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RADICALS OF SYMMETRIC CELLULAR ALGEBRAS

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Abstract. For a symmetric cellular algebra, we study properties of the dual basis of a cellular basis first. Then a nilpotent ideal is constructed. The ideal connects the radicals of cell modules with the radical of the algebra. It also yields some information on the dimensions of simple modules. As a by-product, we obtain some equivalent conditions for a finite-dimensional symmetric cellular algebra to be semisimple.

1. Introduction. Cellular algebras were introduced by Graham and Lehrer [6] in 1996, motivated by previous work of Kazhdan and Lusztig [7]. They were defined by a so-called cellular basis with some nice properties. The theory of cellular algebras provides a systematic framework for studying the representation theory of many important algebras. One can parameterize simple modules for a finite-dimensional cellular algebra by methods of linear algebra. Many classes of algebras from mathematics and physics are found to be cellular, including Hecke algebras of finite type, Ariki–Koike algebras, q-Schur algebras, Brauer algebras, Temperley–Lieb algebras, cyclotomic Temperley–Lieb algebras, Jones algebras, partition algebras, Birman–Wenzl algebras and so on. We refer the reader to [4, 6, 15, 16, 17] for details.

An equivalent basis-free definition of cellular algebras was given by Koenig and Xi [8]. It is useful in dealing with structural problems. Using this definition, Koenig and Xi [9] made explicit an inductive construction of cellular algebras which is called inflation. It can produce all cellular algebras. In [10], Brauer algebras were shown to be iterated inflations of group algebras of symmetric groups. Then more information about Brauer algebras was found.

Recently, Koenig and Xi [11] introduced affine cellular algebras which contain cellular algebras as special cases. Affine Hecke algebras of type A and infinite-dimensional diagram algebras like the affine Temperley–Lieb algebras are affine cellular.

It is an open problem to find explicit formulas for the dimensions of simple modules of a cellular algebra. By the theory of cellular algebras,

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this is equivalent to determining the dimensions of the radicals of bilinear forms associated with cell modules. In [12], for a quasi-hereditary cellular algebra, Lehrer and Zhang found that the radicals of bilinear forms are related to the radical of the algebra. This leads us to studying the radical of a cellular algebra. However, we have no idea for dealing with general cellular algebras now. We will do some work on the radicals of *symmetric* cellular algebras in this paper. Note that Hecke algebras of finite types, Ariki–Koike algebras over any ring containing inverses of the parameters, Khovanov's diagram algebras are all symmetric cellular algebras. The trivial extension of a cellular algebra is also a symmetric cellular algebra. For details, see [1], [13], [18].

We begin with definitions of and some well-known results on symmetric algebras and cellular algebras in Section 2. Then in Section 3, for a symmetric cellular algebra, we study properties of the dual basis of a cellular basis. In Section 4, a nilpotent ideal of a symmetric cellular algebra is constructed. This ideal connects the radicals of cell modules with the radical of the algebra and also yields some information on the dimensions of simple modules. As a by-product, we obtain some equivalent conditions for a finite-dimensional symmetric cellular algebra to be semisimple.

2. Preliminaries. In this section, we start from the definitions of symmetric algebras and cellular algebras (a slightly weaker version due to Goodman) and then recall some well-known results about them.

Let R be a commutative ring with identity and A an associative R-algebra. As an R-module, A is finitely generated and free. Let $f : A\times A \to R$ be an R-bilinear map. We say that f is *non-degenerate* if the determinant of the matrix $(f(a_i, a_j))_{a_i,a_j\in B}$ is a unit in R for some R-basis B of A. We say f is *associative* if $f(ab, c) = f(a, bc)$ for all $a, b, c \in A$, and *symmetric* if $f(a, b) = f(b, a)$ for all $a, b \in A$.

Definition 2.1. An R-algebra A is called *symmetric* if there is a non-degenerate associative symmetric bilinear form f on A. Define an R-linear map $\tau : A \to R$ by $\tau(a) = f(a, 1)$. We call τ a *symmetrizing trace*.

Let A be a symmetric algebra with a basis $B = \{a_i \mid i = 1, \dots, n\}$ and τ a symmetrizing trace. Denote by $D = \{D_i \mid i = i, \dots, n\}$ the basis determined by the requirement that $\tau(D_j a_i) = \delta_{ij}$ for all $i, j = 1, \dots, n$. We will call D the *dual basis* of B. For arbitrary $1 \le i, j \le n$, write $a_i a_j = \sum_k r_{ijk} a_k$, where $r_{ijk} \in R$. Fixing a symmetrizing trace τ for A, we have the following lemma.

Lemma 2.2. *Let A be a symmetric R-algebra with a basis B and dual basis D. Then*

$$a_i D_j = \sum_k r_{kij} D_k, \qquad D_i a_j = \sum_k r_{jki} D_k.$$

Proof. We only prove the first equation. The other one is proved similarly.

Suppose that $a_i D_j = \sum_k r_k D_k$, where $r_k \in R$ for $k = 1, \ldots, n$. Left multiplying by a_{k_0} and then applying τ, we get $\tau(a_{k_0} a_i D_j) = r_{k_0}$. Clearly, $\tau(a_{k_0} a_i D_j) = r_{k_0,i,j}$. This implies that $r_{k_0} = r_{k_0,i,j}$. ∎

Given a symmetric algebra, it is natural to consider the relation between two dual bases determined by two different symmetrizing traces. For this we have the following lemma.

LEMMA 2.3. *Suppose that A is a symmetric R-algebra with a basis $B = \{a_i \mid i = 1, \ldots, n\}$. Let τ, τ' be two symmetrizing traces. Denote by $\{D_i \mid i = 1, \ldots, n\}$ the dual basis of B determined by τ and $\{D'_i \mid i = 1, \ldots, n\}$ the dual basis determined by τ'. Then for $1 \le i \le n$, we have*

$$D'_i = \sum_{j=1}^{n} \tau(a_j D'_i) D_j.$$

Proof. This is proved by a similar method to Lemma 2.2. ∎

Graham and Lehrer [6] introduced the so-called cellular algebras; then Goodman [3] gave a slightly weaker version. Throughout this paper, we will adopt Goodman's definition for the sake of being more general.

DEFINITION 2.4 ([3, Defnition 2.9]). Let R be a commutative ring with identity. An associative unital R-algebra is called a *cellular algebra* with *cell datum* (Λ, M, C, i) if the following conditions are satisfied:

(C1) The finite set Λ is a poset. Associated with each $\lambda \in \Lambda$, there is a finite set $M(\lambda)$. The algebra A has an R-basis $\{C^{\lambda}_{S,T} \mid S, T \in M(\lambda),\ \lambda \in \Lambda\}$.

(C2) The map i is an R-linear anti-automorphism of A with $i^2 = \mathrm{id}$ and
$$i(C^{\lambda}_{S,T}) \equiv C^{\lambda}_{T,S} \pmod{A(<\lambda)}$$
for all $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, where $A(<\lambda)$ is the R-submodule of A generated by $\{C^{\mu}_{S'',T''} \mid S'', T'' \in M(\mu),\ \mu < \lambda\}$.

