COLLOQUIUM MATHEMATICUM

VOL. 133 2013 NO. 1

SHARP SPECTRAL MULTIPLIERS FOR HARDY SPACES ASSOCIATED TO NON-NEGATIVE SELF-ADJOINT OPERATORS SATISFYING DAVIES–GAFFNEY ESTIMATES

BY

PENG CHEN (Adelaide)

Abstract. We consider an abstract non-negative self-adjoint operator L acting on $L^2(X)$ which satisfies Davies–Gaffney estimates. Let $H^p_L(X)$ ($p>0$) be the Hardy spaces associated to the operator L. We assume that the doubling condition holds for the metric measure space X. We show that a sharp Hörmander-type spectral multiplier theorem on $H^p_L(X)$ follows from restriction-type estimates and Davies–Gaffney estimates. We also establish a sharp result for the boundedness of Bochner–Riesz means on $H^p_L(X)$.

1. Introduction. Suppose that L is a non-negative self-adjoint operator acting on $L^2(X,\mu)$, where X is a measure space with measure μ. Then L admits a spectral resolution $E(\lambda)$, and for any bounded Borel function $F:[0,\infty)\to\mathbb{C}$, one can define the operator

$$F(L)=\int_0^\infty F(\lambda)\,dE(\lambda). \tag{1.1}$$

By the spectral theorem, this operator is bounded on $L^2(X)$. Spectral multiplier theorems give sufficient conditions on F and L which imply the boundedness of $F(L)$ on various function spaces defined on X. This is an active topic in harmonic analysis and has been studied extensively. We refer the reader to [A, B, C, COSY, CowS, DeM, DOS, DP1, DY2, GHS, Mi, St] and the references therein.

Before we state our main result, we describe some basic assumptions. Throughout the paper, we assume that (X,d,μ) is a metric measure space with metric d and a non-negative Borel measure μ satisfying the volume doubling condition: there exists a constant $C>0$ such that for all $x\in X$ and for all $r>0$,

$$V(x,2r)\le CV(x,r)<\infty, \tag{1.2}$$

2010 *Mathematics Subject Classification*: Primary 42B15; Secondary 42B20, 42B30, 47F05.
Key words and phrases: spectral multipliers, Hardy spaces, non-negative self-adjoint operator, Davies–Gaffney estimate, restriction type estimate, Bochner–Riesz means, metric measure space.

DOI: 10.4064/cm133-1-4

where $V(x,r)$ is the volume of the ball $B(x,r)$ centered at x of radius r. In particular, X is a space of homogeneous type. See for example [CW].

The doubling condition (1.2) implies that there exist some constants $C, n > 0$ such that

$$V(x,\lambda r) \le C\lambda^n V(x,r) \tag{1.3}$$

uniformly for all $\lambda \ge 1$ and all $x \in X$. In what follows, we shall consider n as small as possible. In the Euclidean space with Lebesgue measure, the smallest such parameter n is the dimension of the space.

The following conditions on the operator L shall be assumed throughout this paper unless otherwise specified:

(H1) The operator L is a non-negative self-adjoint operator acting on $L^2(X)$ and the semigroup $\{e^{-tL}\}_{t>0}$ generated by L satisfies *Davies–Gaffney estimates*: there exist constants $C, c > 0$ such that for all open subsets $U_1, U_2 \subset X$ and all $t > 0$,

$$|\langle e^{-tL} f_1, f_2\rangle| \le C \exp\left(-\frac{\operatorname{dist}(U_1,U_2)^2}{ct}\right) \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)} \tag{DG}$$

for every $f_i \in L^2(X)$ with $\operatorname{supp} f_i \subset U_i$, $i = 1, 2$, where $\operatorname{dist}(U_1,U_2) := \inf_{x\in U_1,\, y\in U_2} d(x,y)$; see for example [D, DL, DY2, HLMMY].

(H2) The operator L satisfies *restriction-type estimates*: Given a subset $E \subset X$, we define the projection operator P_E by multiplying by the characteristic function of E:

$$P_E f(x) := \chi_E(x) f(x).$$

For a function $F : \mathbb{R} \to \mathbb{C}$ and for $R > 0$, we denote by $\delta_R F : \mathbb{R} \to \mathbb{C}$ the function $\lambda \mapsto F(R\lambda)$. Following [COSY], we say that a non-negative self-adjoint operator L satisfies *restriction-type estimates* if, for each $R > 0$ and all Borel functions F such that $\operatorname{supp} F \subset [0, R]$, there exist some p_0 and q satisfying $1 \le p_0 < 2$ and $1 \le q \le \infty$ such that

$$\|F(\sqrt{L}) P_{B(x,r)}\|_{p_0\to 2} \le C V(x,r)^{1/2-1/p_0} (Rr)^{n(1/p_0-1/2)} \|\delta_R F\|_{L^q} \tag{1.4}$$

for all $x \in X$ and all $r \ge 1/R$, where n is the dimension from the doubling condition (1.3). When L is the standard Laplace operator $\Delta = -\sum_{i=1}^n \partial_{x_i}^2$ on $\mathbb{R}^n$, this estimate is equivalent to the classical $(p_0, 2)$ restriction estimate of Stein–Tomas. See [COSY] or Proposition 2.5 below.

