(q) < 0.271789$, and for odd characters the maximum observed was at $q = 111$ where $|L(1, \chi)| - \frac{1}{3} \log(q) < 0.815651$. We conjecture that these upper bounds hold for all q , not just those below $2 \cdot 10^6$.

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*Received 20 June 2013;
 revised 25 July 2013*

(5967)