(C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, then for any $a \in A$, we have
$$aC^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^{\lambda}_{S',T} \pmod{A(<\lambda)},$$
where $r_a(S', S) \in R$ is independent of T.

Applying i to the equation in (C3), we obtain

(C3′) $C^{\lambda}_{T,S} i(a) \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^{\lambda}_{T,S'} \pmod{A(<\lambda)}$.

REMARK 2.5. Graham and Lehrer's [6] original definition requires that $i(C^{\lambda}_{S,T}) = C^{\lambda}_{T,S}$ for all $\lambda \in \Lambda$ and $S, T \in M(\lambda)$. But Goodman pointed out that the results of [6] remained valid with his weaker axiom. In case $2 \in R$

is invertible, these two definitions are equivalent. For details, we refer the reader to [4].

It is easy to check the following lemma by Definition 2.4.

LEMMA 2.6 ([6, Lemma 1.7]). *Let $\lambda \in \Lambda$ and $a \in A$. Then for arbitrary elements $S, T, U, V \in M(\lambda)$, we have*

$$C^{\lambda}_{S,T} a C^{\lambda}_{U,V} \equiv \Phi_a(T, U) C^{\lambda}_{S,V} \pmod{A(<\lambda)},$$

where $\Phi_a(T, U) \in R$ depends only on a, T and U.

We often omit the index a when $a = 1$, that is, write $\Phi(T, U)$ for $\Phi_1(T, U)$.

Let us now recall the definition of cell modules.

DEFINITION 2.7 ([6, Definition 2.1]). Let A be a cellular algebra with cell datum (Λ, M, C, i). For each $\lambda \in \Lambda$, define the left A-module $W(\lambda)$ as follows: $W(\lambda)$ is a free R-module with basis $\{C_S \mid S \in M(\lambda)\}$ and A-action defined by

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S', S) C_{S'} \quad (a \in A,\ S \in M(\lambda)),$$

where $r_a(S', S)$ is the element of R defined in Definition 2.4 (C3).

LEMMA 2.8 ([6, Lemma 2.2]). *There is a natural isomorphism of A-bimodules*

$$C^{\lambda} : W(\lambda) \otimes_R i(W(\lambda)) \to R\text{-span}\{C^{\lambda}_{S,T} \mid S, T \in M(\lambda)\},$$

defined by $(C_S, C_T) \mapsto C^{\lambda}_{S,T}$.

For a cell module $W(\lambda)$, define a bilinear form

$$\Phi_{\lambda} : W(\lambda) \times W(\lambda) \to R$$

by $\Phi_{\lambda}(C_S, C_T) = \Phi(S, T)$. It plays an important role in studying the structure of $W(\lambda)$. It is easy to check that $\Phi(T, U) = \Phi(U, T)$ for arbitrary $T, U \in M(\lambda)$.

Define

$$\operatorname{rad}\lambda := \{x \in W(\lambda) \mid \Phi_{\lambda}(x, y) = 0 \text{ for all } y \in W(\lambda)\}.$$

If $\Phi_{\lambda} \neq 0$, then $\operatorname{rad}\lambda$ is the radical of the A-module $W(\lambda)$. Moreover, if λ is a maximal element in Λ, then $\operatorname{rad}\lambda = 0$.

The following results were proved by Graham and Lehrer [6].

THEOREM 2.9 ([6, Theorem 3.4]). *Let K be a field and A a finite dimensional cellular algebra. For any $\lambda \in \Lambda$, denote the A-module $W(\lambda)/\operatorname{rad}\lambda$ by L_{λ}. Let $\Lambda_0 = \{\lambda \in \Lambda \mid \Phi_{\lambda} \neq 0\}$. Then $\{L_{\lambda} \mid \lambda \in \Lambda_0\}$ is a complete set of (representatives of equivalence classes of) absolutely simple A-modules.*

THEOREM 2.10 ([6, Theorem 3.8]). *Let K be a field and A a cellular K-algebra. Then the following are equivalent:*

(1) *The algebra A is semisimple.*
(2) *The nonzero cell representations $W(\lambda)$ are irreducible and pairwise inequivalent.*
(3) *The form Φ_λ is nondegenerate* (*i.e.* $\operatorname{rad}\lambda = 0$) *for each $\lambda \in \Lambda$.*

For $\lambda \in \Lambda$, fix an order on $M(\lambda)$ and let $M(\lambda) = \{S_1, \ldots, S_{n_\lambda}\}$, where n_λ is the number of elements in $M(\lambda)$. The matrix

$$G(\lambda) = (\Phi(S_i, S_j))_{1 \le i,j \le n_\lambda}$$

is called the *Gram matrix*. It is easy to know that all the determinants of $G(\lambda)$ defined using different orders on $M(\lambda)$ are the same. By the definition of $G(\lambda)$ and $\operatorname{rad}\lambda$, for a finite-dimensional cellular algebra A, it is clear that if $\Phi_\lambda \neq 0$, then $\dim_K L_\lambda = \operatorname{rank} G(\lambda)$.

3. Symmetric cellular algebras. In this section, we study properties of the dual basis of a cellular basis for a symmetric cellular algebra. We will prove that the dual basis is "almost" cellular.

Let A be a symmetric cellular algebra with cell datum (Λ, M, C, i). Throughout, denote the dual basis by $D = \{D^\lambda_{S,T} \mid S, T \in M(\lambda),\ \lambda \in \Lambda\}$, which satisfies

$$\tau(C^\lambda_{S,T} D^\mu_{U,V}) = \delta_{\lambda\mu}\delta_{SV}\delta_{TU}.$$

For any $\lambda, \mu \in \Lambda$, $S, T \in M(\lambda)$, $U, V \in M(\mu)$, write

$$C^\lambda_{S,T} C^\mu_{U,V} = \sum_{\epsilon \in \Lambda,\, X, Y \in M(\epsilon)} r_{(S,T,\lambda),(U,V,\mu),(X,Y,\epsilon)} C^\epsilon_{X,Y}.$$

The lemma below plays an important role throughout this paper.

Lemma 3.1. *Let A be a symmetric cellular algebra with cell datum (Λ, M, C, i) and τ a given symmetrizing trace. For arbitrary $\lambda, \mu \in \Lambda$ and $S, T, P, Q \in M(\lambda)$, $U, V \in M(\mu)$, the following hold:*

(1) $D^\mu_{U,V} C^\lambda_{S,T} = \sum_{\epsilon \in \Lambda, X, Y \in M(\epsilon)} r_{(S,T,\lambda),(Y,X,\epsilon),(V,U,\mu)} D^\epsilon_{X,Y}$.
(2) $C^\lambda_{S,T} D^\mu_{U,V} = \sum_{\epsilon \in \Lambda, X, Y \in M(\epsilon)} r_{(Y,X,\epsilon),(S,T,\lambda),(V,U,\mu)} D^\epsilon_{X,Y}$.
(3) $C^\lambda_{S,T} D^\lambda_{T,Q} = C^\lambda_{S,P} D^\lambda_{P,Q}$.
(4) $D^\lambda_{T,S} C^\lambda_{S,Q} = D^\lambda_{T,P} C^\lambda_{P,Q}$.
(5) $C^\lambda_{S,T} D^\lambda_{P,Q} = 0$ *if* $T \neq P$.
(6) $D^\lambda_{P,Q} C^\lambda_{S,T} = 0$ *if* $Q \neq S$.
(7) $C^\lambda_{S,T} D^\mu_{U,V} = 0$ *if* $\mu \nleq \lambda$.
(8) $D^\mu_{U,V} C^\lambda_{S,T} = 0$ *if* $\mu \nleq \lambda$.