The aim of this paper is to prove a Hörmander-type spectral multiplier theorem for abstract operators satisfying Davies–Gaffney estimates. More precisely, our result shows that restriction-type estimates imply sharp spectral multipliers on Hardy spaces $H^p_L(X)$ for $p > 0$, where $H^p_L(X)$ is a new class of Hardy spaces associated to the operator L (see [ADM, AMR, DL, DP2, DY1, DY2, HLMMY, HM, HMMc, JY] and Section 2 below).

The following theorem is the main result of this paper.

THEOREM 1.1. *Consider a doubling metric measure space* (X, d, μ) *which satisfies* (1.3) *with dimension* n. *Assume that the operator* L *satisfies Davies–Gaffney estimates* (DG) *and restriction-type estimates* (1.4) *for some* p_0, q *satisfying* $1 \le p_0 < 2$ *and* $1 \le q \le \infty$. *Let* ϕ *be a non-trivial smooth function with compact support on* $(0, +\infty)$. *Suppose that* $0 < p \le 1$. *Let* F *be a bounded Borel function for which there exists some constant* $s > n(1/p - 1/2)$ *such that*

$$\sup_{t>0} \|\phi \delta_t F\|_{W^{s,q}(\mathbb{R})} < \infty, \tag{1.5}$$

where $\delta_t F(\lambda) := F(t\lambda)$ *and* $\|F\|_{W^{s,q}(\mathbb{R})} := \|(I - d^2/dx^2)^{s/2} F\|_{L^q(\mathbb{R})}$. *Then the operator* $F(\sqrt{L})$ *is bounded on* $H^p_L(X)$. *That is, there exists a constant* $C > 0$ *such that*

$$\|F(\sqrt{L}) f\|_{H^p_L(X)} \le C \|f\|_{H^p_L(X)}.$$

A standard application of spectral multiplier theorems is to consider the boundedness of Bochner–Riesz means. Recall that *Bochner–Riesz means* $S^\delta_R(L)$ of order $\delta > 0$ for a non-negative self-adjoint operator L are defined by the formula

$$S^\delta_R(L) := \left(I - \frac{L}{R^2}\right)^\delta_+, \qquad R > 0. \tag{1.6}$$

In Theorem 1.1, if one chooses $F(\lambda) = (1 - \lambda^2)^\delta_+$ then $F \in W^{s,q}$ if and only if $\delta > s - 1/q$.

As a consequence of Theorem 1.1, we obtain the boundedness of Bochner–Riesz means for the operator L on the Hardy spaces $H^p_L(X)$.

COROLLARY 1.2. *Assume that the operator* L *satisfies Davies–Gaffney estimates* (DG) *and restriction-type estimates* (1.4) *for some* p_0, q *satisfying* $1 \le p_0 < 2$ *and* $1 \le q \le \infty$. *Suppose that* $0 < p \le 1$. *Then for all* $\delta > n(1/p - 1/2) - 1/q$, *we have*

$$\|S^\delta_R(L)\|_{H^p_L \to H^p_L} = \|(I - L/R^2)^\delta_+\|_{H^p_L \to H^p_L} \le C \tag{1.7}$$

uniformly in $R > 0$.

REMARKS. (a) Theorem 1.1 is a variation of a similar result proved by Duong and Yan [DY2] in which it was assumed that the operator L only satisfies Davies–Gaffney estimates (DG). In this paper, we show that the smoothness condition on the spectral multiplier function can be relaxed if the operator L also satisfies restriction-type estimates (1.4). Namely, instead of measuring the smoothness of the multiplier with the Sobolev space $W^{s,\infty}$ one can use the larger space $W^{s,q}$ for some $q < \infty$ that appears in restriction-type estimates (1.4). Our proof is a fairly technical combination of arguments from [COSY, DY2].

(b) When L is the standard Laplace operator $\Delta = -\sum_{i=1}^{n} \partial_{x_i}^2$ on $\mathbb{R}^n$, restriction-type estimates (1.4) hold for $q = 2$ and $1 \le p_0 \le (2n+2)/(n+3)$. As a consequence of Corollary 1.2, we obtain an alternative proof of the classical results due to Sjölin [Sj] and Stein–Taibleson–Weiss [STW] on the classical Bochner–Riesz means. It is well known that for $p \in (0,1]$ and $\delta > n(1/p - 1/2) - 1/2$, the operator of Bochner–Riesz means $S_R^\delta(\Delta)$ is uniformly bounded on $H^p(\mathbb{R}^n)$; however, for $\delta \le n(1/p - 1/2) - 1/2$, $S_R^\delta(\Delta)$ is not uniformly bounded on $H^p(\mathbb{R}^n)$ (see [Sj, STW]).

Note that when the semigroup e^{-tL} generated by L has a heat kernel $p_t(x,y)$ satisfying Gaussian upper bound estimates, that is,

$$|p_t(x,y)| \le \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right) \tag{1.8}$$

for all $t > 0$ and all $x, y \in X$, then by the observation due to Auscher, Duong and McIntosh [ADM], the Hardy spaces $H_L^p(X), 1 < p < \infty$, coincide with the corresponding $L^p(X)$ spaces (see [ADM, HLMMY]). As a consequence of Theorem 1.1, we obtain the following multiplier result on $L^p(X)$ for $p \ge 1$.