Proof. (1), (2) are corollaries of Lemma 2.2. The equations (5)–(8) are corollaries of (1) and (2). We now prove (3).

By (2), we have

$$C^{\lambda}_{S,T}D^{\lambda}_{T,Q} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} r_{(Y,X,\epsilon),(S,T,\lambda),(Q,T,\lambda)} D^{\epsilon}_{X,Y},$$

$$C^{\lambda}_{S,P}D^{\lambda}_{P,S} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} r_{(Y,X,\epsilon),(S,P,\lambda),(Q,P,\lambda)} D^{\epsilon}_{X,Y}.$$

On the other hand, by (C3) of Definition 2.4 we also have

$$r_{(Y,X,\epsilon),(S,T,\lambda),(Q,T,\lambda)} = r_{(Y,X,\epsilon),(S,P,\lambda),(Q,P,\lambda)}$$

for all $\epsilon \in \Lambda$ and $X, Y \in M(\epsilon)$. This completes the proof of (3).

(4) is proved similarly. ∎

LEMMA 3.2. *Let A be a symmetric cellular algebra with cell datum (Λ, M, C, i), and $D = \{D^{\lambda}_{S,T} \mid S, T \in M(\lambda),\ \lambda \in \Lambda\}$ the dual basis. Then* (C3) *of Definition* 2.4 *holds with respect to the opposite order on Λ. Moreover,*

$$i(D^{\lambda}_{S,T}) \equiv r_{T,S,\lambda} D^{\lambda}_{T,S} \pmod{A_D(>\lambda)},$$

where $r_{T,S,\lambda} \in R$ and where $A_D(>\lambda)$ is the R-submodule of A generated by $\{D^{\mu}_{U,V} \mid U, V \in M(\mu),\ \mu > \lambda\}$.

Proof. Let us prove (C3) holds. For arbitrary $C^{\lambda}_{S,T}$, by Lemma 3.1(2), we have

$$C^{\lambda}_{S,T}D^{\mu}_{U,V} = \sum_{\epsilon\in\Lambda,\ X,Y\in M(\epsilon)} r_{(Y,X,\epsilon),(S,T,\lambda),(V,U,\mu)} D^{\epsilon}_{X,Y}.$$

By (C3) of Definition 2.4, if $\epsilon < \mu$, then $r_{(Y,X,\epsilon),(S,T,\lambda),(V,U,\mu)} = 0$. Therefore,

$$C^{\lambda}_{S,T}D^{\mu}_{U,V} \equiv \sum_{X,Y\in M(\mu)} r_{(Y,X,\mu),(S,T,\lambda),(V,U,\mu)} D^{\mu}_{X,Y} \pmod{A_D(>\mu)},$$

where $A_D(>\mu)$ is the R-submodule of A generated by

$$\{D^{\eta}_{S'',T''} \mid S'', T'' \in M(\lambda),\ \eta > \mu\}.$$

By (C3′) of Definition 2.4, if $Y \neq V$, then $r_{(Y,X,\mu),(S,T,\lambda),(V,U,\mu)} = 0$. So

$$C^{\lambda}_{S,T}D^{\mu}_{U,V} \equiv \sum_{X\in M(\mu)} r_{(V,X,\mu),(S,T,\lambda),(V,U,\mu)} D^{\mu}_{X,V} \pmod{A_D(>\mu)}.$$

Clearly, for arbitrary $X \in M(\mu)$, we have

$$r_{(V,X,\mu),(S,T,\lambda),(V,U,\mu)} = r_{C^{\lambda}_{T,S}}(U, X),$$

which is independent of V. Since $C^{\lambda}_{S,T}$ is arbitrary,

$$aD^{\mu}_{U,V} \equiv \sum_{U'\in M(\mu)} r_{i(a)}(U, U') D^{\mu}_{U',V} \pmod{A_D(>\mu)}$$

for any $a \in A$. By Definition 2.4, $r_{i(a)}(U,U')$ is independent of V. Thus we have completed the proof of (C3).

Let

$$i(D^\lambda_{S,T}) = \sum_{\epsilon\in\Lambda,\, X,Y\in M(\epsilon)} r_{X,Y,\epsilon} D^\epsilon_{X,Y}$$

with $r_{X,Y,\epsilon} \in R$. If there exists $\eta \not\geq \lambda$ such that $r_{P,Q,\eta} \neq 0$ for some $P,Q \in M(\eta)$, then $\tau(i(D^\lambda_{S,T})C^\eta_{Q,P}) = r_{P,Q,\eta} \neq 0$. This implies that $i(D^\lambda_{S,T})C^\eta_{Q,P} \neq 0$. Thus $C^\eta_{P,Q}D^\lambda_{S,T} \neq 0$. But we know $\eta \not\geq \lambda$, so by Lemma 3.1(7), $C^\eta_{P,Q}D^\lambda_{S,T} = 0$, a contradiction. This implies that

$$i(D^\lambda_{S,T}) \equiv \sum_{X,Y\in M(\lambda)} r_{X,Y,\lambda} D^\lambda_{X,Y} \pmod{A_D(>\lambda)}.$$

Now assume $r_{U,V,\lambda} \neq 0$. Then $i(D^\lambda_{S,T})C^\lambda_{V,U} \neq 0$, hence $C^\lambda_{U,V}D^\lambda_{S,T} \neq 0$. By Lemma 3.1(5), $V = S$. We can get $U = T$ similarly. This completes the proof of the lemma. ∎

Remark. Obviously, if $r_{T,S,\lambda} = 1$ for all $\lambda \in \Lambda$ and $S,T \in M(\lambda)$, then the dual basis $D^\lambda_{T,S}$ is again cellular with respect to the opposite order on λ.

It is easy to check that for arbitrary elements $S,T,U,V \in M(\lambda)$,

$$D^\lambda_{S,T}D^\lambda_{U,V} \equiv \Psi(T,U)D^\lambda_{S,V} \pmod{A(>\lambda)},$$

where $\Psi(T,U) \in R$ depends only on T and U. Then we also have Gram matrices $G'(\lambda)$ defined by the dual basis. Now it is natural to ask about the relation between $G(\lambda)$ and $G'(\lambda)$. To study this, we need the following lemma.