Proposition 1.3. *Assume that the heat kernel corresponding to the operator L satisfies Gaussian upper bound estimates* (1.8) *and restriction-type estimates* (1.4) *for some p_0, q satisfying $1 < p_0 < 2$ and $1 \le q \le \infty$. Let $1 \le p_1 \le p_0$. Then for any even bounded Borel function F such that $\sup_{t>0} \|\phi\delta_t F\|_{W^{s,q}} < \infty$ for some $s > n(1/p_1 - 1/2)$, the operator $F(\sqrt{L})$ is bounded on $L^p(X)$ for $p_1 < p < p_1'$. That is, there exists a constant $C > 0$ such that*

$$\|F(\sqrt{L})f\|_{L^p(X)} \le C\|f\|_{L^p(X)}.$$

We should mention that Theorem 1.1 is valid for abstract self-adjoint operators. However, one has to verify (H1) and (H2) before the result can be applied. Usually, it is difficult to verify restriction-type estimates (1.4). We list in Section 4 several examples of operators which satisfy Davies–Gaffney estimates (DG) and restriction-type estimates (1.4). On the other hand, restriction-type estimates (1.4) with $p_0 = 1$ and $q = \infty$ follow from Gaussian upper bound estimates (1.8) for the heat kernel corresponding to the operator (see [COSY, DOS]).

While this paper was being finalized we learned that M. Uhl introduced recently in his Ph.D. thesis [U] a similar condition to our restriction-type estimates and proved a similar spectral multiplier result for the space $H_L^1(X)$ (see also [KU]).

2. Preliminaries. In this section, first we state the finite speed propagation property for the wave equation corresponding to the operator L. Then we state some propositions for the operator L, deduced from restriction-type estimates. These propositions and the finite speed propagation property will be used to deduce the off-diagonal estimates for $F(\sqrt{L})$ in Section 3. At the end of Section 2, we state the definition of the Hardy space $H^p_L(X)$, $0 < p < \infty$, associated to the operator L and state a criterion for the boundedness of spectral multipliers on $H^p_L(X)$.

Let us recall some standard notations. In this paper, we often write B for $B(x, r)$. Given $\lambda > 0$, we write λB for the λ-dilated ball, which is the ball with the same center as B and with radius λr. For $1 \le p \le \infty$, we denote the norm of a function $f \in L^p(X, \mu)$ by $\|f\|_p$. If T is a bounded linear operator from $L^p(X, \mu)$ to $L^q(X, \mu)$ where $p, q \in [1, \infty]$, we write $\|T\|_{p\to q}$ for the operator norm $\|T\|_{L^p\to L^q}$. Let $\phi \in C^\infty_c(0, \infty)$ be a non-negative function such that

$$\operatorname{supp}\phi \subseteq (1/4, 1) \quad \text{and} \quad \sum_{\ell\in\mathbb{Z}} \phi(2^{-\ell}\lambda) = 1 \quad \text{for all } \lambda > 0. \tag{2.1}$$

2.1. Finite speed propagation property for the wave equation. Following [CouS], for $\rho > 0$ we set

$$\mathcal{D}_\rho := \{(x, y) \in X \times X : d(x, y) \le \rho\}.$$

Given an operator T from $L^p(X)$ to $L^q(X)$, we write

$$\operatorname{supp} K_T \subseteq \mathcal{D}_\rho \tag{2.2}$$

if $\langle Tf_1, f_2\rangle = 0$ whenever f_k is in $C(X)$ and $\operatorname{supp} f_k \subseteq B(x_k, \rho_k)$ for $k = 1, 2$, and $\rho_1 + \rho_2 + \rho < d(x_1, x_2)$.

Definition 2.1. One says that the operator L satisfies the *finite speed propagation property* if

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subseteq \mathcal{D}_t \quad \text{for all } t \ge 0. \tag{FS}$$

Proposition 2.2. *Let L be a non-negative self-adjoint operator acting on $L^2(X)$. Then the finite speed propagation property* (FS) *and Davies–Gaffney estimates* (DG) *are equivalent.*

Proof. Consult Theorem 2 in [S1] and Theorem 3.4 in [CouS]. See also [CGT]. ∎

The following lemma gives a straightforward consequence of the finite speed propagation property (FS).

Lemma 2.3. *Assume that L satisfies the finite speed propagation property* (FS) *and that F is an even bounded Borel function with Fourier transform*

$\widehat{F}$ *satisfying* $\operatorname{supp}\widehat{F} \subset [-\rho,\rho]$ *for some* $\rho > 0$. *Then*

$$\operatorname{supp} K_{F(\sqrt{L})} \subseteq \mathcal{D}_\rho.$$

Proof. If F is an even function, then by the Fourier inversion formula,

$$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{F}(t)\cos(t\sqrt{L})\,dt.$$

Note that $\operatorname{supp}\widehat{F} \subset [-\rho,\rho]$. Then Lemma 2.3 follows from (FS). ∎

2.2. Restriction-type estimates. Let us recall the following result. For its proof, we refer the reader to [COSY, Proposition 2.3].