Lemma 3.3. *Let A be a symmetric cellular algebra with cell datum (Λ, M, C, i). For every $\lambda \in \Lambda$ and $S,T,U,V,P \in M(\lambda)$, we have*

$$C^\lambda_{S,T}D^\lambda_{T,U}C^\lambda_{U,V}D^\lambda_{V,P} = \sum_{Y\in M(\lambda)} \Phi(Y,V)\Psi(Y,V)C^\lambda_{S,T}D^\lambda_{T,P}.$$

Proof. By Lemma 3.1(1), we have

$$\begin{aligned} C^\lambda_{S,T}D^\lambda_{T,U}C^\lambda_{U,V}D^\lambda_{V,P} &= C^\lambda_{S,T}(D^\lambda_{T,U}C^\lambda_{U,V})D^\lambda_{V,P} \\ &= \sum_{\epsilon\in\Lambda,\, X,Y\in M(\epsilon)} r_{(U,V,\lambda),(Y,X,\epsilon),(U,T,\lambda)} C^\lambda_{S,T}D^\epsilon_{X,Y}D^\lambda_{V,P}. \end{aligned}$$

If $\varepsilon > \lambda$, then by Lemma 3.1(7), $C^\lambda_{S,T}D^\epsilon_{X,Y} = 0$; if $\varepsilon < \lambda$, then by Definition 2.4 (C3), $r_{(U,V,\lambda),(Y,X,\epsilon),(U,T,\lambda)} = 0$. This implies that

$$\sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} r_{(U,V,\lambda),(Y,X,\epsilon),(U,T,\lambda)} C^{\lambda}_{S,T} D^{\epsilon}_{X,Y} D^{\lambda}_{V,P} = \sum_{X,Y\in M(\lambda)} r_{(U,V,\lambda),(Y,X,\lambda),(U,T,\lambda)} C^{\lambda}_{S,T} D^{\lambda}_{X,Y} D^{\lambda}_{V,P}.$$

By Definition 2.4 (C3), if $X \neq T$, then $r_{(U,V,\lambda),(Y,X,\lambda),(U,T,\lambda)} = 0$. Hence,

$$\sum_{X,Y\in M(\lambda)} r_{(U,V,\lambda),(Y,X,\lambda),(U,T,\lambda)} C^{\lambda}_{S,T} D^{\lambda}_{X,Y} D^{\lambda}_{V,P} = \sum_{Y\in M(\lambda)} r_{(U,V,\lambda),(Y,T,\lambda),(U,T,\lambda)} C^{\lambda}_{S,T} D^{\lambda}_{T,Y} D^{\lambda}_{V,P}.$$

Note that

$$D^{\lambda}_{T,Y} D^{\lambda}_{V,P} \equiv \Psi(Y,V) D^{\lambda}_{T,P} \pmod{A_D(>\lambda)}.$$

Moreover, by Lemma 3.1(7), if $\epsilon > \lambda$, then $C^{\lambda}_{S,T} D^{\epsilon}_{X,Y} = 0$. Thus

$$\sum_{Y\in M(\lambda)} r_{(U,V,\lambda),(Y,T,\lambda),(U,T,\lambda)} C^{\lambda}_{S,T} D^{\lambda}_{T,Y} D^{\lambda}_{V,P} = \sum_{Y\in M(\lambda)} \Phi(Y,V)\Psi(Y,V) C^{\lambda}_{S,T} D^{\lambda}_{T,P}.$$

This completes the proof. ∎

By Lemma 3.1, $C^{\lambda}_{U,V} D^{\lambda}_{V,P}$ is independent of the choice of V, hence so is $\sum_{Y\in M(\lambda)} \Phi(Y,V)\Psi(Y,V)$. Then for any $\lambda \in \Lambda$, we can define a constant $k_{\lambda,\tau}$ as follows.

Definition 3.4. Keep the notation above. For $\lambda \in \Lambda$, take an arbitrary $V \in M(\lambda)$. Define

$$k_{\lambda,\tau} = \sum_{X\in M(\lambda)} \Phi(X,V)\Psi(X,V).$$

It is helpful to note that $\{k_{\lambda,\tau} \mid \lambda \in \Lambda\}$ is not independent of the choice of symmetrizing trace. Fixing a symmetrizing trace τ, we often write $k_{\lambda,\tau}$ as k_λ. The following lemma gives the relation among $G(\lambda)$, $G'(\lambda)$ and k_λ.

Lemma 3.5. *Let A be a symmetric cellular algebra with cell datum (Λ, M, C, i). For any $\lambda \in \Lambda$, fix an order on the set $M(\lambda)$. Then $G(\lambda)G'(\lambda) = k_\lambda E$, where E is the identity matrix.*

Proof. For an arbitrary $\lambda \in \Lambda$, according to the definition of $G(\lambda)$, $G'(\lambda)$ and k_λ, we only need to show that $\sum_{Y\in M(\lambda)} \Phi(Y,U)\Psi(Y,V) = 0$ for arbitrary $U, V \in M(\lambda)$ with $U \neq V$.

In fact, on one hand, for arbitrary $S \in M(\lambda)$, by Lemma 3.1(5), $U \neq V$ implies that $C^{\lambda}_{S,U}D^{\lambda}_{V,S} = 0$. Then $C^{\lambda}_{S,U}D^{\lambda}_{U,S}C^{\lambda}_{S,U}D^{\lambda}_{V,S} = 0$. On the other hand, by a similar method to the proof of Lemma 3.3,

$$\begin{aligned}
C^{\lambda}_{S,U}D^{\lambda}_{U,S}C^{\lambda}_{S,U}D^{\lambda}_{V,S} &= \sum_{\epsilon\in\Lambda,\, X,Y\in M(\epsilon)} r_{(S,U,\lambda),(Y,X,\epsilon),(S,U,\lambda)}C^{\lambda}_{S,U}D^{\epsilon}_{X,Y}D^{\lambda}_{V,S} \\
&= \sum_{Y\in M(\lambda)} r_{(S,U,\lambda),(Y,U,\lambda),(S,U,\lambda)}C^{\lambda}_{S,U}D^{\lambda}_{U,Y}D^{\lambda}_{V,S} \\
&= \sum_{Y\in M(\lambda)} \Phi(Y,U)\Psi(Y,V)C^{\lambda}_{S,U}D^{\lambda}_{U,S}.
\end{aligned}$$

Then $\sum_{Y\in M(\lambda)} \Phi(Y,U)\Psi(Y,V)C^{\lambda}_{S,U}D^{\lambda}_{U,S} = 0$. This implies that

$$\tau\Big(\sum_{Y\in M(\lambda)} \Phi(Y,U)\Psi(Y,V)C^{\lambda}_{S,U}D^{\lambda}_{U,S}\Big) = 0.$$

Since $\tau(C^{\lambda}_{S,U}D^{\lambda}_{U,S}) = 1$, we have $\sum_{Y\in M(\lambda)} \Phi(Y,U)\Psi(Y,V) = 0$. ∎

COROLLARY 3.6. *Let A be a symmetric cellular algebra over an integral domain R. Then $k_\lambda = 0$ for any $\lambda \in \Lambda$ with* $\operatorname{rad}\lambda \neq 0$.

Proof. Since $|G(\lambda)| = 0$ is equivalent to $\operatorname{rad}\lambda \neq 0$, Lemma 3.5 shows that $\operatorname{rad}\lambda \neq 0$ implies $k_\lambda = 0$. ∎

4. Radicals of symmetric cellular algebras. To study radicals of symmetric cellular algebras, we need the following lemma.

LEMMA 4.1. *Let A be a symmetric cellular algebra. Then for any $\lambda \in \Lambda$, the elements of the form $\sum_{S,U\in M(\lambda)} r_{SU}C^{\lambda}_{S,V}D^{\lambda}_{V,U}$ with $r_{SU} \in R$ form an ideal of A.*

Proof. Denote the set of elements of the above form by I^λ. Then we claim that $C^{\eta}_{P,Q}C^{\lambda}_{S,V}D^{\lambda}_{V,U} \in I^\lambda$ for any $\eta \in \Lambda$, $P,Q \in M(\eta)$, and $S,U \in M(\lambda)$. In fact, by (C3) of Definition 2.4 and Lemma 3.1(7),

$$\begin{aligned}
C^{\eta}_{P,Q}C^{\lambda}_{S,V}D^{\lambda}_{V,U} &= \sum_{\epsilon\in\Lambda,\, X,Y\in M(\epsilon)} r_{(P,Q,\eta),(S,V,\lambda),(X,Y,\epsilon)}C^{\epsilon}_{X,Y}D^{\lambda}_{V,U} \\
&= \sum_{X\in M(\lambda)} r_{(P,Q,\eta),(S,V,\lambda),(X,V\lambda)}C^{\lambda}_{X,V}D^{\lambda}_{V,U}.
\end{aligned}$$

That $C^{\lambda}_{S,V}D^{\lambda}_{V,U}C^{\eta}_{P,Q} \in I^\lambda$ is proved similarly. ∎

We will denote $\sum_{\lambda\in\Lambda,\, k_\lambda=0} I^\lambda$ by I^Λ. Similarly, for each $\lambda \in \Lambda$, the elements of the form $\sum_{S,U\in M(\lambda)} r_{U,S}D^{\lambda}_{U,V}C^{\lambda}_{V,S}$ with $r_{U,S} \in R$ also form an ideal I^λ_D of A. Denote $\sum_{\lambda\in\Lambda,\, k_\lambda=0} I^\lambda_D$ by I^Λ_D and set $I = I^\Lambda + I^\Lambda_D$.

Define

$$\Lambda_1 = \{\lambda \in \Lambda \mid \operatorname{rad}\lambda = 0\}, \quad \Lambda_2 = \Lambda_0 - \Lambda_1,$$
$$\Lambda_3 = \Lambda - \Lambda_0, \quad \Lambda_4 = \{\lambda \in \Lambda_1 \mid k_\lambda = 0\}.$$

Now we are in a position to give the main result of this paper.

THEOREM 4.2. *Suppose that R is an integral domain and that A is a symmetric cellular algebra with a cellular basis $C = \{C^\lambda_{S,T} \mid S,T \in M(\lambda),\ \lambda \in \Lambda\}$. Let τ be a symmetrizing trace on A and let $\{D^\lambda_{T,S} \mid S,T \in M(\lambda),\ \lambda \in \Lambda\}$ be the dual basis of C with respect to τ. Then*

(1) *$I \subseteq \operatorname{rad} A$ and $I^3 = 0$.*
(2) *I is independent of the choice of τ.*

Moreover, if R is a field, then

(3) $\dim_R I \geq \sum_{\lambda\in\Lambda_2} (n_\lambda + \dim_R \operatorname{rad}\lambda) \dim_R L_\lambda + \sum_{\lambda\in\Lambda_4} n_\lambda^2$,
where n_λ is the number of elements in $M(\lambda)$.

(4) $\sum_{\lambda\in\Lambda_2} (\dim_K L_\lambda)^2 - \sum_{\lambda\in\Lambda_3} n_\lambda^2 \leq \sum_{\lambda\in\Lambda_2} (\dim_K \operatorname{rad}\lambda)^2 - \sum_{\lambda\in\Lambda_4} n_\lambda^2.$

Proof. (1) $I \subseteq \operatorname{rad} A$ and $I^3 = 0$.

Firstly, we prove $(I^\Lambda)^2 = 0$. Obviously, by the definition of I^Λ, every element of $(I^\Lambda)^2$ can be written as a linear combination of elements of the form $C^\lambda_{S_1,T} D^\lambda_{T,S_2} C^\mu_{U_1,V} D^\mu_{V,U_2}$(we omit the coefficient here) with $k_\lambda = 0$ and $k_\mu = 0$.

If $\mu < \lambda$, then $C^\lambda_{S_1,T} D^\lambda_{T,S_2} C^\mu_{U_1,V} D^\mu_{V,U_2} = 0$ by Lemma 3.1(8).

If $\mu > \lambda$, then by Lemma 3.1(1)&(7),

$$C^\lambda_{S_1,T} D^\lambda_{T,S_2} C^\mu_{U_1,V} D^\mu_{V,U_2} = \sum_{Y\in M(\lambda)} r_{(U_1,V,\mu),(Y,T,\lambda),(S_2,T,\lambda)} C^\lambda_{S_1,T} D^\lambda_{T,Y} D^\mu_{V,U_2}.$$

However, by Lemma 3.2, every $D^\eta_{P,Q}$ with nonzero coefficient in the expansion of $D^\lambda_{T,Y} D^\mu_{V,U_2}$ satisfies $\eta \geq \mu$. Since $\mu > \lambda$, we have $\eta > \lambda$. Now, by Lemma 3.1(7), we get $C^\lambda_{S_1,T} D^\eta_{P,Q} = 0$, that is, $C^\lambda_{S_1,T} D^\lambda_{T,S_2} C^\mu_{U_1,V} D^\mu_{V,U_2} = 0$ if $\mu > \lambda$.

If $\lambda = \mu$, by Lemma 3.1(3)&(4), we only need to consider elements of the form

$$C^\lambda_{S_1,T_1} D^\lambda_{T_1,S_2} C^\lambda_{S_2,T_2} D^\lambda_{T_2,S_3}.$$

By Lemmas 3.3 and 3.6,

$$C^\lambda_{S_1,T_1} D^\lambda_{T_1,S_2} C^\lambda_{S_2,T_2} D^\lambda_{T_2,S_3} = k_\lambda C^\lambda_{S_1,T_1} D^\lambda_{T_1,S_3} = 0.$$

Then we see that all the elements of the form $C^\lambda_{S_1,T} D^\lambda_{T,S_2} C^\mu_{U_1,V} D^\mu_{V,U_2}$ are zero, that is, $(I^\Lambda)^2 = 0$.

Similarly, we get $(I^\Lambda_D)^2 = 0$.