PROPOSITION 2.4. *Suppose that* (X,d,μ) *satisfies the doubling property* (1.3) *with dimension* n. *Let* $1 \le p_0 < 2$ *and* $N > n(1/p - 1/2)$. *Then the following statements are equivalent:*

(i) *Restriction-type estimates* (1.4) *hold with* $q = \infty$.
(ii) *For all* $x > 0$ *and all* r, t *with* $r \ge t > 0$ *we have*

$$(\mathrm{G}_{p_0,2}) \qquad \|e^{-t^2L}P_{B(x,r)}\|_{p_0\to 2} \le CV(x,r)^{1/2-1/p_0}(r/t)^{n(1/p_0-1/2)}.$$

(iii) *For all* $x \in X$ *and all* r, t *with* $r \ge t > 0$ *we have*

$$(\mathrm{E}_{p_0,2}) \qquad \|(1+t\sqrt{L})^{-N}P_{B(x,r)}\|_{p_0\to 2} \le CV(x,r)^{1/2-1/p_0}(r/t)^{n(1/p_0-1/2)}.$$

Following [GHS], we say that the operator L satisfies L^{p_0} *to* $L^{p_0'}$ *restriction estimates* if the spectral measure $dE_{\sqrt{L}}(\lambda)$ maps $L^{p_0}(X)$ to $L^{p_0'}(X)$ for some p_0 satisfying $1 \le p_0 \le 2n/(n+1)$, with an operator norm estimate

$$(\mathrm{R}_{p_0}) \qquad \|dE_{\sqrt{L}}(\lambda)\|_{p_0\to p_0'} \le C\lambda^{n(1/p_0-1/p_0')-1}$$

for all $\lambda > 0$.

PROPOSITION 2.5. *Suppose that there exist positive constants* $0 < C_1 \le C_2 < \infty$ *such that* $C_1r^n \le V(x,r) \le C_2r^n$ *for every* $x \in X$ *and* $r > 0$. *Then* L^{p_0} *to* $L^{p_0'}$ *restriction estimates* (R_{p_0}) *and restriction-type estimates* (1.4) *with* $q = 2$ *are equivalent.*

Proof. See [COSY, Proposition 2.4]. ∎

2.3. Hardy spaces $H^p_L(X)$. Assume that the operator L satisfies Davies–Gaffney estimates (DG). Following [AMR], one can define the L^2 adapted Hardy space

$$H^2(X) := \overline{R(L)}, \tag{2.3}$$

that is, the closure of the range of L in $L^2(X)$. Then $L^2(X)$ is the orthogonal sum of $H^2(X)$ and the null space $N(L)$. Consider the following quadratic operators associated to L:

$$S_K f(x) := \left(\int_0^\infty \int_{d(x,y)<t} |(t^2L)^K e^{-t^2L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}, \tag{2.4}$$

where $x \in X$, $f \in L^2(X)$ and K is a natural number. For each $K \geq 1$ and $0 < p < \infty$, we now define

$$D_{K,p} := \{f \in H^2(X) : S_K f \in L^p(X)\}.$$

DEFINITION 2.6. Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying (DG).

(i) For each $p \in (0, 2]$, the Hardy space $H^p_L(X)$ associated to L is the completion of the space $D_{1,p}$ in the norm

$$\|f\|_{H^p_L(X)} := \|S_1 f\|_{L^p(X)}.$$

(ii) For each $p \in (2, \infty)$, the Hardy space $H^p_L(X)$ associated to L is the completion of the space $D_{K_0,p}$ in the norm

$$\|f\|_{H^p_L(X)} := \|S_{K_0} f\|_{L^p(X)}, \qquad \text{where } K_0 = [n/4] + 1.$$

Under the assumption of Gaussian upper bound estimates (1.8), by the observation due to Auscher, Duong and McIntosh [ADM], Hardy spaces $H^p_L(X), 1 < p < \infty$, coincide with the corresponding $L^p(X)$ spaces (see [ADM, HLMMY]). Note that in this paper, we only assume Davies–Gaffney estimates on the heat kernel of L, and hence for $1 < p < \infty$, $p \neq 2$, $H^p_L(X)$ may or may not coincide with the space $L^p(X)$. However, it can be verified that $H^2_L(X) = H^2(X)$ and the dual of $H^p_L(X)$ is $H^{p'}_L(X)$ with $1/p + 1/p' = 1$ (see [HLMMY, Proposition 9.4]).

2.4. A criterion for boundedness of spectral multipliers on $H^p_L(X)$. We now state a criterion from [DY2] that allows us to derive estimates on Hardy spaces $H^p_L(X)$. This criterion generalizes the classical Calderón–Zygmund theory. We would like to emphasize that the conditions imposed involve the multiplier operator and its action on functions, but not its kernel.

LEMMA 2.7. *Let L be a non-negative self-adjoint operator acting on $L^2(X)$ and satisfying Davies–Gaffney estimates* (DG). *Let m be a bounded Borel function. Suppose that $0 < p \leq 1$ and $M > (n/2)(1/p - 1/2)$. Assume that there exist constants $s > n(1/p - 1/2)$ and $C > 0$ such that, for every $j = 2, 3, \ldots$,*

$$\|F(L)(I - e^{-r_B^2 L})^M f\|_{L^2(2^j B \setminus 2^{j-1}B)} \leq C 2^{-js} \|f\|_{L^2(B)} \tag{2.5}$$

for every ball B with radius r_B and for all $f \in L^2(X)$ with supp $f \subset B$. *Then the operator $F(L)$ extends to a bounded operator on $H^p_L(X)$. More* *precisely, there exists a constant* $C > 0$ *such that for all* $f \in H^p_L(X)$,

$$\|F(L)f\|_{H^p_L(X)} \leq C\|f\|_{H^p_L(X)}. \tag{2.6}$$

Proof. We refer the reader to [DY2, Theorem 3.1]. ∎

3. Proof of Theorem 1.1. In order to prove Theorem 1.1, we will need an auxiliary lemma about some estimates for the operator $F(\sqrt{L})$ away from the diagonal which are deduced from restriction-type estimates and the finite speed propagation property. In the following, for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, $B^{p,q}_s(\mathbb{R})$ denotes the usual Besov space (see for example [BL]).

LEMMA 3.1. *Assume that the operator* L *satisfies the finite speed propagation property* (FS) *and restriction-type estimates* (1.4) *for some* p_0, q *satisfying* $1 \leq p_0 < 2$ *and* $1 \leq q \leq \infty$. *Also assume that the function* F *is even and supported on* $[-R, R]$. *Then for each* $s > \max\{n(1/p_0 - 1/2) - 1, 0\}$, *there exists a constant* C_s *such that for each ball* $B = B(x, r)$ *and for every* $j = 1, 2, \ldots,$ *we have the following estimates:*

(i) *for* $rR \geq 1$,

$$\begin{aligned}\|P_{B(x,2^j r)^c} F(\sqrt{L}) P_{B(x,r)}\|_{p_0\to 2} \\ \leq C_s V(x,r)^{1/2-1/p_0}(Rr)^{n(1/p_0-1/2)}(2^j rR)^{-s}\|\delta_R F\|_{B^{q,1}_s(\mathbb{R})};\end{aligned} \tag{3.1}$$

(ii) *for* $rR < 1$,

$$\begin{aligned}\|P_{B(x,2^j r)^c} F(\sqrt{L}) P_{B(x,r)}\|_{p_0\to 2} \\ \leq C_s V(x,R^{-1})^{1/2-1/p_0}(2^j rR)^{-s}\|\delta_R F\|_{B^{q,1}_s(\mathbb{R})}.\end{aligned} \tag{3.2}$$

Proof. For all $r, R > 0$ and every $j = 1, 2, \ldots,$ we can see that if $2^{j-5}rR \leq 1$, then estimates (3.1) and (3.2) follow from restriction-type estimates (1.4) immediately. Thus in the rest of the proof, we fix r, j and R such that $2^{j-5}rR > 1$.

Let ϕ_0 and ϕ_k be smooth even functions supported in $[-4,4]$ and $[2^k, 2^{k+2}] \cup [-2^{k+2}, -2^k]$ respectively, such that $\phi_0(\lambda) + \sum_{k\geq 1}\phi_k(\lambda) = 1$ for all $\lambda > 0$ and $\phi_0 = 1$ on $[-2,2]$. Set $\psi(\lambda) = \phi_0(\lambda/(2^{j-3}r))$ and $\psi_0(\lambda) = \phi_0(\lambda/(2^{j-3}rR))$. Define T_ϕ by $\widehat{T_\phi F} := \phi\widehat{F}$. Since $\operatorname{supp}\psi \subset [-2^{j-1}r, 2^{j-1}r]$, it follows from Lemma 2.3 that

$$\operatorname{supp} K_{T_\psi F(\sqrt{L})} \subset \{(z,y) \in X \times X : d(z,y) \leq 2^{j-1}r\}.$$

This implies

$$K_{F(\sqrt{L})}(z,y) = K_{[F - T_\psi F](\sqrt{L})}(z,y)$$

for all z, y such that $d(z,y) > 2^{j-1}r$. Hence,

$$\|P_{B(x,2^j r)^c} F(\sqrt{L}) P_{B(x,r)}\|_{p_0\to 2} \leq \|[F - T_\psi F](\sqrt{L}) P_{B(x,r)}\|_{p_0\to 2}. \tag{3.3}$$

Since $\operatorname{supp} F \subset [-R, R]$, one can write

$$F - T_\psi F = \delta_{R^{-1}}(\phi_0)[F - T_\psi F] - (1 - \delta_{R^{-1}}(\phi_0))T_\psi F, \tag{3.4}$$

which gives

$$\begin{aligned} \|P_{B(x,2^j r)^c} F(\sqrt{L}) P_{B(x,r)}\|_{p_0 \to 2} & \\ &\le \|\delta_{R^{-1}}(\phi_0)[F - T_\psi F](\sqrt{L}) P_{B(x,r)}\|_{p_0\to 2} \\ &\quad + \|(1 - \delta_{R^{-1}}(\phi_0))T_\psi F(\sqrt{L}) P_{B(x,r)}\|_{p_0 \to 2} \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{3.5}$$

For (i), i.e. $rR \ge 1$, we note that $\operatorname{supp} \delta_{R^{-1}}(\phi_0) \subset [-4R, 4R]$. By restriction-type estimates (1.4), it follows that

$$\mathrm{I} \le CV(x, r)^{1/2-1/p_0} (Rr)^{n(1/p_0-1/2)} \|\phi_0 \delta_R [F - T_\psi F]\|_{L^q}. \tag{3.6}$$

Note that $\phi_i(\lambda)(1 - \psi_0(\lambda)) = \phi_i(\lambda)(1 - \phi_0(\lambda/(2^{j-3} rR))) = 0$ for all $\lambda \in \mathbb{R}$ unless $2^i \ge 2^{j-4} rR$. Consequently, $T_{\phi_i}[I - T_{\psi_0}]\delta_R F = 0$ unless $i \ge i_0$, where $i_0 = \log_2(2^{j-4} rR)$. This implies that

$$\begin{aligned} \|\phi_0 \delta_R [F - T_\psi F]\|_{L^q} & \\ &\le \|\delta_R F - T_{\psi_0}(\delta_R F)\|_{L^q} = \Big\| \sum_{i\ge 0} T_{\phi_i}[I - T_{\psi_0}]\delta_R F \Big\|_{L^q} \\ &\le \sum_{i \ge i_0} \|T_{\phi_i}[I - T_{\psi_0}]\delta_R F\|_{L^q} \le \sum_{i\ge i_0} \|T_{\phi_i} \delta_R F\|_{L^q} \\ &\le 2^{-i_0 s} \sum_{i \ge i_0} 2^{is} \|T_{\phi_i}\delta_R F\|_{L^q} \le C(2^j rR)^{-s} \|\delta_R F\|_{B^{q,1}_s(\mathbb{R})}. \end{aligned} \tag{3.7}$$