To prove $I^3 = 0$, we now only need to consider elements in $I^{\Lambda} I_D^{\Lambda} I^{\Lambda}$ and $I_D^{\Lambda} I^{\Lambda} I_D^{\Lambda}$. For $\lambda, \mu, \eta \in \Lambda$ with $k_\lambda = k_\mu = k_\eta = 0$ and $S, T, M \in M(\lambda)$, $U, V, N \in M(\mu)$, $P, Q, W \in M(\eta)$, suppose that

$$C^{\lambda}_{S,T} D^{\lambda}_{T,M} D^{\mu}_{U,V} C^{\mu}_{V,N} C^{\eta}_{P,Q} D^{\eta}_{Q,W} \neq 0.$$

If $\lambda > \mu$, then any $D^{\epsilon}_{X,Y}$ with nonzero coefficient in the expansion of $D^{\lambda}_{T,M} D^{\mu}_{U,V}$ satisfies $\epsilon \geq \lambda$, so $\epsilon > \mu$; this implies that $D^{\epsilon}_{X,Y} C^{\mu}_{V,N} = 0$ by Lemma 3.1, a contradiction. If $\lambda < \mu$, then any $D^{\epsilon}_{X,Y}$ with nonzero coefficient in the expansion of $D^{\lambda}_{T,M} D^{\mu}_{U,V}$ satisfies $\epsilon \geq \mu$, so $\epsilon > \lambda$; this implies that $C^{\lambda}_{S,T} D^{\epsilon}_{X,Y} = 0$ by Lemma 3.1, a contradiction. Thus $\lambda = \mu$. Similarly, we get $\eta = \mu$. By direct computation, we can also get $C^{\lambda}_{S,T} D^{\lambda}_{T,M} D^{\mu}_{U,V} C^{\mu}_{V,N} C^{\eta}_{P,Q} D^{\eta}_{Q,W} = 0$. This implies that $I^{\Lambda} I_D^{\Lambda} I^{\Lambda} = 0$. Similarly $I_D^{\Lambda} I^{\Lambda} I_D^{\Lambda} = 0$ is proved. Then $I^3 = 0$ follows.

Now it is clear that $I \subseteq \operatorname{rad} A$ for I is a nilpotent ideal of A.

(2) *I is independent of the choice of τ.*

Let τ and τ' be two symmetrizing traces and D, d the dual bases determined by τ and τ' respectively. By Lemma 2.3, for arbitrary $d^{\lambda}_{U,V} \in d$,

$$d^{\lambda}_{U,V} = \sum_{\varepsilon \in \Lambda,\, X, Y \in M(\varepsilon)} \tau(C^{\varepsilon}_{X,Y} d^{\lambda}_{U,V}) D^{\varepsilon}_{Y,X}.$$

Then for arbitrary $S \in M(\lambda)$,

$$C^{\lambda}_{S,U} d^{\lambda}_{U,V} = \sum_{\varepsilon \in \Lambda,\, X, Y \in M(\varepsilon)} \tau(C^{\varepsilon}_{X,Y} d^{\lambda}_{U,V}) C^{\lambda}_{S,U} D^{\varepsilon}_{Y,X}.$$

By Lemma 3.1(7)&(8), if $\varepsilon < \lambda$, then $C^{\varepsilon}_{X,Y} d^{\lambda}_{U,V} = 0$; if $\varepsilon > \lambda$, then $C^{\lambda}_{S,U} D^{\varepsilon}_{Y,X} = 0$. This implies that

$$C^{\lambda}_{S,U} d^{\lambda}_{U,V} = \sum_{X, Y \in M(\lambda)} \tau(C^{\lambda}_{X,Y} d^{\lambda}_{U,V}) C^{\lambda}_{S,U} D^{\lambda}_{Y,X}.$$

By Lemma 3.1(5), if $Y \neq U$, then $C^{\lambda}_{S,U} D^{\lambda}_{Y,X} = 0$. Hence

$$C^{\lambda}_{S,U} d^{\lambda}_{U,V} = \sum_{X \in M(\lambda)} \tau(C^{\lambda}_{X,U} d^{\lambda}_{U,V}) C^{\lambda}_{S,U} D^{\lambda}_{U,X}.$$

Note that $\tau(C^{\lambda}_{X,U} d^{\lambda}_{U,V}) = \tau(d^{\lambda}_{U,V} C^{\lambda}_{X,U})$. Then it follows from Lemma 3.1 that $d^{\lambda}_{U,V} C^{\lambda}_{X,U} = 0$ if $X \neq V$. Thus

$$C^{\lambda}_{S,U} d^{\lambda}_{U,V} = \tau(C^{\lambda}_{V,U} d^{\lambda}_{U,V}) C^{\lambda}_{S,U} D^{\lambda}_{U,V}.$$

Similarly, we obtain

$$C^\lambda_{S,U}D^\lambda_{U,V} = \tau'(C^\lambda_{V,U}D^\lambda_{U,V})C^\lambda_{S,U}d^\lambda_{U,V},$$
$$d^\lambda_{V,U}C^\lambda_{U,S} = \tau(C^\lambda_{V,U}d^\lambda_{U,V})D^\lambda_{V,U}C^\lambda_{U,S},$$
$$D^\lambda_{V,U}C^\lambda_{U,S} = \tau'(C^\lambda_{V,U}D^\lambda_{U,V})d^\lambda_{V,U}C^\lambda_{U,S}.$$

The above four formulas imply that I is independent of the choice of symmetrizing trace.

(3) $\dim_R I \geq \sum_{\lambda\in\Lambda_2}(n_\lambda + \dim_R \operatorname{rad}\lambda)\dim_R L_\lambda + \sum_{\lambda\in\Lambda_4} n_\lambda^2$.

For any $\lambda \in \Lambda_2$ and $S, T \in M(\lambda)$, it follows from Lemma 3.1 that

$$C^\lambda_{S,T}D^\lambda_{T,T} \equiv \sum_{X\in M(\lambda)} \Phi(X,S)D^\lambda_{X,T} \pmod{A_D(>\lambda)},$$
$$D^\lambda_{T,T}C^\lambda_{T,S} \equiv \sum_{Y\in M(\lambda)} \Phi(Y,S)D^\lambda_{T,Y} \pmod{A_D(>\lambda)}.$$

Let V be the R-space generated by the union of the sets

$$\Big\{\sum_{X\in M(\lambda)} \Phi(X,S)D^\lambda_{X,T} \;\Big|\; S,T \in M(\lambda)\Big\},$$
$$\Big\{\sum_{Y\in M(\lambda)} \Phi(Y,S)D^\lambda_{T,Y} \;\Big|\; S,T \in M(\lambda)\Big\}.$$

Then it is easy to deduce from the definition of I^λ and I^λ_D that

$$\dim_R(I^\lambda + I^\lambda_D) \geq \dim V.$$

On the other hand, since $\Phi_\lambda \neq 0$ and $\operatorname{rank} G_\lambda = \dim_R L_\lambda$, we have

$$\dim V = 2n_\lambda \dim_R L_\lambda - (\dim_R L_\lambda)^2,$$

that is, $\dim V = \dim_R L_\lambda \times (n_\lambda + \dim_R \operatorname{rad}\lambda)$. Thus

$$\dim_R(I^\lambda + I^\lambda_D) \geq \dim_R L_\lambda \times (n_\lambda + \dim_R \operatorname{rad}\lambda).$$

Clearly, the above inequality holds true for any $\lambda \in \Lambda_4$, so

$$\dim_R(I^\lambda + I^\lambda_D) \geq n_\lambda^2$$

for any $\lambda \in \Lambda_4$. It is clear by Lemma 3.2 that

$$\dim_R I \geq \sum_{\lambda\in\Lambda_2} \dim_R(I^\lambda + I^\lambda_D) + \sum_{\lambda\in\Lambda_4} n_\lambda^2,$$

and thus item (3) follows.

(4) $\sum_{\lambda\in\Lambda_2}(\dim_K L_\lambda)^2 - \sum_{\lambda\in\Lambda_3} n_\lambda^2 \leq \sum_{\lambda\in\Lambda_2}(\dim_K \operatorname{rad}\lambda)^2$.