Combining estimates (3.6) and (3.7), we have

$$\mathrm{I} \le CV(x, r)^{1/2-1/p_0} (Rr)^{n(1/p_0-1/2)} (2^j rR)^{-s} \|\delta_R F\|_{B^{q,1}_s(\mathbb{R})}.$$

To estimate the term II, we let $\check{f}$ denote the inverse Fourier transform of a function f. Observe that $|\lambda - y| \approx |\lambda|$ if $|\lambda| \ge 2R$ and $|y| \le R$, and then

$$\begin{aligned} &\sup_\lambda (1 - \delta_{R^{-1}}(\phi_0)(\lambda)) T_\psi F(\lambda)(1 + R^{-1}|\lambda|)^{s+1} \\ &\quad \le \sup_\lambda (1 - \phi_0(\lambda/R)) \Big| \int_{-R}^{R} F(y)\check{\psi}(\lambda - y)\, dy \Big| (1 + |\lambda|/R)^{s+1} \\ &\quad \le C \sup_\lambda (1 - \phi_0(\lambda/R)) 2^{j-3} rR(1 + 2^{j-3} r|\lambda|)^{-s-1} (1 + |\lambda|/R)^{s+1} \|\delta_R F\|_{L^q} \\ &\quad \le C(2^j rR)^{-s} \|\delta_R F\|_{L^q}. \end{aligned}$$

This, together with Proposition 2.4, shows for each $s > \max\{n(1/p_0 - 1/2) - 1, 0\}$

that

$$\mathrm{II} \le \sup_{\lambda} |(1-\delta_{R^{-1}}(\phi_0)(\lambda))T_\psi F(\lambda)(1+R^{-1}|\lambda|)^{s+1}| \times \|(I+R^{-1}\sqrt{L})^{-s-1}P_{B(x,r)}\|_{p_0\to 2} \le CV(x,r)^{1/2-1/p_0}(Rr)^{n(1/p_0-1/2)}(2^j rR)^{-s}\|\delta_R F\|_{L^q} \tag{3.8}$$

as desired.

Combining estimates of I and II, we obtain estimate (3.1) for each $s > \max\{n(1/p_0-1/2)-1, 0\}$.

For (ii), i.e. $rR < 1$, we use estimate (3.5) and $r < R^{-1}$ to write

$$\|P_{B(x,2^j r)^c}F(\sqrt{L})P_{B(x,r)}\|_{p_0\to 2} \le \|\delta_{R^{-1}}(\phi_0)[F-T_\psi F](\sqrt{L})P_{B(x,R^{-1})}\|_{p_0\to 2} + \|(1-\delta_{R^{-1}}(\phi_0))T_\psi F(\sqrt{L})P_{B(x,R^{-1})}\|_{p_0\to 2}.$$

Replacing $B(x,r)$ by $B(x,R^{-1})$ in (3.6) and (3.8), a similar argument to that for (i) shows (3.2). We omit the details. ∎

Proof of Theorem 1.1. We will apply Lemma 2.7. It suffices to verify condition (2.5). Recall that ϕ is a non-negative C_0^∞ function such that

$$\operatorname{supp}\phi \subseteq (1/4,1) \quad \text{and} \quad \sum_{\ell\in\mathbb{Z}} \phi(2^{-\ell}\lambda) = 1 \quad \text{for all } \lambda > 0.$$

Then

$$F(\lambda) = \sum_{\ell\in\mathbb{Z}} \phi(2^{-\ell}\lambda)F(\lambda) =: \sum_{\ell\in\mathbb{Z}} F_\ell(\lambda) \quad \text{for all } \lambda > 0.$$

For every $\ell\in\mathbb{Z}$ and $r>0$, set $F^\ell_{r,M} := F_\ell(\lambda)(1-e^{-r^2\lambda^2})^M$. So for every ball $B = B(x,r)$ and $f\in L^2(X)$,

$$\|F(\sqrt{L})(I-e^{-r^2L})^M f\|_{L^2(2^jB\setminus 2^{j-1}B)} \le \sum_{\ell\in\mathbb{Z}} \|F^\ell_{r,M}(\sqrt{L})f\|_{L^2(2^jB\setminus 2^{j-1}B)}. \tag{3.9}$$