By (1) and (3),

$$\dim_R \operatorname{rad} A \geq \sum_{\lambda\in\Lambda_2} (n_\lambda + \dim_R \operatorname{rad}\lambda) \dim_R L_\lambda + \sum_{\lambda\in\Lambda_4} n_\lambda^2.$$

By the formula

$$\dim_R \operatorname{rad} A = \dim_R A - \sum_{\lambda\in\Lambda_0} (\dim_R L_\lambda)^2,$$

we have

$$\dim_R A - \sum_{\lambda\in\Lambda_0} (\dim_R L_\lambda)^2 \geq \sum_{\lambda\in\Lambda_2} (n_\lambda + \dim_R \operatorname{rad}\lambda) \dim_R L_\lambda + \sum_{\lambda\in\Lambda_4} n_\lambda^2.$$

That is,

$$\sum_{\lambda\in\Lambda_3} n_\lambda^2 + \sum_{\lambda\in\Lambda_0} n_\lambda^2 - \sum_{\lambda\in\Lambda_0} (\dim_R L_\lambda)^2 \\ \geq \sum_{\lambda\in\Lambda_2} (n_\lambda + \dim_R \operatorname{rad}\lambda) \dim_R L_\lambda + \sum_{\lambda\in\Lambda_4} n_\lambda^2,$$

or

$$\sum_{\lambda\in\Lambda_3} n_\lambda^2 + \sum_{\lambda\in\Lambda_2} n_\lambda^2 - \sum_{\lambda\in\Lambda_2} (\dim_R L_\lambda)^2 \\ \geq \sum_{\lambda\in\Lambda_2} (n_\lambda + \dim_R \operatorname{rad}\lambda) \dim_R L_\lambda + \sum_{\lambda\in\Lambda_4} n_\lambda^2,$$

or

$$\sum_{\lambda\in\Lambda_2} (\dim_K L_\lambda)^2 - \sum_{\lambda\in\Lambda_3} n_\lambda^2 \\ \leq \sum_{\lambda\in\Lambda_2} n_\lambda^2 - \sum_{\lambda\in\Lambda_2} (n_\lambda + \dim_R \operatorname{rad}\lambda) \dim_R L_\lambda - \sum_{\lambda\in\Lambda_4} n_\lambda^2.$$

According to $\dim_R L_\lambda = n_\lambda - \dim_R \operatorname{rad}\lambda$, the right side of the above inequality is $\sum_{\lambda\in\Lambda_2} (\dim_K \operatorname{rad}\lambda)^2 - \sum_{\lambda\in\Lambda_4} n_\lambda^2$. This completes the proof. ∎

COROLLARY 4.3. *Let R be an integral domain and A a symmetric cellular algebra. Let λ be the minimal element in Λ. If $\operatorname{rad}\lambda \neq 0$, then R-span$\{C^\lambda_{S,T} \mid S, T \in M(\lambda)\} \subset \operatorname{rad} A$.*

Proof. If $a = \sum_{X,Y\in M(\lambda)} r_{X,Y} C^\lambda_{X,Y}$ is not in $\operatorname{rad} A$, then there exists some $D^\mu_{U,V}$ such that $aD^\mu_{U,V} \notin \operatorname{rad} A$. If $\mu \neq \lambda$, then $aD^\mu_{U,V} = 0$ by Lemma 3.1, so $aD^\mu_{u,V}$ is in $\operatorname{rad} A$. If $\mu = \lambda$, then $aD^\mu_{U,V} \in \operatorname{rad} A$ by Theorem 4.2. This is a contradiction. ∎

Corollary 4.4. *Let A be a finite-dimensional symmetric cellular algebra and $r \in \operatorname{rad} A$. Assume that $\lambda \in \Lambda$ satisfies:*

(1) *There exist $S, T \in M(\lambda)$ such that $C^{\lambda}_{S,T}$ appears in the expansion of r with nonzero coefficient.*
(2) *For any $\mu > \lambda$ and $U, V \in M(\mu)$, the coefficient of $C^{\mu}_{U,V}$ in the expansion of r is zero.*

Then $k_\lambda = 0$.

Proof. Since $r = \sum_{\varepsilon\in\Lambda,\, X,Y\in M(\varepsilon)} r_{X,Y,\varepsilon} C^{\varepsilon}_{X,Y} \in \operatorname{rad} A$, we have $rD^{\lambda}_{T,S} \in \operatorname{rad} A$. The conditions (1) and (2) imply that

$$rD^{\lambda}_{T,S} = \sum_{X\in M(\lambda)} r_{X,T,\lambda} C^{\lambda}_{X,T} D^{\lambda}_{T,S}.$$

It is easy to check that $(rD^{\lambda}_{T,S})^n = (k_\lambda r_{S,T,\lambda})^{n-1} rD^{\lambda}_{T,S}$. Applying τ on both sides of this equation, we get $\tau((rD^{\lambda}_{T,S})^n) = (k_\lambda r_{S,T,\lambda})^{n-1} r_{S,T,\lambda}$. If $k_\lambda \neq 0$, then $\tau((rD^{\lambda}_{T,S})^n) \neq 0$. Hence $rD^{\lambda}_{T,S}$ is not nilpotent and so $rD^{\lambda}_{T,S} \notin \operatorname{rad} A$, a contradiction. This implies that $k_\lambda = 0$. ∎

Example 4.5. Consider the group algebra $\mathbb{F}_3 S_3$, where $\mathbb{F}_3$ is the field of integers modulo 3. The algebra has a basis

$$\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}.$$

A cellular basis is

$$\begin{aligned}
C^{(3)}_{1,1} &= 1 + s_1 + s_2 + s_1s_2 + s_2s_1 + s_1s_2s_1,\\
C^{(2,1)}_{1,1} &= 1 + s_1, \qquad & C^{(2,1)}_{1,2} &= s_2 + s_1s_2,\\
C^{(2,1)}_{2,1} &= s_2 + s_2s_1, \qquad & C^{(2,1)}_{2,2} &= 1 + s_1s_2s_1,\\
C^{(1^3)}_{1,1} &= 1.
\end{aligned}$$

The corresponding dual basis is

$$\begin{aligned}
D^{(3)}_{1,1} &= -s_2 + s_1s_2 + s_2s_1,\\
D^{(2,1)}_{1,1} &= s_1 + s_2 - s_1s_2 - s_2s_1, \qquad & D^{(2,1)}_{2,1} &= s_2 - s_1s_2,\\
D^{(2,1)}_{1,2} &= s_2 - s_2s_1, \qquad & D^{(2,1)}_{2,2} &= s_2 - s_1s_2 - s_2s_1 + s_1s_2s_1,\\
D^{(1^3)}_{1,1} &= 1 - s_1 - s_2 + s_1s_2 + s_2s_1 - s_1s_2s_1.
\end{aligned}$$

It is easy to see that $\Lambda_3 = (3)$ and $\Lambda_1 = (1^3)$. Thus $\dim \operatorname{rad} A = 4$. Now we compute I. We have

$$\begin{aligned}
C_{1,1}^{(3)} D_{1,1}^{(3)} &= 1 + s_1 + s_2 + s_1s_2 + s_2s_1 + s_1s_2s_1,\\
C_{1,2}^{(2,1)} D_{2,1}^{(2,1)} &= 1 + s_1 - s_2 - s_1s_2s_1,\\
C_{1,2}^{(2,1)} D_{2,2}^{(2,1)} &= s_2 + s_1s_2 - s_2s_1 - s_1s_2s_1,\\
C_{2,1}^{(2,1)} D_{1,2}^{(2,1)} &= 1 - s_1 - s_1s_2 + s_1s_2s_1,\\
C_{2,1}^{(2,1)} D_{1,1}^{(2,1)} &= s_2 + s_2s_1 - s_1 - s_1s_2.
\end{aligned}$$

Thus $\dim I = 4$. This implies that $I = \operatorname{rad} A$.