Fix $f\in L^2(X)$ with $\operatorname{supp} f\subset B$ and take $j\ge 2$. Note that $\operatorname{supp} F^\ell_{r,M} \subset [-2^\ell, 2^\ell]$. So if $r2^\ell < 1$, it follows by Lemma 3.1 that for each $s > \max\{n(1/p_0-1/2)-1, 0\}$,

$$\begin{aligned}
&\|F^\ell_{r,M}(\sqrt{L})f\|_{L^2(2^jB\setminus 2^{j-1}B)} \\
&\le \|P_{2^jB\setminus 2^{j-1}B}F^\ell_{r,M}(\sqrt{L})P_B\|_{p_0\to 2}\|f\|_{p_0} \\
&\le CV(x,2^{-\ell})^{1/2-1/p_0}(2^j r2^\ell)^{-s}\|\delta_{2^\ell}F^\ell_{r,M}\|_{B^{q,1}_s(\mathbb{R})}\|f\|_{p_0} \\
&\le CV(x,2^{-\ell})^{1/2-1/p_0}(2^j r2^\ell)^{-s}\|\delta_{2^\ell}F^\ell_{r,M}\|_{B^{q,1}_s(\mathbb{R})}V(x,r)^{1/p_0-1/2}\|f\|_2 \\
&\le C2^{-js}(2^\ell r)^{2M-s}\|\phi\delta_{2^\ell}F\|_{B^{q,1}_s(\mathbb{R})}\|f\|_2.
\end{aligned} \tag{3.10}$$

If $r2^\ell \ge 1$, then

$$\begin{aligned}(3.11)\quad &\|F^\ell_{r,M}(\sqrt{L})f\|_{L^2(2^jB\setminus 2^{j-1}B)}\\ &\le \|P_{2^jB\setminus 2^{j-1}B}F^\ell_{r,M}(\sqrt{L})P_B\|_{p_0\to 2}\|f\|_{p_0}\\ &\le CV(x,r)^{1/2-1/p_0}(2^\ell r)^{n(1/p_0-1/2)}(2^jr2^\ell)^{-s}\|\delta_{2^\ell}F^\ell_{r,M}\|_{B^{q,1}_s(\mathbb{R})}\|f\|_{p_0}\\ &\le C(2^\ell r)^{n(1/p_0-1/2)}(2^jr2^\ell)^{-s}\|\delta_{2^\ell}F^\ell_{r,M}\|_{B^{q,1}_s(\mathbb{R})}\|f\|_2\\ &\le C2^{-js}(2^\ell r)^{n(1/p_0-1/2)-s}\|\phi\delta_{2^\ell}F\|_{B^{q,1}_s(\mathbb{R})}\|f\|_2.\end{aligned}$$

Note that for each $\varepsilon > 0$, $\|F\|_{B^{q,1}_{s-\varepsilon}(\mathbb{R})} \le C_\varepsilon\|F\|_{W^{s,q}(\mathbb{R})}$ (see for example [BL]). Choosing s such that $M > s > n(1/p - 1/2)$, it follows from (1.5) and (3.9)–(3.11) that

$$\|F(\sqrt{L})(I-e^{-r^2L})^Mf\|_{L^2(2^jB\setminus 2^{j-1}B)} \le C2^{-js}\|f\|_2.$$

This proves condition (2.5). Hence, by Lemma 2.7, $F(\sqrt{L})$ can be extended to a bounded operator on $H^p_L(X)$. ■

Proof of Proposition 1.3. We observe that for $p_1 = p_0$, Proposition 1.3 follows from [COSY, Theorem 4.1]. By Theorem 1.1, Proposition 1.3 holds for $p_1 = 1$. We now use an idea from [Mi] to construct a family $\{F_z : z \in \mathbb{C}, 0 \le \operatorname{Re} z \le 1\}$ of spectral multipliers as follows:

$$F_z(\lambda) = \sum_{j=-\infty}^{\infty}\eta(2^{-j}\lambda)\left(1-2^{2j}\frac{d^2}{d\lambda^2}\right)^{(z-\theta)n(1-1/p_0)/2}(F(\lambda)\phi(2^{-j}\lambda)),$$

where $\theta = (1-1/p_1)/(1-1/p_0)$ and $\eta \in C^\infty_c([1/4,4])$, $\phi \in C^\infty_c([1/2,2])$, $\eta = 1$ on $[1/2,2]$ and $\sum_j \eta(2^{-j}\lambda) = \sum_j \phi(2^{-j}\lambda) = 1$ for all $\lambda > 0$. Observe that if $z = 1+iy$, then

$$\sup_{t>0}\|\phi\delta_tF_{1+iy}\|_{W^{s_1,q}} \le C\sup_{t>0}\|\phi\delta_tF\|_{W^{s,q}}(1+|y|)^{n/2}$$

for some $s_1 > n(1/p_0 - 1/2)$. On the other hand, if $z = iy$, then

$$\sup_{t>0}\|\phi\delta_tF_{iy}\|_{W^{s_2,q}} \le C\sup_{t>0}\|\phi\delta_tF\|_{W^{s,q}}(1+|y|)^{n/2}$$

for some $s_2 > n/2$. It follows by [COSY, Theorem 4.1] that $F_{1+iy}(\sqrt{L})$ is bounded on $H^p_L(X)$ for $p_0 < p < p_0'$, and by Theorem 1.1 that $F_{iy}(\sqrt{L})$ is bounded on $H^1_L(X)$. Applying the three-line theorem, we conclude that $F_\theta(\sqrt{L}) = F(\sqrt{L})$ is bounded on $H^p_L(X)$, that is, $F(\sqrt{L})$ is bounded on $L^p(X)$ for $p_1 < p < p_1'$. ■

4. Applications. In this section, we discuss several examples of operators which satisfy the Davies–Gaffney estimates (H1) and restriction-type estimates (H2), and then we apply our main results to these operators.