Of course, for a symmetric cellular algebra A, the ideal I may not be equal to $\operatorname{rad} A$. Here is an example.

EXAMPLE 4.6. Let K be a field and Q be the quiver

$$\begin{tikzcd}
\underset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \underset{2}{\bullet} \arrow[l, shift left, "\alpha_1'"] \arrow[r, shift left, "\alpha_2"] & \underset{3}{\bullet} \arrow[l, shift left, "\alpha_2'"]
\end{tikzcd}$$

with relations ρ given as follows:

- all paths of length ≥ 3;
- $\alpha_1'\alpha_1 - \alpha_2\alpha_2'$;
- $\alpha_1\alpha_2$, $\alpha_2'\alpha_1'$.

Let $A = K(Q, \rho)$. Define τ by

$$\begin{aligned}
\tau(e_1) = \tau(e_2) &= \tau(e_3) = 0;\\
\tau(\alpha_i\alpha_i') = \tau(\alpha_i'\alpha_i) &= 1, \qquad i = 1, 2;\\
\tau(\alpha_i) = \tau(\alpha_i') &= 0.
\end{aligned}$$

Then A is a symmetric cellular algebra with a cellular basis

$$e_1; \quad \begin{matrix} \alpha_1\alpha_1' & \alpha_1 \\ \alpha_1' & e_2 \end{matrix}; \quad \begin{matrix} \alpha_2\alpha_2' & \alpha_2 \\ \alpha_2' & e_3 \end{matrix}.$$

The dual basis is

$$\alpha_1\alpha_1'; \quad \begin{matrix} e_1 & \alpha_1' \\ \alpha_1 & \alpha_1'\alpha_1 \end{matrix}; \quad \begin{matrix} e_2 & \alpha_2' \\ \alpha_2 & \alpha_2'\alpha_2 \end{matrix}.$$

It is easy to see that $\dim(\operatorname{rad} A) = 6$ and $\dim I = 2$.

As a by-product of the results on radicals, we will give some equivalent conditions for a finite-dimensional symmetric cellular algebra to be semisimple.

COROLLARY 4.7. *Let A be a finite-dimensional symmetric cellular algebra. Then the following are equivalent:*

(1) *A is semisimple.*

(2) $k_\lambda \neq 0$ *for all* $\lambda \in \Lambda$.
(3) $\{C^\lambda_{S,T} D^\lambda_{T,T} \mid \lambda \in \Lambda,\ S,T \in M(\lambda)\}$ *is a basis of* A.
(4) *For any* $\lambda \in \Lambda$, *there exist* $S,T \in M(\lambda)$ *such that* $(C^\lambda_{S,T} D^\lambda_{T,S})^2 \neq 0$.
(5) *For any* $\lambda \in \Lambda$ *and arbitrary* $S,T \in M(\lambda)$, $(C^\lambda_{S,T} D^\lambda_{T,S})^2 \neq 0$.

Proof. (2)⇒(1). If $k_\lambda \neq 0$ for all $\lambda \in \Lambda$, then $\operatorname{rad} \lambda = 0$ for all $\lambda \in \Lambda$ by Corollary 3.6. This implies that A is semisimple by Theorem 2.10.

(1)⇒(2). Assume that there exists some $\lambda \in \Lambda$ such that $k_\lambda = 0$. Then it is easy to check that I^λ is a nilpotent ideal of A. Obviously, $I^\lambda \neq 0$ because at least $C^\lambda_{U,V} D^\lambda_{V,U} \neq 0$. This implies that $I^\lambda \subseteq \operatorname{rad} A$. But A is semisimple, a contradiction. This implies that $k_\lambda \neq 0$ for all $\lambda \in \Lambda$.

(2)⇒(3). Suppose

$$\sum_{\lambda \in \Lambda,\, S,T \in M(\lambda)} k_{S,T,\lambda} C^\lambda_{S,T} D^\lambda_{T,T} = 0.$$

Take a maximal element $\lambda_0 \in \Lambda$. For arbitrary $X, Y \in M(\lambda_0)$,

$$C^{\lambda_0}_{X,X} D^{\lambda_0}_{X,Y} \Big(\sum_{\lambda \in \Lambda, S,T \in M(\lambda)} k_{S,T,\lambda} C^\lambda_{S,T} D^\lambda_{T,T} \Big) = k_{\lambda_0} \sum_{T \in M(\lambda_0)} k_{Y,T,\lambda_0} C^{\lambda_0}_{X,T} D^{\lambda_0}_{T,T} = 0.$$

This implies that

$$\tau\Big(k_{\lambda_0} \sum_{T \in M(\lambda_0)} k_{Y,T,\lambda_0} C^{\lambda_0}_{X,T} D^{\lambda_0}_{T,T} \Big) = 0,$$

i.e., $k_{\lambda_0} k_{Y,X,\lambda_0} = 0$. Since $k_{\lambda_0} \neq 0$, we get $k_{Y,X,\lambda_0} = 0$.

Repeating this process, we conclude that all the $k_{S,T,\lambda}$ are zeros.

(3)⇒(2). Since $\{C^\lambda_{S,T} D^\lambda_{T,T} \mid \lambda \in \Lambda,\ S,T \in M(\lambda)\}$ is a basis of A,

$$1 = \sum_{\lambda \in \Lambda,\, S,T \in M(\lambda)} k_{S,T,\lambda} C^\lambda_{S,T} D^\lambda_{T,T}.$$

For arbitrary $\mu \in \Lambda$ and $U, V \in M(\mu)$, we have

$$\begin{aligned} C^\mu_{U,V} D^\mu_{V,V} &= \sum_{\lambda \in \Lambda,\, S,T \in M(\lambda)} k_{S,T,\lambda} C^\lambda_{S,T} D^\lambda_{T,T} C^\mu_{U,V} D^\mu_{V,V} \\ &= k_\mu \sum_{X \in M(\mu)} k_{X,U,\mu} C^\mu_{X,V} D^\mu_{V,V}. \end{aligned}$$

This implies that $k_\mu \neq 0$ since $C^\mu_{U,V} D^\mu_{V,V} \neq 0$. As μ is arbitrary, this shows that $k_\lambda \neq 0$ for all $\lambda \in \Lambda$.

(2)⇔(4) and (2)⇔(5) are clear by Lemma 3.3. ∎

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