4.1. Sub-Laplacians on homogeneous groups. Let G be a homogeneous Lie group of polynomial growth with homogeneous dimension n (see for example [C, DeM, FS]), and let $X_1, \dots, X_k$ be a system of left-invariant vector fields on G satisfying the Hörmander condition. We define the sub-Laplace operator L acting on $L^2(\mathrm{G})$ by the formula

$$L = -\sum_{i=1}^{k} X_i^2. \tag{4.1}$$

PROPOSITION 4.1. *Let L be the homogeneous sub-Laplacian defined by* (4.1) *acting on a homogeneous group* G. *Then condition* (1.4) *holds for $p_0 = 1$ and $q = 2$, and hence the conclusions of Theorem* 1.1 *and Corollary* 1.2 *hold for $q = 2$.*

Proof. It is well known that the heat kernel corresponding to L satisfies Davies–Gaffney estimates (DG). It is not difficult to check that for some constant $C > 0$,

$$\|F(\sqrt{L})\|^2_{L^2(X)\to L^\infty(X)} = C \int_0^\infty |F(t)|^2 t^{n-1}\, dt.$$

See for example, [DOS, (7.1)], or [C, Proposition 10]. It is known that the above equality implies condition (1.4) with $p_0 = 1$ and $q = 2$ (see [COSY, Section 12]). Then Theorem 1.1 and Corollary 1.2 imply Proposition 4.1. ∎

Proposition 4.1 can be extended to "quasi-homogeneous" operators acting on homogeneous groups; see [S2] and [DOS].

4.2. Schrödinger operators on asymptotically conic manifolds. Asymptotically conic manifolds (see [Me]) are defined as the interior of a compact manifold M with boundary, such that the metric g is smooth on the interior, and in a collar neighborhood of the boundary it has the form

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2},$$

where x is a defining function of the smooth boundary and $h(x)$ is a smooth family of metrics on the boundary.

PROPOSITION 4.2. *Let (M, g) be a non-trapping asymptotically conic manifold of dimension $n \geq 3$, and let x be a defining function of the smooth boundary ∂M. Let $L := -\Delta + V$ be a Schrödinger operator with $V \in x^3 C^\infty(M)$, and assume that L is a positive operator and* 0 *is neither an eigenvalue nor a resonance. Then restriction-type estimates* (1.4) *hold with $q = 2$ for all $1 \leq p_0 \leq (2n+2)/(n+3)$, and hence the conclusions of Theorem* 1.1 *and Corollary* 1.2 *hold for $q = 2$.*

Proof. It was proved in [GHS, Theorem 1.3] that condition (R_{p_0}) is satisfied for L when $1 \leq p_0 \leq (2n+2)/(n+3)$. By Proposition 2.5, Theorem 1.1 and Corollary 1.2, we obtain Proposition 4.2. ∎

4.3. Schrödinger operators with the inverse-square potential. Now we consider Schrödinger operator $L = -\Delta + V$ on $L^2(\mathbb{R}^n, dx)$, where $V(x) = c/|x|^2$. We assume that $n > 2$ and $c > -(n-2)^2/4$. The classical Hardy inequality

$$-\Delta \geq \frac{(n-2)^2}{4}|x|^{-2} \tag{4.2}$$

shows that the self-adjoint operator L is non-negative if $c > -(n-2)^2/4$. Set $p_c^* = n/\sigma$ and $\sigma = \max\{(n-2)/2 - \sqrt{(n-2)^2/4 + c}, 0\}$. If $c \geq 0$, then the semigroup $\exp(-tL)$ is pointwise bounded by the Gaussian upper bound (1.8) and hence acts on all L^p spaces with $1 \leq p \leq \infty$. If $c < 0$, then $\exp(-tL)$ acts as a uniformly bounded semigroup on $L^p(\mathbb{R}^n)$ for $p \in ((p_c^*)', p_c^*)$ and the range $((p_c^*)', p_c^*)$ is optimal (see for example [LSV]).

For these Schrödinger operators, we have the following proposition.

PROPOSITION 4.3. *Assume that $n > 2$ and let $L = -\Delta + V$ be a Schrödinger operator on $L^2(\mathbb{R}^n, dx)$, where $V(x) = c/|x|^2$ and $c > -(n-2)^2/4$. Suppose that $p_0 \in ((p_c^*)', 2n/(n+2)]$ where $p_c^* = n/\sigma$, $(p_c^*)'$ is its dual exponent and $\sigma = \max\{(n-2)/2 - \sqrt{(n-2)^2/4 + c}, 0\}$. Then restriction-type estimates* (1.4) *hold with $q = 2$, and hence the conclusions of Theorem* 1.1 *and Corollary* 1.2 *hold for $q = 2$.*

Proof. It was proved in [COSY, Section 10] that L satisfies restriction estimates (R_{p_0}) for all $p_0 \in ((p_c^*)', 2n/(n+2)]$. If $c \geq 0$, then $p = (p_c^*)' = 1$ is included. By Proposition 2.5, (R_{p_0}) and (1.4) with $q = 2$ are equivalent. Now Proposition 4.3 follows from Theorem 1.1 and Corollary 1.2. ■

Acknowledgements. This project was supported by Australian Research Council Discovery Grant DP 110102488. The author would like to thank the referee for helpful comments and suggestions. The author thanks X. T. Duong, A. Sikora, L. A. Ward and L. X. Yan for fruitful discussions.

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Peng Chen
School of Information Technology and Mathematical Sciences
University of South Australia
Adelaide, SA, 5095, Australia
E-mail: achenpeng1981@163.com

Received 14 October 2012;
revised 14 August 2013 (5787)